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# 时间尺度上非完整系统 Herglotz 型 Vacco 动力学及其 Noether 定理*

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**摘要**　时间尺度定义为实数域的任意非空闭子集,统一了连续与离散系统的处理方式.文章将非完整系统的 Herglotz 型 Vacco 动力学推广到时间尺度上,研究其 Noether 对称性与守恒律.首先,基于时间尺度上 Herglotz 变分原理,建立时间尺度上 Herglotz 型 Vacco 动力学方程.其次,根据时间尺度上的 Hamilton-Herglotz 作用量在无限小变换下的不变性,定义了时间尺度上非完整系统 Herglotz 型 Vacco 动力学的 Noether 对称性,给出相应的 Noether 等式.最后,证明时间尺度上非完整系统 Herglotz 型 Vacco 动力学的 Noether 定理并给出相应的守恒量.文末举例说明理论分析结果.

**关键词**　Vacco 动力学,　Noether 定理,　时间尺度,　Herglotz 变分原理

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# Herglotz-Type Vacco Dynamics and its Noether's Theorem for Nonholonomic Systems on Time Scales*

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**Abstract**　Time scales are defined as any non-empty closed subset of the real number field, which unifies the treatment of continuous and discrete systems. In this paper, the Herglotz-type Vacco dynamics of nonholonomic systems are extended to the time scales, and its Noether symmetry and conservation law are investigated. Firstly, based on the Herglotz variational principle on time scales, the Herglotz-type Vacco dynamics equations on time scales are established. Secondly, according to the invariance of Hamilton-Herglotz action on time scales under infinitesimal transformations, the Noether symmetry of Herglotz-type Vacco dynamics of nonholonomic systems on time scales is defined, and the corresponding Noether identities are presented. Finally, the Noether's theorem of Herglotz-type Vacco dynamics for nonholonomic systems on time scales is proven, and the corresponding conserved quantities are provided. At the conclusion of the paper, two examples are presented to demonstrate the results of theoretical analysis.

**Key words**　Vacco dynamics,　Noether's theorem,　time scale,　variational principle of Herglotz type

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## 引言

变分原理是分析力学的基础之一,主要分为积分变分原理与微分变分原理[1]. Herglotz 变分原理突破了经典变分原理仅适用于保守系统的局限,为处理一类非保守动力学问题提供了一个新方法[2]. Georgieva 等[3,4]提出含单个和多个独立变量的 Herglotz 变分原理及其 Noether 定理. 此后,Herglotz 型 Noether 定理被以多种形式推广,如含时滞系统[5,6]、分数阶系统[7,8]、Hamilton 系统[9,10]和 Birkhoff 系统[11,12]. 最近,文献[13,14]将 Herglotz 变分原理应用于非完整系统,给出了包含非保守非完整系统以及广义 Chaplygin 系统在内的 Herglotz 型 Noether 对称性及其守恒律. 此外, de León 等[15,16]将 Herglotz 变分原理作为一个最优控制问题来处理,利用 Pontryagin 最大值原理给出了广义 Euler-Lagrange 方程新的证明,并通过预辛几何和接触几何探讨了 Herglotz 最优控制理论及其与 Vacco 动力学的关系.

经典非完整力学一般以 d'Alembert-Lagrange 原理为基础,采用 Appell-Chetaev 模型处理受理想非完整约束的动力学建模[17]. 1982 年, Kozlov[18]通过对不可积分约束系统动力学的研究,导出了非完整系统中的一类新模型——Vacco 模型. 非完整系统中,Appell-Chetaev 模型与 Vacco 模型的并存引起了国内外学者的兴趣与讨论[19-22],进而促进了 Vacco 动力学及其对称性的研究与发展. 1993 年,张解放[23]研究了 Vacco 动力学的 Noether 定理. 随后,Vacco 动力学的 Lie 对称性[24]、Mei 对称性[25]、联合对称性[26]、高阶 Vacco 动力学[27]相继被研究. 最近,文献[28,29]对非完整系统的 Herglotz 型 Vacco 动力学进行了初步讨论,但仅涉及连续系统,关于 Herglotz 型离散 Vacco 动力学研究尚未见报道.

时间尺度上动力系统理论是连续系统和离散系统理论的统一和推广. 德国学者 Hilger[30]最早提出时间尺度理论. Bohner[31,32]对时间尺度上动力学方程与变分原理进行了系统的研究,给出了与时间尺度有关的基本概念和性质. Bartosiewicz 和 Torres[33]将 Noether 定理推广到任意时间尺度模型. Martins 和 Torres[34]运用 Caputo[35]提出的对偶原理,从 delta 导数出发推导出了基于 nabla 导数的时间尺度上 Noether 定理. Anerot 等[36]通过数值模拟发现 Bartosiewicz 和 Torres 等关于时间尺度上 Noether 守恒量的结果有误,并加以修正. 田雪等[37]研究了时间尺度上 Herglotz 变分原理及其 Noether 定理,进而推广至 Hamilton 力学[38]、Birkhoff 力学[39]. 然而, 这些研究都仅限于完整约束力学系统,关于时间尺度上非完整系统 Herglotz 型 Noether 定理的研究仍是一个开放的课题.

本文研究时间尺度上非完整约束 Vacco 动力学系统, 将建立时间尺度上的 Herglotz 型 Vacco 动力学方程, 同时利用 Herglotz 型 Noether 对称变换的定义, 证明时间尺度上非完整系统 Herglotz 型 Vacco 动力学的 Noether 等式与 Noether 定理,给出相应的守恒量.

## 1　预备知识

下面简单介绍一些时间尺度上的基本概念与性质,关于时间尺度的具体介绍可以参考文献[31].

设 $\mathbb{T}$是时间尺度, $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$ 是前跳算子,$\rho(t)=\sup\{s\in\mathbb{T}:s<t\}$ 是后跳算子; $\mu(t)=\sigma(t)-t$ 和 $\nu(t)=t-\rho(t)$ 是步差函数.

如果 $\mathbb{T}=\infty$,定义$\mathbb{T}^k=\mathbb{T}$;如果 $\sup\mathbb{T}<\infty$,则定义$\mathbb{T}^k=\mathbb{T}/[\rho(\sup\mathbb{T}),\sup\mathbb{T}]$.

设函数 $f:\mathbb{T}\to\mathbb{R}$,$t\in\mathbb{T}$,定义 $f^\sigma:\mathbb{T}\to\mathbb{R}$ 为 $f^\sigma(t)=f[\sigma(t)]$. 若对任意 $\varepsilon>0$,存在 $t$ 邻域 $U=(t-\delta,t+\delta)\cap\mathbb{T}$,使得对所有的 $s\in U$ 有 $|f[\sigma(t)]-f(s)-f^\Delta(t)[\sigma(t)-s]|\leqslant\varepsilon|\sigma(t)-s|$ 成立,则称函数 $f$ 在点 $t$ 处是 $\Delta$ 可微的,记作 $f^\Delta$. 如果对所有 $t\in\mathbb{T}^k$, $f^\Delta$ 都存在,则称 $f$ 在$\mathbb{T}^k$上是 $\Delta$ 可微的,且有 $f^\sigma(t)=f(t)+\mu(t)f^\Delta(t)$.

