

# 耦合 Hirota 方程高阶孤子解的动力学特性分析\*

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**摘要** 基于二分量耦合 Hirota 方程, 研究该方程的高阶孤子解并分析其动力学特性. 利用 Lax 对和广义 Darboux 变换, 对该方程的特征函数进行泰勒展开, 推导出 Hirota 方程二阶、三阶孤子解的表达式. 分情况讨论谱参数  $\lambda$  的实部和虚部, 选取不同的自由参数, 通过数值模拟得到孤子间相互作用演化图; 分析不同参数对孤子振幅变化和传播方向的影响. 结果表明, 对参数的不同取值会影响孤子之间的相互作用以及传播方向.

**关键词** 耦合 Hirota 方程, 广义 Darboux 变换, Lax 对方程, 孤子解

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## Dynamical Characteristics Analysis of Higher-Order Soliton Solutions for the Coupled Hirota Equation\*

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**Abstract** On the basis of the two-component coupled Hirota equation, the higher-order solitons solution of the equation is investigated, and dynamical characteristics is analyzed. Based on the Lax pair and generalized Darboux transformation, the Taylor expansion of the characteristic function is carried out, and the expressions of the second- and third-order soliton solutions of Hirota equation are derived. The real and imaginary parts of spectral parameter  $\lambda$  are discussed in different cases. Under the influence of the value of different free parameters, the evolution graphs of soliton interaction are obtained via numerical simulation. The effects of different parameters on the amplitude change and propagation direction of solitons are analyzed. The results show that different values of parameters affect the interaction and propagation direction of solitons.

**Key words** coupled Hirota equation, generalized Darboux transformation, Lax pair equation, soliton solution

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## 引言

非线性科学起源于20世纪60年代,是研究非线性现象共性的一门学科,孤子理论是非线性科学的分支之一<sup>[1,2]</sup>.一般情况下,孤子在相互碰撞后,传播轨迹和速度能够保持不变.正因为孤子具有这种特殊性质,在物理学<sup>[3]</sup>、光纤孤子通信<sup>[4]</sup>、生物科学<sup>[5]</sup>和海洋科学<sup>[6]</sup>等领域中得到广泛的应用.光孤子在光纤中生成,可作为信息传输的载体,构建了光孤子通信<sup>[7]</sup>.生物学中利用孤子分析了呼吸捕获机制,玻色-爱因斯坦凝聚也发现了孤子解<sup>[8]</sup>.目前,孤子现象能够合理解释很多相关自然现象,因此对孤子的研究具有重要意义.

随着学者们对于孤子解的深入研究,逐渐提出一些经典方法,如逆散射方法<sup>[9]</sup>、Darboux变换法<sup>[10,11]</sup>、Hirota双线性方法<sup>[12]</sup>、修正辅助方程法<sup>[13]</sup>、相似变换<sup>[14]</sup>等.本文采用广义Darboux变换<sup>[15]</sup>,是在经典Darboux变换的基础上,通过引入一个小参数,利用泰勒展开与极限思想得到广义Darboux变换.主要思想是基于Lax对方程,构造Darboux矩阵,选取零解作为种子解,得到Lax对的特解,从而求出孤子解的迭代表达式.与经典Darboux变换相比,广义Darboux变换只需要考虑一个谱参数,很大程度简化了运算过程,提高了运算效率.非线性薛定谔方程常用于研究孤子解,Hirota方程是薛定谔方程的可积推广,也是分析孤子解动力学特性的一个重要模型.