记 $\mathbb{T}$上 $\Delta$ 可微且其 $\Delta$ 导数右稠密连续的函数集合为 $C^1_{\mathrm{rd}}$.

设函数 $F:\mathbb{T}\to\mathbb{R}$ 为 $f:\mathbb{T}\to\mathbb{R}$ 的一个原函数,且对所有的 $t\in\mathbb{T}^k$,有 $F^\Delta(t)=f(t)$,那么$\int f(t)\Delta t=F(t)+c$,其中 $c$ 是任意常数;$f$ 的定积分定义为

$$\int_a^b f(t)\Delta t=F(b)-F(a)$$

其中 $a,b\in\mathbb{T}$,$a<b$.

定义指数函数为

$$e_p(t,s)=\exp\left\{\int_s^t \xi_{\mu(\theta)}[p(\theta)]\Delta\theta\right\}$$

其中, $p\in R(\mathbb{T})$,$\xi_h(x)=\dfrac{1}{h}\mathrm{Log}(1+xh)$,$x\in$

$\mathbb{C}_h$，$\mathbb{C}_h=\left\{x\in\mathbb{C}:x\neq-\dfrac{1}{h}\right\}$，Log 是主对数函数. 特别地，当 $h=0$，定义 $\xi_0(x)=x, x\in\mathbb{C}$.

**引理 1**[31]： 设 $f,g:\mathbb{T}\to\mathbb{R}$，$t\in\mathbb{T}^k$ 是 Δ 可微的，对于 $a,b\in\mathbb{T}$，时间尺度上 Δ 导数有以下性质：

$$(f+g)^{\Delta}(t)=f^{\Delta}(t)+g^{\Delta}(t) \tag{1}$$

$$(kf)^{\Delta}(t)=kf^{\Delta}(t) \tag{2}$$

$$\begin{aligned}(fg)^{\Delta}(t)&=f^{\Delta}(t)g(t)+f^{\sigma}(t)g^{\Delta}(t)\\&=f^{\Delta}(t)g^{\sigma}(t)+f(t)g^{\Delta}(t)\end{aligned} \tag{3}$$

$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g^{\sigma}(t)} \tag{4}$$

**引理 2**[31]： 若 $t,s,r\in\mathbb{T}$，$t_0$ 固定且 $t_0\in\mathbb{T}$，指数函数的性质有

$$e_p^{\Delta}(t,t_0)=p(t)e_p(t,t_0) \tag{5}$$

$$\begin{aligned}e_p^{\sigma}(t,s)&=e_p[\sigma(t),s]\\&=[1+\mu(t)p(t)]e_p(t,s)\end{aligned} \tag{6}$$

$$\frac{1}{e_p(t,s)}=e_p(s,t) \tag{7}$$

$$e_p(t,s)e_p(s,y)=e_p(t,y) \tag{8}$$

$$[e_p(s,t)]^{\Delta}=\frac{-p}{e_p^{\sigma}(t,s)} \tag{9}$$

**引理 3**[31]： 假设 $e_p(t,s)$ 是回归的，令 $t_0\in\mathbb{T}$，$y_0\in\mathbb{R}$，则初值问题

$$y^{\Delta}=p(t)y+f(t),\ y(t_0)=y_0 \tag{10}$$

存在唯一解

$$y(t)=e_p(t,t_0)y_0+\int_{t_0}^{t}e_p[t,\sigma(\theta)]f(\theta)\Delta\theta \tag{11}$$

**引理 4**[31]： （Dubois-Reymond 引理）令 $g\in C_{\mathrm{rd}}$，$g:[a,b]\to\mathbb{R}^n$，如果对一切 $\eta\in C_{\mathrm{rd}}^1$，且有 $\eta(a)=\eta(b)=0$，都成立 $\int_a^b g^{\mathrm{T}}\eta^{\Delta}(t)\Delta t=0$，则 $g(t)=c$，其中常数 $c\in\mathbb{R}^n$.

关于 Δ 导数与等时变分，有关系[37]

$$\frac{\Delta}{\Delta t}(\delta q)=\delta\frac{\Delta}{\Delta t}(q)=\delta q^{\Delta},(\delta q)^{\sigma}=\delta q^{\sigma} \tag{12}$$

等时变分与非等时变分的关系为

$$\tilde{\Delta}q=\delta q+q^{\Delta}\tilde{\Delta}t \tag{13}$$

其中 $\tilde{\Delta}$ 表示时间尺度上的非等时变分.

## 2 时间尺度上 Herglotz 型 Vacco 动力学方程

设力学系统的位形由 $n$ 个广义坐标 $q_s(s=1,2,\cdots,n)$ 确定，运动受有 $g$ 个彼此独立的 Vacco 型非完整约束

$$\varphi_{\beta}[t,q_s^{\sigma}(t),q_s^{\Delta}(t)]=0$$
$$(\beta=1,2,\cdots,g;s=1,2,\cdots,n) \tag{14}$$

时间尺度上非完整系统 Vacco 动力学的 Herglotz 变分问题可表述为：由微分方程

$$\begin{aligned}z^{\Delta}(t)=&L[t,q_s^{\sigma}(t),q_s^{\Delta}(t),z(t)]+\\&\lambda_{\beta}(t)\varphi_{\beta}[t,q_s^{\sigma}(t),q_s^{\Delta}(t)]\end{aligned} \tag{15}$$

定义的泛函 $z(t)$，在给定的边界条件

$$q_s(t)\,|_{t=a}=q_{sa},\ q_s(t)\,|_{t=b}=q_{sb} \tag{16}$$

和初始条件

$$z(t)\,|_{t=a}=z_a \tag{17}$$

下，$z(b)\to$extr.. 其中泛函 $z(t)$ 是时间尺度上的 Hamilton-Herglotz 作用量，$L:\mathbb{T}^k\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}$ 为时间尺度上 Herglotz 型 Lagrange 函数，$\lambda_{\beta}(t)$ 为 Lagrange 乘子，$q_s^{\sigma}(t)=(q_s\circ\sigma)(t)$，$t\in\mathbb{T}$，$q_s^{\Delta}(t)$ 为广义速度，$q_{sa}$、$q_{sb}$ 和 $z_a$ 均为固定常数，上述函数均属于 $C_{\mathrm{rd}}^1(\mathbb{T})$.

对方程(15)进行等时变分运算，有

$$\begin{aligned}\delta z^{\Delta}=&\frac{\partial L}{\partial q_s^{\sigma}}\delta q_s^{\sigma}+\frac{\partial L}{\partial q_s^{\Delta}}\delta q_s^{\Delta}+\frac{\partial L}{\partial z}\delta z+\\&\lambda_{\beta}\left(\frac{\partial\varphi_{\beta}}{\partial q_s^{\sigma}}\delta q_s^{\sigma}+\frac{\partial\varphi_{\beta}}{\partial q_s^{\Delta}}\delta q_s^{\Delta}\right)\end{aligned} \tag{18}$$

利用时间尺度上 Δ 导数与等时变分的关系式

$$\frac{\Delta}{\Delta t}(\delta q_s)=\delta q_s^{\Delta} \tag{19}$$

式(18)可进一步写成

$$(\delta z)^{\Delta}=\Omega+\frac{\partial L}{\partial z}\delta z \tag{20}$$

其中

$$\begin{aligned}\Omega=&\frac{\partial L}{\partial q_s^{\sigma}}\delta q_s^{\sigma}+\frac{\partial L}{\partial q_s^{\Delta}}(\delta q_s)^{\Delta}+\\&\lambda_{\beta}\left[\frac{\partial\varphi_{\beta}}{\partial q_s^{\sigma}}\delta q_s^{\sigma}+\frac{\partial\varphi_{\beta}}{\partial q_s^{\Delta}}(\delta q_s)^{\Delta}\right]\end{aligned} \tag{21}$$

令 $p=\dfrac{\partial L}{\partial z}$，由引理 3 知方程(20)的解为

$$\begin{aligned}\delta z(t)=&\delta z(a)e_p(t,a)+\int_a^t e_p[t,\sigma(\theta)]\left\{\frac{\partial L}{\partial q_s^{\sigma}}\delta q_s^{\sigma}+\right.\\&\left.\frac{\partial L}{\partial q_s^{\Delta}}(\delta q_s)^{\Delta}+\lambda_{\beta}\left[\frac{\partial\varphi_{\beta}}{\partial q_s^{\sigma}}\delta q_s^{\sigma}+\frac{\partial\varphi_{\beta}}{\partial q_s^{\Delta}}(\delta q_s)^{\Delta}\right]\right\}\Delta\theta\end{aligned} \tag{22}$$

由初始条件(17)与指数函数性质(5)～(9)，式(22)进一步写成

$$\delta z(t)=e_p(t,a)\int_a^t e_p[a,\sigma(\theta)]\left\{\frac{\partial L}{\partial q_s^{\Delta}}(\delta q_s)^{\Delta}+\right.$$

$$\frac{\partial L}{\partial q_s^\sigma}\delta q_s^\sigma+\lambda_\beta\left[\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\delta q_s^\sigma+\frac{\partial\varphi_\beta}{\partial q_s^\Delta}(\delta q_s)^\Delta\right]\Bigg\}\Delta\theta \tag{23}$$

令 $t=b$,考虑到 $z(b)\to\text{extr.}$,可得

$$\int_a^b e_p[a,\sigma(t)]\left[\left(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\right)\delta q_s^\sigma+\left(\frac{\partial L}{\partial q_s^\Delta}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\right)(\delta q_s)^\Delta\right]\Delta t=0 \tag{24}$$

对式(24)中含 $q_s^\sigma$ 的项进行分部积分,且考虑边界条件(16)得

$$\begin{aligned}&\int_a^b e_p[a,\sigma(t)]\left(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\right)\delta q_s^\sigma\Delta t\\&=\int_a^b\left(\left(\left\{\int_a^t e_p[a,\sigma(\theta)]\left(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\right)\Delta\theta\right\}\delta q_s\right)^\Delta-\left\{\int_a^t e_p[a,\sigma(\theta)]\left(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\right)\Delta\theta\right\}(\delta q_s)^\Delta\right)\Delta t\\&=\left(\left\{\int_a^t e_p[a,\sigma(\theta)]\left(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\right)\Delta\theta\right\}\delta q_s\right)\Bigg|_a^b-\int_a^b\left\{\int_a^t e_p[a,\sigma(\theta)]\left(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\right)\Delta\theta\right\}(\delta q_s)^\Delta\Delta t\\&=-\int_a^b\left\{\int_a^t e_p[a,\sigma(\theta)]\left(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\right)\Delta\theta\right\}(\delta q_s)^\Delta\Delta t\end{aligned} \tag{25}$$

将式(25)代入式(24)得

$$0=\int_a^b\left(e_p[a,\sigma(t)]\left(\frac{\partial L}{\partial q_s^\Delta}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\right)-\left\{\int_a^t e_p[a,\sigma(\theta)]\left(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\right)\Delta\theta\right\}\right)(\delta q_s)^\Delta\Delta t \tag{26}$$

由引理 4 知

$$e_p[a,\sigma(t)]\left(\frac{\partial L}{\partial q_s^\Delta}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\right)-\int_a^t e_p[a,\sigma(\theta)]\left(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\right)\Delta\theta=c \tag{27}$$

其中 $c$ 为常数.

对式(27)两边关于 $t$ 求 $\Delta$ 导数可得

$$\frac{\Delta}{\Delta t}\left\{e_p[a,\sigma(t)]\left(\frac{\partial L}{\partial q_s^\Delta}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\right)\right\}-e_p[a,\sigma(t)]\left(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\right)=0 \tag{28}$$

称方程(28)为时间尺度上非完整约束系统 Vacco 动力学的 Herglotz 型方程.

**注:** 若取$\mathbb{T}=\mathbb{R}$,则有前跳算子,$\sigma(t)=t$,步差函数 $\mu(t)=\sigma(t)-t=0$,指数函数 $e_p[a,\sigma(t)]=\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)$,于是式(28)给出经典意义下非完整约束系统 Vacco 动力学的 Herglotz 型方程[29]

$$\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot q_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial\dot q_s}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s}-\lambda_\beta\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\varphi_\beta}{\partial\dot q_s}-\dot\lambda_\beta\frac{\partial\varphi_\beta}{\partial\dot q_s}+\lambda_\beta\frac{\partial L}{\partial z}\frac{\partial\varphi_\beta}{\partial\dot q_s}\right)=0\ (s=1,2,\cdots,n) \tag{29}$$

## 3　时间尺度上 Herglotz 型 Vacco 动力学的 Noether 定理

引入时间尺度上的单参数无限小变换

$$t^*=t+\tilde\Delta t,\ q_s^*(t^*)=q_s(t)+\tilde\Delta q_s \tag{30}$$

或其展开式

$$\begin{aligned}&t^*=t+\varepsilon\tau(t,q_k^\sigma,q_k^\Delta,z),\\&q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,q_k^\sigma,q_k^\Delta,z)\\&(s,k=1,2,\cdots,n)\end{aligned} \tag{31}$$

其中 $\varepsilon$ 为无限小参数,$\tau$、$\xi_s$ 为无限小变换的生成元.

在无限小变换(30)的作用下,Hamilton-Herglotz 作用量相应地变成

$$z^*(t^*)=z(t)+\tilde\Delta z(t) \tag{32}$$

**定义 1:** 对于时间尺度上非完整系统 Vacco 动力学的 Herglotz 变分问题,如果作用量 $z$ 在变换(30)作用下,对于任意的子区间 $[t_a,t_b]\subseteq[a,b]$ 成立

$$\tilde\Delta z(t_b)=0 \tag{33}$$

则称变换(30)为时间尺度上非完整系统 Herglotz 型 Vacco 动力学的 Noether 对称变换.