本文研究二分量耦合Hirota方程,该方程可用来模拟双折射光纤中两个超短光纤场的波的传播<sup>[16,17]</sup>

$$\begin{aligned} i q_{1,t} + q_{1,xx} + 2(|q_1|^2 + |q_2|^2)q_1 + i\epsilon(q_{1,xxx} + 6|q_1|^2 q_{1,x} + 3|q_2|^2 q_{1,x} + 3q_2^* q_{2,x} q_1) &= 0 \\ i q_{2,t} + q_{2,xx} + 2(|q_1|^2 + |q_2|^2)q_2 + i\epsilon(q_{2,xxx} + 6|q_2|^2 q_{2,x} + 3|q_1|^2 q_{2,x} + 3q_1^* q_{1,x} q_2) &= 0 \end{aligned} \quad (1)$$

其中 $q_1(x,t), q_2(x,t)$ 表示两个复场函数,\*为共轭, $x$ 为演化距离, $t$ 为演化时间, $\epsilon$ 为实常数.

很多学者对方程(1)进行了研究,杨丹玉等<sup>[18]</sup>利用Darboux变换得到该方程的亮-暗孤子;孙文荣等<sup>[19]</sup>研究了该方程的高阶怪波,证明了在基带调制不稳定性条件下可以激发怪波;Liu<sup>[20]</sup>通过研究该方程的黎曼-希尔伯特问题获得了该方程

的渐近性;基于广义Darboux变换,Chai等<sup>[21]</sup>研究了该方程的局域波,得到了呼吸子到孤子的转换;Tasgal等<sup>[22]</sup>通过逆散射方法,构建了显式孤子解,系统分析了线性耦合项,而关于该方程的孤子解问题研究较少.本文利用广义Darboux变换得到方程(1)的高阶孤子解迭代表达式,进一步分析其孤子解动力学特性.

## 1 广义Darboux变换

方程(1)具有如下形式的Lax对方程

$$\begin{aligned} \Phi_x &= U\Phi, \\ \Phi_t &= V\Phi = (V_3\lambda^3 + V_2\lambda^2 + V_1\lambda + V_0 + cE)\Phi \end{aligned} \quad (2)$$

其中

$$\begin{aligned} U &= \begin{pmatrix} i\lambda & q_1 & q_2 \\ -q_1^* & -i\lambda & 0 \\ -q_2^* & 0 & -i\lambda \end{pmatrix}, \\ V_3 &= \begin{pmatrix} 4i\epsilon & 0 & 0 \\ 0 & -4i\epsilon & 0 \\ 0 & 0 & -4i\epsilon \end{pmatrix}, \\ V_2 &= \begin{pmatrix} 2i & 4\epsilon q_1 & 4\epsilon q_2 \\ -4\epsilon q_1^* & 2i & 0 \\ -4\epsilon q_2^* & 0 & 2i \end{pmatrix}, \\ V_1 &= 2 \begin{pmatrix} -i\epsilon(|q_1|^2 + |q_2|^2) & -q_1 - i\epsilon q_{1,x} & -q_2 - i\epsilon q_{2,x} \\ q_1^* - i\epsilon q_{1,x}^* & i\epsilon|q_1|^2 & i\epsilon q_1^* q_2 \\ q_2^* - i\epsilon q_{2,x}^* & i\epsilon q_1 q_2^* & i\epsilon|q_2|^2 \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} V_0 &= \begin{pmatrix} v_1 + v_2 & v_3 & v_4 \\ -v_3^* & -v_1 & v_5 \\ -v_4^* & -v_5^* & -v_2 \end{pmatrix}, \\ v_1 &= i|q_1|^2 - \epsilon q_{1,x} q_1^2 + \epsilon q_{1,x}^* q_1, \\ v_2 &= i|q_2|^2 - \epsilon q_{2,x} q_2^* + \epsilon q_{2,x}^* q_2, \\ v_3 &= i q_{1,x} - \epsilon q_{1,xx} - 2\epsilon q_1(|q_1|^2 + |q_2|^2), \\ v_4 &= i q_{2,x} - \epsilon q_{2,xx} - 2\epsilon q_2(|q_1|^2 + |q_2|^2), \\ v_5 &= \epsilon q_{2,x} q_1^* - \epsilon q_{1,x}^* q_2 - i q_{1,x}^* q_2. \end{aligned}$$

$\Phi = [\varphi(x,t), \phi(x,t), \chi(x,t)]^T$ 为方程(1)的本征波函数, $\lambda$ 为光谱参数, $T$ 为向量转置,\*为共轭.方程(2)可由相容性条件 $U_t - V_x + [U, V]$ 得到.

Darboux矩阵满足

$$T[1] = \lambda I - H\Lambda_1 H^{-1}$$

其中

$$\mathbf{H} = \begin{pmatrix} \varphi_1 & \varphi_1^* & \chi_1^* \\ \varphi_1 & -\varphi_1^* & 0 \\ \chi_1 & 0 & -\varphi_1^* \end{pmatrix}, \mathbf{\Lambda}_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1^* & 0 \\ 0 & 0 & \lambda_1^* \end{pmatrix},$$

$\mathbf{I}$  是  $3 \times 3$  的单位矩阵,  $\lambda = \lambda_1$  为光谱参数, 是一个复常数,  $\lambda_1^*$  为  $\lambda_1$  的共轭复数.

假设方程(1)的种子解为  $q_1[0] = q_2[0] = 0$ , 代入 Lax 对方程(2)中, 得出与  $\lambda = \lambda_1$  和  $\lambda = \lambda_3$  对应的基解矩阵

$$\begin{aligned} \Phi_a[0] &= \phi \begin{pmatrix} \varphi_1[0] \\ \phi_1[0] \\ \chi_1[0] \end{pmatrix} = \begin{pmatrix} h_{11} e^{i\lambda_1(x + 4\beta\lambda_1^2 t + 2\lambda_1 t)} \\ h_{21} e^{-i\lambda_1(x + 4\beta\lambda_1^2 t - 2\lambda_1 t)} \\ h_{31} e^{-i\lambda_1(x + 4\beta\lambda_1^2 t - 2\lambda_1 t)} \end{pmatrix}, \\ \Phi_b[0] &= \phi \begin{pmatrix} \varphi_2[0] \\ \phi_2[0] \\ \chi_2[0] \end{pmatrix} = \begin{pmatrix} h_{12} e^{i\lambda_2(x + 4\beta\lambda_2^2 t + 2\lambda_2 t)} \\ h_{22} e^{-i\lambda_2(x + 4\beta\lambda_2^2 t - 2\lambda_2 t)} \\ h_{32} e^{-i\lambda_2(x + 4\beta\lambda_2^2 t - 2\lambda_2 t)} \end{pmatrix}. \end{aligned} \quad (3)$$

其中  $h_{j1}, h_{j2} (j=1, 2, 3)$  是任意复参数.

根据 Darboux 矩阵推导出经典 Darboux 变换:  $\lambda = \lambda_k, \Phi_k = (\varphi_k, \phi_k, \chi_k)^T, (k=1, 2, \dots, N)$ ,

$$\Phi_N[N-1] = \mathbf{T}[N-1]\mathbf{T}[N-2] \cdots \mathbf{T}[1]\Phi_N,$$

$$q_1[N] = q_1[0] + 2i \sum_{k=1}^N (\lambda_k - \lambda_k^*) \frac{\varphi_k[k-1]\phi_k^*[k-1]}{|\varphi_k[k-1]|^2 + |\phi_k[k-1]|^2 + |\chi_k[k-1]|^2},$$

$$q_2[N] = q_2[0] + 2i \sum_{k=1}^N (\lambda_k - \lambda_k^*) \frac{\varphi_k[k-1]\chi_k^*[k-1]}{|\varphi_k[k-1]|^2 + |\phi_k[k-1]|^2 + |\chi_k[k-1]|^2}.$$