**定理 1**(Noether 等式): 对于由式(28)和式(14)确定的时间尺度上非完整系统 Herglotz 型 Vacco 动力学,作用量 $z$ 在变换(30)作用下保持不变的充分必要条件为对一切 $t\in[a,b]$成立

$$\left(\frac{\partial L}{\partial t}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial t}+\lambda_\beta^\Delta\varphi_\beta^\sigma\right)\tau+\left(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\right)\xi_s^\sigma+\left(\frac{\partial L}{\partial q_s^\Delta}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\right)\xi_s^\Delta+\left[(L+\lambda_\beta\varphi_\beta)^\sigma-\left(\frac{\partial L}{\partial q_s^\Delta}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\right)q_s^{\Delta\sigma}\right]\tau^\Delta=0 \tag{34}$$

**证明:** 对式(15)进行非等时变分运算

$$\tilde\Delta z^\Delta=\frac{\partial L}{\partial t}\tilde\Delta t+\frac{\partial L}{\partial q_s^\sigma}\tilde\Delta q_s^\sigma+\frac{\partial L}{\partial q_s^\Delta}\tilde\Delta q_s^\Delta+\frac{\partial L}{\partial z}\tilde\Delta z+\lambda_\beta^\Delta\varphi_\beta^\sigma\tilde\Delta t+\lambda_\beta\left(\frac{\partial\varphi_\beta}{\partial t}\tilde\Delta t+\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\tilde\Delta q_s^\sigma+\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\tilde\Delta q_s^\Delta\right) \tag{35}$$

由式(12)与式(13),易得关系式

$$\tilde{\Delta} q_s^\Delta = \frac{\Delta}{\Delta t}(\tilde{\Delta} q_s) - q_s^{\Delta\sigma} \frac{\Delta}{\Delta t}(\tilde{\Delta} t),$$

$$\tilde{\Delta} z^\Delta = \frac{\Delta}{\Delta t}(\tilde{\Delta} z) - z^{\Delta\sigma} \frac{\Delta}{\Delta t}(\tilde{\Delta} t) \tag{36}$$

则式(35)可进一步写成

$$\frac{\Delta}{\Delta t}(\tilde{\Delta} z) = \frac{\partial L}{\partial t}\tilde{\Delta} t + \frac{\partial L}{\partial q_s^\sigma}\tilde{\Delta} q_s^\sigma + \frac{\partial L}{\partial q_s^\Delta}\left[\frac{\Delta}{\Delta t}(\tilde{\Delta} q_s) - q_s^{\Delta\sigma}\frac{\Delta}{\Delta t}(\tilde{\Delta} t)\right] + \frac{\partial L}{\partial z}\tilde{\Delta} z + \lambda_\beta^\Delta \varphi_\beta^\sigma \tilde{\Delta} t + \lambda_\beta\left\{\frac{\partial \varphi_\beta}{\partial t}\tilde{\Delta} t + \frac{\partial \varphi_\beta}{\partial q_s^\sigma}\tilde{\Delta} q_s^\sigma + \frac{\partial \varphi_\beta}{\partial q_s^\Delta}\left[\frac{\Delta}{\Delta t}(\tilde{\Delta} q_s) - q_s^{\Delta\sigma}\frac{\Delta}{\Delta t}(\tilde{\Delta} t)\right]\right\} + (L + \lambda_\beta \varphi_\beta)^\sigma \frac{\Delta}{\Delta t}(\tilde{\Delta} t) \tag{37}$$

考虑初值条件(17),解上述方程得

$$\tilde{\Delta} z = e_p(t,t_a)\int_{t_a}^{t} e_p[t_a,\sigma(\theta)]\left\{\left(\frac{\partial L}{\partial \theta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial \theta}\right)\tilde{\Delta}\theta + \left(\frac{\partial L}{\partial q_s^\sigma} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\sigma}\right)\tilde{\Delta} q_s^\sigma + \left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)\left[\frac{\Delta}{\Delta \theta}(\tilde{\Delta} q_s) - q_s^{\Delta\sigma}\frac{\Delta}{\Delta \theta}(\tilde{\Delta}\theta)\right] + \lambda_\beta^\Delta\varphi_\beta^\sigma\tilde{\Delta}\theta + (L+\lambda_\beta\varphi_\beta)^\sigma\frac{\Delta}{\Delta\theta}(\tilde{\Delta}\theta)\right\}\Delta\theta \tag{38}$$

由定义5,当 $t=t_b$ 时,有

$$0 = \int_{t_a}^{t_b} e_p[t_a,\sigma(t)]\left\{\left(\frac{\partial L}{\partial t} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial t}\right)\tilde{\Delta} t + \left(\frac{\partial L}{\partial q_s^\sigma} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\sigma}\right)\tilde{\Delta} q_s^\sigma + \left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)\times\left[\frac{\Delta}{\Delta t}(\tilde{\Delta} q_s) - q_s^{\Delta\sigma}\frac{\Delta}{\Delta t}(\tilde{\Delta} t)\right] + \lambda_\beta^\Delta\varphi_\beta^\sigma\tilde{\Delta} t + (L+\lambda_\beta\varphi_\beta)^\sigma\frac{\Delta}{\Delta t}(\tilde{\Delta} t)\right\}\Delta t \tag{39}$$

将式(31)代入式(39),整理得

$$0 = \int_{t_a}^{t_b} \varepsilon\left\{e_p[t_a,\sigma(t)]\left[\left(\frac{\partial L}{\partial t} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial t} + \lambda_\beta^\Delta\varphi_\beta^\sigma\right)\tau + \left(\frac{\partial L}{\partial q_s^\sigma} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\sigma}\right)\xi_s^\sigma + \left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)\xi_s^\Delta + (L+\lambda_\beta\varphi_\beta)^\sigma\tau^\Delta - \left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)q_s^{\Delta\sigma}\tau^\Delta\right]\right\}\Delta t \tag{40}$$

考虑积分区间的任意性得

$$0 = e_p[t_a,\sigma(t)]\left\{\left(\frac{\partial L}{\partial t} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial t} + \lambda_\beta^\Delta\varphi_\beta^\sigma\right)\tau + \left(\frac{\partial L}{\partial q_s^\sigma} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\sigma}\right)\xi_s^\sigma + \left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)\xi_s^\Delta + \left[(L+\lambda_\beta\varphi_\beta)^\sigma - \left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)q_s^{\Delta\sigma}\right]\tau^\Delta\right\} \tag{41}$$

即

$$\left(\frac{\partial L}{\partial t} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial t} + \lambda_\beta^\Delta\varphi_\beta^\sigma\right)\tau + \left(\frac{\partial L}{\partial q_s^\sigma} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\sigma}\right)\xi_s^\sigma + \left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)\xi_s^\Delta + \left[(L+\lambda_\beta\varphi_\beta)^\sigma - \left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)q_s^{\Delta\sigma}\right]\tau^\Delta = 0 \tag{42}$$

定理得证.

称式(34)为时间尺度上非完整系统 Herglotz 型 Vacco 动力学的 Noether 等式.