其中

$$\mathbf{T}[k] = \lambda_{k+1} \mathbf{I} - \mathbf{H}[k-1]\mathbf{\Lambda}[k]\mathbf{H}[k-1]^{-1}, \Phi_k[k-1] = (\mathbf{T}[k-1]\mathbf{T}[k-2] \cdots \mathbf{T}[1])|_{\lambda=\lambda_k} \Phi_k,$$

$$\mathbf{H}[k-1] = \begin{pmatrix} \varphi_i[k-1] & \phi_i^*[k-1] & \chi_i^*[k-1] \\ \phi_i[k-1] & -\varphi_i^*[k-1] & 0 \\ \chi_i[k-1] & 0 & \varphi_i^*[k-1] \end{pmatrix}, \mathbf{\Lambda}[k] = \begin{pmatrix} \lambda_k & 0 & 0 \\ 0 & \lambda_k^* & 0 \\ 0 & 0 & \lambda_k^* \end{pmatrix}, \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

基于以上分析, 得到方程(1)一阶孤子解表达式

$$q_1[1] = q_1[0] + 2i(\lambda_1 - \lambda_1^*) \frac{\varphi_1[0]\phi_1^*[0]}{|\varphi_1[0]|^2 + |\phi_1[0]|^2 + |\chi_1[0]|^2}, \quad (4)$$

$$q_2[1] = q_2[0] + 2i(\lambda_1 - \lambda_1^*) \frac{\varphi_1[0]\chi_1^*[0]}{|\varphi_1[0]|^2 + |\phi_1[0]|^2 + |\chi_1[0]|^2}. \quad (5)$$

根据上述经典 Darboux 变换, 构造该方程广义 Darboux 变换获取方程(1)的精确解.

设  $\Psi(\lambda_3 + \eta) = \Phi_3[2]|_{\lambda=\lambda_3+\eta}$  是方程(2)相对于谱参量  $\lambda = \lambda_3 + \eta$  的一个特解,  $\eta$  为扰动的一个小参量, 在  $\eta = 0$  处对  $\Psi(\lambda_3 + \eta)$  进行泰勒展开, 有

$$\Psi(\lambda_3 + \eta) = \Psi_0 + \Psi_1\eta + \Psi_2\eta^2 + \cdots, \quad (6)$$

$$\Phi_1[0] = (\varphi_{23}[0] \ \phi_{23}[0] \ \chi_{23}[0])^T = \Psi_0$$

$$\Phi_1[1] = (\varphi_{24}[0] \ \phi_{24}[0] \ \chi_{24}[0])^T = \Psi_0 + \mathbf{T}[2]\Psi_1,$$

$$q_1[N] = q_1[N-1] + 2i(\lambda_1 - \lambda_1^*) \frac{\varphi_1[N+1]\phi_1^*[N+1]}{|\varphi_1[N+1]|^2 + |\phi_1[N+1]|^2 + |\chi_1[N+1]|^2},$$

$$q_2[N] = q_2[N-1] + 2i(\lambda_1 - \lambda_1^*) \frac{\varphi_1[N+1]\chi_1^*[N+1]}{|\varphi_1[N+1]|^2 + |\phi_1[N+1]|^2 + |\chi_1[N+1]|^2}. \quad (7)$$

## 2 二阶和三阶孤子的动力学特性

当  $N=2$  时, 在给定谱参数的前提下, 对自由参数  $h_{1i}, h_{2i}$  和  $h_{3i} (i=1, 2)$  取值, 讨论二阶孤子的动力学特性.

(1) 当  $\text{Re}(\lambda_1) \neq \text{Re}(\lambda_3), \text{Im}(\lambda_1) = \text{Im}(\lambda_3)$  时,

其中  $\Psi_0 = (\varphi_{23}[0] \ \phi_{23}[0] \ \chi_{23}[0])^T,$

$$\Psi_1 = (\varphi_{23}[1] \ \phi_{23}[1] \ \chi_{23}[1])^T.$$

利用 Maple 可求得泰勒展开式中的系数, 由于表达式较复杂, 此处省略.