**定理2**(Noether 定理): 对于由式(28)和式(14)确定的时间尺度上非完整系统 Herglotz 型 Vacco 动力学,如果变换(30)是 Noether 对称的,则系统存在守恒量

$$I_N = \int_{t_a}^{t}\left(e_p[t_a,\sigma(\theta)]\left\{(L+\lambda_\beta\varphi_\beta)^\sigma\tau^\Delta + \left[\left(\frac{\partial L}{\partial \theta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial \theta} + \lambda_\beta^\Delta\varphi_\beta^\sigma\right) + \left(\frac{\partial L}{\partial q_s^\sigma} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\sigma}\right)q_s^\Delta + \left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)q_s^{\Delta\Delta}\right]\tau\right\} + \frac{\Delta}{\Delta\theta}\left\{e_p[t_a,\sigma(\theta)]\times\left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)\right\}\mu(\theta)\frac{\Delta}{\Delta\theta}(q_s^\Delta\tau)\right)\Delta\theta + e_p[t_a,\sigma(t)]\left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)(\xi_s - q_s^\Delta\tau) = \text{const.} \tag{43}$$

证明:

$$\frac{\Delta}{\Delta t}I_N = e_p[t_a,\sigma(t)]\left\{(L+\lambda_\beta\varphi_\beta)^\sigma\tau^\Delta + \left[\left(\frac{\partial L}{\partial t} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial t} + \lambda_\beta^\Delta\varphi_\beta^\sigma\right) + \left(\frac{\partial L}{\partial q_s^\sigma} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\sigma}\right)q_s^\Delta + \left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)q_s^{\Delta\Delta}\right]\tau\right\} + \frac{\Delta}{\Delta t}\left\{e_p[t_a,\sigma(t)]\left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)\right\}\mu(t)\frac{\Delta}{\Delta t}(q_s^\Delta\tau) + \frac{\Delta}{\Delta t}\left\{e_p[t_a,\sigma(t)]\left(\frac{\partial L}{\partial q_s^\Delta} + \lambda_\beta\frac{\partial \varphi_\beta}{\partial q_s^\Delta}\right)(\xi_s - q_s^\Delta\tau)\right\} \tag{44}$$

利用关系

$$(fg)^\Delta(t) = f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t),$$

$$f^{\sigma}(t)=f(t)+\mu(t)f^{\Delta}(t) \tag{45}$$

可得

$$\begin{aligned}
\frac{\Delta}{\Delta t}I_N = & e_p[t_a,\sigma(t)]\Big\{(L+\lambda_\beta\varphi_\beta)^\sigma\tau^\Delta+\tau\Big[\Big(\frac{\partial L}{\partial t}+\\
&\lambda_\beta\frac{\partial\varphi_\beta}{\partial t}+\lambda_\beta^\Delta\varphi_\beta^\sigma\Big)+\Big(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\Big)q_s^\Delta+\\
&\Big(\frac{\partial L}{\partial q_s^\Delta}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\Big)q_s^{\Delta\Delta}\Big]\Big\}+\frac{\Delta}{\Delta t}\Big\{e_p[t_a,\sigma(t)]\Big(\frac{\partial L}{\partial q_s^\Delta}+\\
&\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\Big)\Big\}\times[(q_s^\Delta\tau)^\sigma-q_s^\Delta\tau]+\\
&\frac{\Delta}{\Delta t}\Big\{e_p[t_a,\sigma(t)]\Big(\frac{\partial L}{\partial q_s^\Delta}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\Big)\Big\}\times\\
&[\xi_s^\sigma-(q_s^\Delta\tau)^\sigma]+e_p[t_a,\sigma(t)]\Big(\frac{\partial L}{\partial q_s^\Delta}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\Big)\times\\
&(\xi_s^\Delta-q_s^{\Delta\Delta}\tau-q_s^{\Delta\sigma}\tau^\Delta)\\
=&e_p[t_a,\sigma(t)]\Big[\Big(\frac{\partial L}{\partial t}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial t}+\lambda_\beta^\Delta\varphi_\beta^\sigma\Big)\tau+\\
&\Big(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\Big)\xi_s^\sigma+\Big(\frac{\partial L}{\partial q_s^\Delta}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\Big)\xi_s^\Delta+\\
&(L+\lambda_\beta\varphi_\beta)^\sigma\tau^\Delta-\Big(\frac{\partial L}{\partial q_s^\Delta}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\Big)q_s^{\Delta\sigma}\tau^\Delta\Big]+\\
&\Big(\frac{\Delta}{\Delta t}\Big\{e_p[t_a,\sigma(t)]\Big(\frac{\partial L}{\partial q_s^\Delta}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\Delta}\Big)\Big\}-\\
&e_p[t_a,\sigma(t)]\Big(\frac{\partial L}{\partial q_s^\sigma}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s^\sigma}\Big)\Big)(\xi_s^\sigma-q_s^\Delta\tau)
\end{aligned} \tag{46}$$

由式(34)和定理 1 知

$$\frac{\Delta}{\Delta t}I_N=0 \tag{47}$$

两边同时积分，定理得证.

**注：** 若取 $\mathbb{T}=\mathbb{R}$，则有前跳算子 $\sigma(t)=t$，步差函数 $\mu(t)=\sigma(t)-t=0$，指数函数 $e_p[a,\sigma(t)]=\exp\Big(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\Big)$，于是式(34)给出经典意义下非完整系统 Herglotz 型 Vacco 动力学的 Noether 等式[29]

$$\begin{aligned}
&\Big[\Big(\frac{\partial L}{\partial t}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial t}\Big)\tau+\Big(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial q_s}\Big)\xi_s+\dot\lambda_\beta\varphi_\beta\tau+\\
&\Big(\frac{\partial L}{\partial\dot q_s}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial\dot q_s}\Big)(\dot\xi_s-\dot q_s\dot\tau)+(L+\lambda_\beta\varphi_\beta)\dot\tau\Big]\\
&=0
\end{aligned} \tag{48}$$

而守恒量式(43)则成为经典意义下非完整系统 Herglotz 型 Vacco 动力学的 Noether 守恒量[29]

$$\begin{aligned}
I_N=&\exp\Big(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\Big)\Big[(L+\lambda_\beta\varphi_\beta)\tau+\\
&\Big(\frac{\partial L}{\partial\dot q_s}+\lambda_\beta\frac{\partial\varphi_\beta}{\partial\dot q_s}\Big)(\xi_s-\dot q_s\tau)\Big]=\text{const.}
\end{aligned} \tag{49}$$

## 4 算例

**例 1** 在时间尺度上研究 Appell-Hamel 例，并考虑系统受黏滞阻尼作用，阻尼系数取为 1. Herglotz 型 Lagrange 函数为

$$L=\frac{1}{2}m[(q_1^\Delta)^2+(q_2^\Delta)^2+(q_3^\Delta)^2]-mgq_3^\sigma-z \tag{50}$$

运动受到的非线性非完整约束为

$$\varphi=(q_3^\Delta)^2-c[(q_1^\Delta)^2+(q_2^\Delta)^2]=0 \tag{51}$$

其中 $c$ 是一个常数.

将式(50)和式(51)代入方程(15)中，则泛函 $z$ 满足如下方程：

$$\begin{aligned}
z^\Delta=&\frac{1}{2}m[(q_1^\Delta)^2+(q_2^\Delta)^2+(q_3^\Delta)^2]-mgq_3^\sigma-\\
&z+\lambda[(q_3^\Delta)^2-c(q_1^\Delta)^2-c(q_2^\Delta)^2]
\end{aligned} \tag{52}$$

根据时间尺度上 Vacco 动力学的 Herglotz 型方程(28)，可得

$$\begin{aligned}
&\frac{\Delta}{\Delta t}\{e_p[a,\sigma(t)]\times(m-2c\lambda)q_1^\Delta\}=0,\\
&\frac{\Delta}{\Delta t}\{e_p[a,\sigma(t)]\times(m-2c\lambda)q_2^\Delta\}=0,\\
&\frac{\Delta}{\Delta t}\{e_p[a,\sigma(t)]\times(m+2\lambda)q_3^\Delta\}+\\
&\quad e_p[a,\sigma(t)]mg=0
\end{aligned} \tag{53}$$

其中 $p=\frac{\partial L}{\partial z}=-1$.

Noether 等式(34)给出

$$\begin{aligned}
&\lambda^\Delta[(q_3^\Delta)^2-c(q_1^\Delta)^2-c(q_2^\Delta)^2]^\sigma\tau-mg\xi_3^\sigma+\\
&(m-2c\lambda)(q_1^\Delta\xi_1^\Delta+q_2^\Delta\xi_2^\Delta)+(m+2c\lambda)q_3^\Delta\xi_3^\Delta+\\
&\frac{1}{2}m[(q_1^\Delta)^2+(q_2^\Delta)^2+(q_3^\Delta)^2]^\sigma\tau^\Delta-mgq_3^{\sigma\sigma}\tau^\Delta-\\
&z^\sigma\tau^\Delta+[\lambda(q_3^\Delta)^2-c\lambda(q_1^\Delta)^2-c\lambda(q_2^\Delta)^2]^\sigma\tau^\Delta-\\
&[(m-2c\lambda)q_1^\Delta q_1^{\Delta\sigma}+(m-2c\lambda)q_2^\Delta q_2^{\Delta\sigma}+\\
&(\frac{1}{2}m+c\lambda)(q_3^\Delta+q_3^{\Delta\sigma})q_3^{\Delta\sigma}]\tau^\Delta=0
\end{aligned} \tag{54}$$

上述方程有解：

$$\tau=0,\ \xi_1=1,\ \xi_2=1,\ \xi_3=0 \tag{55}$$

根据定理 2，将式(55)代入式(43)，可得系统相应的 Noether 型守恒量为

$$\begin{aligned}
I_N=&e_p[t_a,\sigma(t)](m-2c\lambda)(q_1^\Delta+q_2^\Delta)\\
&=\text{const.}
\end{aligned} \tag{56}$$

如果 $\mathbb{T}=\mathbb{R}$,则有 $\sigma(t)=t$, $e_p[t_a,\sigma(t)]=e^{t-t_a}$,由式(53)和式(51)所描述的时间尺度上非完整系统退化为经典的连续形式的非完整系统,相应的守恒量式(56)变为

$$I_N=e^{t-t_a}(m-2c\lambda)(\dot{q}_1+\dot{q}_2)=\text{const.}\tag{57}$$

**例 2** 在时间尺度 $\mathbb{T}=\{2^n,n\in\mathbb{N}\}$ 上研究匀质圆球在粗糙水平面上纯滚动问题[14]. Herglotz 型 Lagrange 函数为

$$\begin{aligned}L=&\frac{1}{5}mr^2[(q_1^\Delta)^2+(q_2^\Delta)^2+(q_3^\Delta)^2+\\&2q_1^\Delta q_3^\Delta\cos q_2^\sigma]+\frac{1}{2}m[(q_4^\Delta)^2+(q_5^\Delta)^2]-\gamma z\end{aligned}\tag{58}$$

其中,质量 $m$、半径 $r$ 和阻尼系数 $\gamma$ 均为常数.

运动受到的非完整约束为

$$\begin{aligned}&\varphi_1=q_4^\Delta+r(q_3^\Delta\sin q_2^\sigma\cos q_1^\sigma-q_2^\Delta\sin q_1^\sigma)=0,\\&\varphi_2=q_5^\Delta+r(q_3^\Delta\sin q_2^\sigma\sin q_1^\sigma+q_2^\Delta\cos q_1^\sigma)=0\end{aligned}\tag{59}$$

由于 $\mathbb{T}=\{2^n,n\in\mathbb{N}\}$,则前跳算子 $\sigma(t)=2t$,步差函数 $\mu(t)=\sigma(t)-t=t$. 根据时间尺度上 Vacco 动力学的 Herglotz 型方程(28),可得

$$\begin{aligned}&\frac{\Delta}{\Delta t}\{e_p(a,2t)\times\frac{2}{5}mr^2[q_1^\Delta+q_3^\Delta\cos q_2(2t)]\}+\\&\quad e_p(a,2t)\{\lambda_1 r[q_3^\Delta\sin q_2(2t)\sin q_1(2t)+\\&\quad q_2^\Delta\cos q_1(2t)]+\lambda_2 r[q_2^\Delta\sin q_1(2t)-\\&\quad q_3^\Delta\sin q_2(2t)\cos q_1(2t)]\}=0,\\&\frac{\Delta}{\Delta t}\{e_p(a,2t)[\frac{2}{5}mr^2q_2^\Delta-\\&\quad\lambda_1 r\sin q_1(2t)+\lambda_2 r\cos q_1(2t)]\}-\\&\quad e_p(a,2t)[-\frac{2}{5}mr^2q_1^\Delta q_3^\Delta\sin q_2(2t)+\\&\quad\lambda_1 rq_3^\Delta\cos q_2(2t)\cos q_1(2t)+\\&\quad\lambda_2 rq_3^\Delta\cos q_2(2t)\sin q_1(2t)]=0,\\&\frac{\Delta}{\Delta t}(e_p(a,2t)\{\frac{2}{5}mr^2[q_3^\Delta+q_1^\Delta\cos q_2(2t)]+\\&\quad\lambda_1 r\sin q_2(2t)\cos q_1(2t)+\\&\quad\lambda_2 r\sin q_2(2t)\sin q_1(2t)\})=0,\\&\frac{\Delta}{\Delta t}[e_p(a,2t)(mq_4^\Delta+\lambda_1)]=0,\\&\frac{\Delta}{\Delta t}[e_p(a,2t)(mq_5^\Delta+\lambda_2)]=0.\end{aligned}\tag{60}$$

其中 $p=\dfrac{\partial L}{\partial z}=-\gamma$.