方程(1)的  $N-1$  阶广义 Darboux 变换定义如下 ( $N=2, 3$ )

分量  $q_1[2], q_2[2]$  中两个孤子发生弹性碰撞, 孤子在处碰撞发生了波动, 碰撞前后孤子的传播方向和轨迹都没有发生变化, 以原来的振幅和方向继续向前运动, 如图 1 所示.

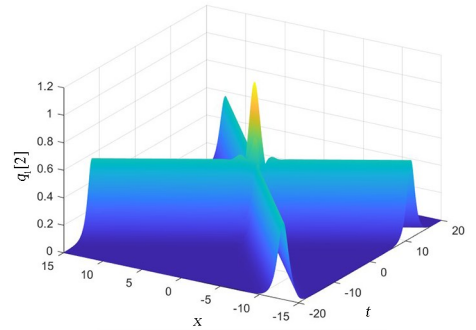
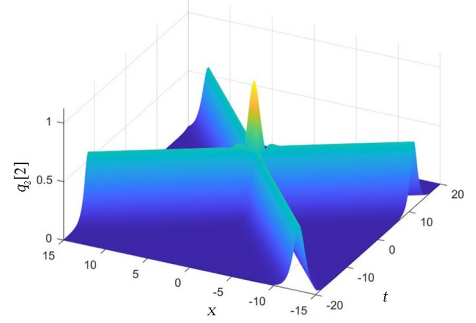
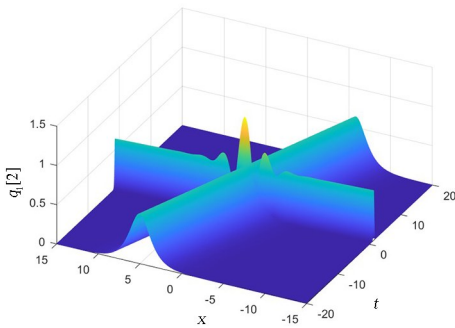
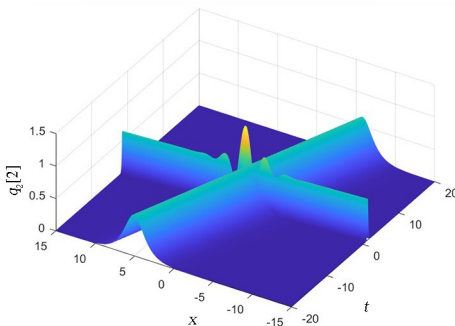
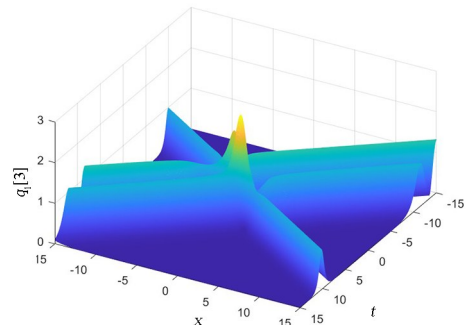
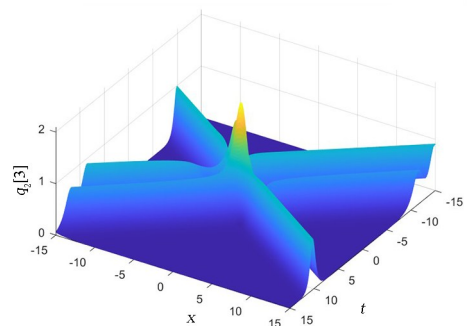
(2) 当  $\text{Re}(\lambda_1) = \text{Re}(\lambda_3), \text{Im}(\lambda_1) \neq \text{Im}(\lambda_3)$  时, 分量  $q_1[2], q_2[2]$  中二阶孤子发生弹性碰撞. 与图

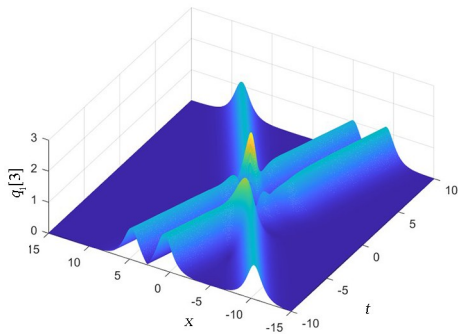
1 相比,两个孤子运动轨迹间的夹角发生改变,在碰撞瞬间能量发生变化,孤子振幅达到最大,碰撞后传播方向和能量没有发生改变,如图 2 所示。

当  $N=3$  时,分析三阶孤子的动力学特性. 比较谱参数和的实部  $\lambda_1$  和  $\lambda_3$  虚部,分以下两种情况:

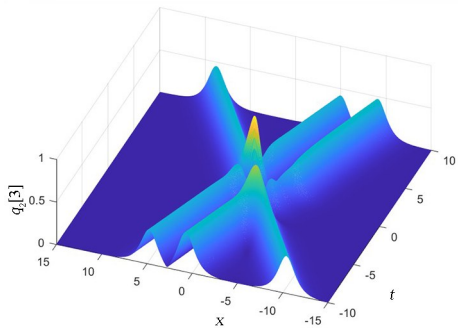
(1)当  $\text{Re}(\lambda_1) \neq \text{Re}(\lambda_3)$ ,  $\text{Im}(\lambda_1) = \text{Im}(\lambda_3)$  时,分量  $q_1[3]$ 、 $q_2[3]$  表现为三个孤子之间的弹性碰撞,碰撞后三个孤子的振幅和传播方向未发生改变,如图 3 所示,分量  $q_1[3]$  的振幅比分量  $q_2[3]$  的振幅高,在碰撞瞬间振幅达到最大,能量在  $x=0$  处达到最大。

(2)当  $\text{Re}(\lambda_1) \neq \text{Re}(\lambda_3)$ ,  $\text{Im}(\lambda_1) = \text{Im}(\lambda_3)$  时,分量  $q_1[3]$ 、 $q_2[3]$  中的三个孤子发生弹性碰撞,在碰撞过后,孤子能量发生波动,如图 4 所示.  $q_1[3]$  和  $q_2[3]$  振幅和传播轨迹未发生改变,如图 4(a)、图 4(b) 所示,  $q_1[3]$  和  $q_2[3]$  的动力学特性相同. 保持其他参数不变,调整参数  $h_{12}$  和  $h_{32}$  值,此时,  $q_1[3]$  和  $q_2[3]$  中的三个孤子发生非弹性碰撞. 在碰撞瞬间,有两个孤子退化为一个孤子. 与弹性碰撞相比,非弹性碰撞在碰撞后振幅变化明显降低,但传播方向未发生改变,如图 4(c)、图 4(d) 所示。