Noether 等式(34)给出

$$\begin{aligned}&\lambda_1^\Delta\{q_4^\Delta+r[q_3^\Delta\sin q_2(2t)\cos q_1(2t)-\\&\quad q_2^\Delta\sin q_1(2t)]\}^\sigma\tau+\lambda_2^\Delta\{q_5^\Delta+r[-q_2^\Delta\cos q_1(2t)+\\&\quad q_3^\Delta\sin q_2(2t)\sin q_1(2t)]\}^\sigma\tau+\\&\quad\{\lambda_1 r[-q_3^\Delta\sin q_2(2t)\sin q_1(2t)-q_2^\Delta\cos q_1(2t)]+\\&\quad\lambda_2 r[q_3^\Delta\sin q_2(2t)\cos q_1(2t)-q_2^\Delta\sin q_1(2t)]\}\xi_1^\sigma+\\&\quad[-\frac{2}{5}mr^2q_1^\Delta q_3^\Delta\sin q_2(2t)+\\&\quad\lambda_1 rq_3^\Delta\cos q_2(2t)\cos q_1(2t)+\\&\quad\lambda_2 rq_3^\Delta\cos q_2(2t)\sin q_1(2t)]\xi_2^\sigma+\\&\quad\frac{2}{5}mr^2[q_1^\Delta+q_3^\Delta\cos q_2(2t)]\xi_1^\Delta+\\&\quad[\frac{2}{5}mr^2q_2^\Delta-\lambda_1 r\sin q_1(2t)+\lambda_2 r\cos q_1(2t)]\xi_2^\Delta+\\&\quad\{\frac{2}{5}mr^2[q_3^\Delta+q_1^\Delta\cos q_2(2t)]+\\&\quad\lambda_1 r\sin q_2(2t)\cos q_1(2t)+\\&\quad\lambda_2 r\sin q_2(2t)\sin q_1(2t)\}\xi_3^\Delta+\\&\quad(mq_4^\Delta+\lambda_1)\xi_4^\Delta+(mq_5^\Delta+\lambda_2)\xi_5^\Delta+\{\frac{1}{5}mr^2[(q_1^\Delta)^2+\\&\quad(q_2^\Delta)^2+(q_3^\Delta)^2+2q_1^\Delta q_3^\Delta\cos q_2(2t)]+\\&\quad\frac{1}{2}m[(q_4^\Delta)^2+(q_5^\Delta)^2]-\gamma z+\lambda_1[q_4^\Delta+\\&\quad rq_3^\Delta\sin q_2(2t)\cos q_1(2t)-q_2^\Delta\sin q_1(2t)]+\\&\quad\lambda_2[q_5^\Delta+r(q_3^\Delta\sin q_2(2t)\sin q_1(2t)+\\&\quad q_2^\Delta\cos q_1(2t)]\}^\sigma\tau^\Delta-\\&\quad(\frac{2}{5}mr^2[q_1^\Delta+q_3^\Delta\cos q_2(2t)]q_1^\Delta(2t)+\\&\quad[\frac{2}{5}mr^2q_2^\Delta-\lambda_1 r\sin q_1(2t)+\\&\quad\lambda_2 r\cos q_1(2t)]q_2^\Delta(2t)+\{\frac{2}{5}mr^2[q_3^\Delta+\\&\quad q_1^\Delta\cos q_2(2t)]+\lambda_1 r\sin q_2(2t)\cos q_1(2t)+\\&\quad\lambda_2 r\sin q_2(2t)\sin q_1(2t)\}q_3^\Delta(2t)+(mq_4^\Delta+\\&\quad\lambda_1)q_4^\Delta(2t)+(mq_5^\Delta+\lambda_2)q_5^\Delta(2t))\tau^\Delta=0\end{aligned}\tag{61}$$

上述方程有解:

$$\tau=0,\ \xi_1=0,\ \xi_2=0,\ \xi_3=1,\ \xi_4=1,\ \xi_5=1\tag{62}$$

根据定理 2,将式(62)代入式(43),可得系统相应的 Noether 型守恒量为

$$\begin{aligned}I_N=&e_p(t_a,2t)\{\frac{2}{5}mr^2[q_3^\Delta+q_1^\Delta\cos q_2(2t)]+\\&r\sin q_2(2t)[\lambda_1\cos q_1(2t)+\lambda_2\sin q_1(2t)]+\\&m(q_4^\Delta+q_5^\Delta)+\lambda_1+\lambda_2\}=\text{const.}\end{aligned}\tag{63}$$

## 5 结论

文章基于时间尺度上的 Herglotz 变分原理研究了非完整约束系统的 Herglotz 型 Vacco 动力学及其 Noether 定理.主要结果如下:首先,基于时间尺度上 Herglotz 变分原理建立了非完整约束系统 Vacco 动力学的 Herglotz 型方程(27);其次,根据时间尺度上 Hamilton-Herglotz 作用量在无限小变换下的不变性,导出了时间尺度上非完整约束系统 Herglotz 型 Vacco 动力学的 Noether 等式(定理 1);最后,证明了时间尺度上非完整约束系统 Vacco 动力学的 Herglotz 型 Noether 定理(定理 2).当选取时间尺度时,文中的方程(27)、定理 1 与定理 2 均可退化为通常意义下非完整约束系统的相应结果[29].本文的研究方法和结果具有一般性,可以进一步推广至时间尺度上高阶非完整约束系统.

## 参考文献

[1] 梅凤翔,吴惠彬,李彦敏. 分析力学史略[M]. 北京:科学出版社,2019.
MEI F X,WU H B,LI Y M. A brief history of analytical mechanics [M]. Beijing: Science Press, 2019. (in Chinese)

[2] GUENTHER R B,GUENTHER C M,GOTTSCH J A. The Herglotz lectures on contact transformations and Hamiltonian systems [R]. Torun, Poland: Juliusz Center for Nonlinear Studies, 1996.

[3] GEORGIEVA B,GUENTHER R B. First Noether-type theorem for the generalized variational principle of Herglotz [J]. Topological Methods in Nonlinear Analysis, 2002, 20(2): 261-273.

[4] GEORGIEVA B,GUENTHER R,BODUROV T. Generalized variational principle of Herglotz for several independent variables. First Noether-type theorem [J]. Journal of Mathematical Physics, 2003, 44(9): 3911-3927.

[5] SANTOS S P S,MARTINS N,TORRES D F M. Variational problems of Herglotz type with time delay: DuBois-Reymond condition and Noether's first theorem [J]. Discrete and Continuous Dynamical Systems, 2015, 35(9): 4593-4610.

[6] SANTOS S P S,MARTINS N,TORRES D F M. Noether currents for higher-order variational problems of Herglotz type with time delay [J]. Discrete and Continuous Dynamical Systems, Series S, 2018, 11(1): 91-102.

[7] ALMEIDA R,MALINOWSKA A B. Fractional variational principle of Herglotz [J]. Discrete and Continuous Dynamical Systems, Series A, 2017, 19(8): 2367-2381.

[8] DENG Y Y,ZHANG Y. Noether's theorem of Herglotz type for fractional Lagrange system with nonholonomic constraints [J]. Fractal and Fractional, 2024, 8(5): 296.

[9] 张毅. 相空间中非保守系统的 Herglotz 广义变分原理及其 Noether 定理[J]. 力学学报,2016,48(6):1382-1389.
ZHANG Y. Generalized variational principle of Herglotz type for nonconservative system in phase space and Noether's theorem [J]. Chinese Journal of Theoretical and Applied Mechanics, 2016, 48(6): 1382-1389. (in Chinese)

[10] DONG X C,ZHANG Y. Herglotz-type principle and first integrals for nonholonomic systems in phase space [J]. Acta Mechanica, 2023, 234(12): 6083-6095.

[11] ZHANG Y,TIAN X. Conservation laws for Birkhoffian systems of Herglotz type [J]. Chinese Physics B, 2018, 27(9): 090502.

[12] DING J J,ZHANG Y. Noether's theorem for fractional Birkhoffian system of Herglotz type with time delay [J]. Chaos, Solitons & Fractals, 2020, 138: 109913.

[13] 董欣畅,张毅. 非保守非完整系统的 Herglotz 型 Noether 定理[J]. 动力学与控制学报,2024,22(5):1-7.
DONG X C,ZHANG Y. Herglotz-type Noether theorem for nonconservative nonholonomic system [J]. Journal of Dynamics and Control, 2024, 22(5): 1-7. (in Chinese)

[14] 张毅. 非保守广义 Chaplygin 系统的 Herglotz 型 Noether 定理[J]. 力学学报,2024,56(9):2695-2702.
ZHANG Y. Noether's theorem of Herglotz form for generalized chaplygin systems with nonconservative forces [J]. Chinese Journal of Theoretical and Applied Mechanics, 2024, 56(9): 2695-2702. (in Chinese)

[15] DE LEÓN M,LAINZ M,MUÑOZ-LECANDA M C. The Herglotz principle and vakonomic dynamics

[C]//Geometric Science of Information. Cham: Springer, 2021: 183－190.

[16] DE LEÓN M, LAINZ M, MUÑOZ-LECANDA M C. Optimal control, contact dynamics and Herglotz variational problem [J]. Journal of Nonlinear Science, 2022, 33(1): 9.

[17] 梅凤翔. 关于非完整力学——分析力学札记之二十六[J]. 力学与实践, 2013, 35(5): 79－81.
Mei F X. On nonholonomic mechanics—Notes on analytical mechanics, XXVI [J]. Mechanics in Engineering, 2015, 37(5): 630－634. (in Chinese)

[18] KOZLOV V V. Dynamics of systems with non-integrable constraints I [J]. Moscow University Mechanics Bulletinl, 1982, 3: 92－100.

[19] 郭仲衡, 高普云. 关于经典非完整力学[J]. 力学学报, 1990, 22(2): 185－190.
GUO Z H, GAO P Y. On the classic nonholonomic dynamics [J]. Chinese Journal of Theoretical and Applied Mechanics, 1990, 22(2): 185－190. (in Chinese)

[20] 陈滨. 关于经典非完整力学的一个争议[J]. 力学学报, 1991, 23(3): 379－384.
CHEN B. A contention to the classic nonholonomic dynamics [J]. Chinese Journal of Theoretical and Applied Mechanics, 1991, 23(3): 379－384. (in Chinese)

[21] 郭永新, 赵喆, 刘世兴, 等. 非完整系统 Chetaev 动力学和 vakonomic 动力学的等价条件[J]. 物理学报, 2006, 55(8): 3838－3844.
GUO Y X, ZHAO Z, LIU S X, et al. Conditions for Chetaev dynamics to be equivalent to vakonomic dynamics in nonholonomic systems [J]. Acta Physica Sinica, 2006, 55(8): 3838－3844. (in Chinese)

[22] LEMOS N A. Complete inequivalence of nonholonomic and vakonomic mechanics [J]. Acta Mechanica, 2022, 233(1): 47－56.

[23] 张解放. Vacco 动力学的 Noether 理论[J]. 应用数学和力学, 1993(7): 635－641.
ZHANG J F. Noether's theory of Vacco dynamics [J]. Applied Mathematics and Mechanics, 1993 (7): 635－641. (in Chinese)

[24] DING N, FANG J H. Lie symmetry and conserved quantities for nonholonomic vacco dynamical systems [J]. Communications in Theoretical Physics, 2006, 46(2): 265－268.

[25] 顾书龙, 张宏彬. Vacco 动力学方程的 Mei 对称性、Lie 对称性和 Noether 对称性[J]. 物理学报, 2005, 54(9): 3983－3986.
GU S L, ZHANG H B. Mei symmetry, Noether symmetry and Lie symmetry of a Vacco system [J]. Acta Physica Sinica, 2005, 54(9): 3983－3986. (in Chinese)

[26] LI Y C, JING H X, XIA L L, et al. Unified symmetry of Vacco dynamical systems [J]. Chinese Physics, 2007, 16(8): 2154－2158.

[27] 陈立群. 高阶非完整系统的 Vacco 动力学[J]. 鞍山钢铁学院学报, 1992, 15(1): 34－39.
CHEN L Q. The Vacco dynamics for high-order nonholonomic sustems [J]. Journal of University of Science and Technology Liaoning, 1992, 15(1): 34－39. (in Chinese)

[28] HUANG L Q, ZHANG Y. Herglotz-type vakonomic dynamics and Noether theory of nonholonomic systems with delayed arguments [J]. Chaos, Solitons & Fractals, 2024, 182: 114854.

[29] HUANG L Q, ZHANG Y. Herglotz-type vakonomic dynamics and its Noether symmetry for nonholonomic constrained systems [J]. Journal of Mathematical Physics, 2024, 65(7): 072901.

[30] HILGER S. Analysis on measure chains: a unified approach to continuous and discrete calculus [J]. Results in Mathematics, 1990, 18(1): 18－56.

[31] BOHNER M, PETERSON A C. Dynamic equations on time scales [M]. Boston: Birkhäuser, 2001.

[32] BOHNER M. Calculus of variations in time scales [J]. Dynamics System and Applications, 2004, 13 (12): 351－379.

[33] BARTOSIEWICZ Z, TORRES D F M. Noether's theorem on time scales [J]. Journal of Mathematical Analysis and Applications, 2008, 342(2): 1220－1226.

[34] MARTINS N, TORRES D F M. Noether's symmetry theorem for nabla problems of the calculus of variations [J]. Applied Mathematics Letters, 2010, 23(12): 1432－1438.

[35] CAPUTO M C. Time scales: from nabla calculus to delta calculus and vice versa via duality [J]. International Journal of Difference Equations, 2010, 5(1): 25－40.

[36] ANEROT B, CRESSON J, HARIZ BELGACEM K, et al. Noether's-type theorems on time scales [J]. Journal of Mathematical Physics, 2020, 61 (11): 113502.

[37] 田雪，张毅. 时间尺度上 Herglotz 变分原理及其 Noether 定理[J]. 力学季刊，2018，39(2)：237－248.
TIAN X，ZHANG Y. Variational principle of Herglotz type and its Noether's theorem on time scales [J]. Chinese Quarterly of Mechanics，2018，39(2)：237－248.（in Chinese）

[38] TIAN X，ZHANG Y. Noether symmetry and conserved quantity for Hamiltonian system of Herglotz type on time scales [J]. Acta Mechanica，2018，229(9)：3601－3611.

[39] TIAN X，ZHANG Y. Time-scales Herglotz type Noether theorem for delta derivatives of Birkhoffian systems [J]. Royal Society Open Science，2019，6(11)：191248.