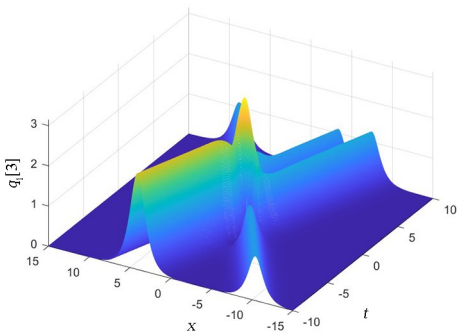
(a) 分量  $q_1$  孤子间弹性碰撞(a) Elastic collisions between solitons  $q_1$ (b) 分量  $q_2$  孤子间弹性碰撞(b) Elastic collisions between solitons  $q_2$ 图 2  $h_{11}=h_{21}=h_{31}=h_{12}=h_{22}=h_{32}=i$ ;  $\lambda_1=0.4+0.4i$ ;  
 $\lambda_3=0.4+0.8i$  时,二阶孤子间的相互作用演化图Fig. 2 Second-order solitons at  $h_{11}=h_{21}=h_{31}=h_{12}=h_{22}=h_{32}=i$ ;  
 $\lambda_1=0.4+0.4i$ ;  $\lambda_3=0.4+0.8i$ (a) 分量  $q_1$  孤子间弹性碰撞(a) Elastic collisions between solitons  $q_1$ (b) 分量  $q_2$  孤子间弹性碰撞(b) Elastic collisions between solitons  $q_2$ 图 1  $h_{11}=h_{21}=h_{31}=h_{12}=h_{22}=h_{32}=i$ ;  $\lambda_1=-1+0.45i$ ;  
 $\lambda_3=0.3+0.45i$  时,二阶孤子间的相互作用演化图Fig. 1 Second-order solitons at  $h_{11}=h_{21}=h_{31}=h_{12}=h_{22}=h_{32}=i$ ;  
 $\lambda_1=-1+0.45i$ ;  $\lambda_3=0.3+0.45i$ (a) 分量  $q_1$  孤子间弹性碰撞(a) Elastic collisions between solitons  $q_1$ (b) 分量  $q_2$  孤子间弹性碰撞(b) Elastic collisions between solitons  $q_2$ 图 3  $h_{11}=h_{21}=h_{31}=h_{12}=h_{22}=h_{32}=i$ ;  $\lambda_1=0.2+0.7i$ ;  
 $\lambda_3=0.4+0.4i$  时,三阶孤子间的相互作用演化图Fig. 3 Third-order solitons at  $h_{11}=h_{21}=h_{31}=h_{12}=h_{22}=h_{32}=i$ ;  
 $\lambda_1=0.2+0.7i$ ;  $\lambda_3=0.4+0.4i$



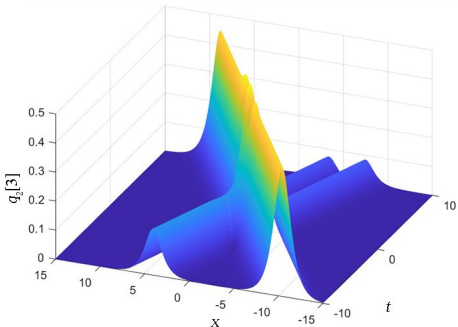
(a) 分量 $q_1$ 孤子间弹性碰撞  
(a) Elastic collisions between solitons  $q_1$



(b) 分量 $q_2$ 孤子间弹性碰撞  
(b) Elastic collisions between solitons  $q_2$



(c) 分量 $q_1$ 孤子间非弹性碰撞  
(c) Inelastic collisions between solitons  $q_1$



(d) 分量 $q_2$ 孤子间非弹性碰撞  
(d) Inelastic collisions between solitons  $q_2$

图 4  $h_{11}=h_{31}=2i;h_{22}=5;\lambda_3=0.4+0.6i$  时,三阶孤子之间的相互作用演化图 (a), (b)  $h_{12}=i, h_{32}=0$   
(c), (d)  $h_{32}=1+2i; h_{12}=0$

Fig. 4 Third-order solitons at  $h_{11}=h_{31}=2i; h_{22}=5;$   
 $\lambda_3=0.4+0.6i$  and (a), (b)  $h_{12}=i, h_{32}=0$   
(c), (d)  $h_{32}=1+2i; h_{12}=0$

### 3 小结

本文利用广义 Darboux 变换研究二分量耦合 Hirota 方程高阶孤子解的动力学特性. 基于二分量耦合 Hirota 方程的 Lax 对方程, 选取种子解得到 Lax 对方程的特解; 利用泰勒展开, 得到二阶、三阶孤子解的表达式, 分情况讨论谱参数, 调整自由参数  $\lambda$  的取值, 通过数值模拟得到该方程的孤子解二阶和三阶相互作用的演化图, 分析孤子的弹性碰撞以及非弹性碰撞的动力学特性. 所得结果进一步丰富了 Hirota 方程解的研究, 对于理解非线性领域中的孤子现象具有重要意义.

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