

新分数阶 jerk 混沌系统的有效数值模拟^{*}

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摘要 目前大多采用预测校正法来计算分数阶动力系统, 其它数值方法使用较少. 本文利用拉格朗日多项式插值, 详细阐述了一种用于计算分数阶动力系统的数值方法. 并利用此方法成功获得了新分数阶 jerk 混沌系统的有效数值解, 所得结果与预测校正法的结果进行对比, 发现两者十分吻合, 验证了方法的有效性. 进一步探讨了新分数阶 jerk 混沌系统在不同系统阶数 α 下的最大 Lyapunov 指数, 得出当系统阶数 $\alpha=0.85$, $\alpha=0.95$ 时, 系统是不稳定的, 进一步展示了在 Caputo 意义下的系统相图. 研究结果表明, 不同的系统阶数 α 对新分数阶 jerk 混沌系统的动力学行为有显著影响.

关键词 新分数阶 jerk 混沌系统, 数值模拟, 稳定性, 动力学行为

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Effective Numerical Simulation of New Fractional-order Chaotic jerk System^{*}

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Abstract Most of the current research methods use the prediction correction method, but other numerical methods are less frequently used. In this paper, a numerical method for computing fractional-order dynamical systems is elaborated using Lagrange polynomial interpolation. And this is used to successfully obtain an effective numerical solution for a new fractional-order jerk chaotic system. The results obtained are compared with those of the prediction correction method and found to be in good agreement with each other, verifying the validity of this method. The maximum Lyapunov exponent of the new fractional-order jerk chaotic system is further explored for different system orders, and it is concluded that the system exhibits instability when the system order $\alpha=0.85$, $\alpha=0.95$, and the phase diagram of the system in Caputo's sense is further demonstrated. The results show that different system order α have a significant effect on the dynamical behaviour of new fractional-order jerk chaotic system.

Key words new fractional-order chaotic jerk system, effective numerical simulation, stability, dynamical behavior

引言

随着数学领域的不断扩展, 20 世纪出现的分

数阶微积分逐渐成为人们关注的热点^[1,2]. 分数阶微积分源于德国数学家 Leibniz 和法国数学家 IFopital 对导数阶数为 1/2 时意义的探讨. 随后, 分

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数阶微分方程在理论上迅猛发展^[3,4]. 近年来, 动力系统在建模和控制领域取得了巨大的进展, 此进展对于理解和优化各种实际系统的行为至关重要. 然而, 传统的整数阶微分方程在描述某些复杂系统的行为时会存在局限性, 将分数阶微积分和分数阶微分方程引入到动力系统的建模中是解决局限性的有效办法. 分数阶微积分不仅将传统整数阶微积分中的导数和积分概念推广到分数阶, 还提供了一种更灵活、更通用地描述动态系统行为的方法, 还可以捕捉到一些传统方法所无法描述的现象, 从而为科学研究和工程实践提供了全新的视角和方法.

分数阶动力系统的研究已经涵盖了多个领域, 包括控制理论^[5]、信号处理^[6]、电力系统^[7]、生物医学工程^[8]、金融学^[9]等. 例如, 在控制领域^[10], 分数阶控制器已经被成功应用于机器人控制、自动驾驶系统和智能电网等领域. 在生物医学工程领域^[11], 分数阶动力系统的模型被用来研究生物体内的生理过程和疾病发展机制. 在金融学领域^[12], 分数阶动力系统被用来建立更准确的金融市场模型, 从而更好地预测市场变化和制定投资策略.

2004 年, Li 等^[13]研究了分数阶 Chen 系统的混沌行为, 并设计了一种简单有效的线性反馈控制器, 以稳定分数阶 Chen 混沌系统. 2009 年, Yu 等^[14]对一类分数阶 Lorenz 混沌系统的动力学行为进行了广泛的研究, 给出解的存在唯一性的严格证明, 讨论了分数阶 Lorenz 系统与整数阶 Lorenz 系统相比的稳定性. 2016 年, 毛北行等^[15]根据分数阶系统的相关理论研究了一类分数阶复杂网络混沌系统的投影同步问题, 给出了分数阶复杂网络以及分数阶时滞复杂网络系统实现投影同步的充分性条件. 2020 年, Chi 等^[16]建立了描述煤层气吸附过程的时间分数阶传热模型, 引入时间有限差分法和空间 Legendre 配置法的谱配置格式, 得到了该模型的数值解. 2021 年, 司辉等^[17]利用分数阶 Lyapunov 稳定性理论, 导出了分数阶混沌系统同步的一些新的充分条件. 2023 年, 王荣浩等^[18]针对分数阶时变切换系统, 借助 Caputo 分数阶导数引理和切换 Lyapunov 函数, 利用矩阵不等式技术提出了分数阶时变切换系统有限时间稳定的充分条件.

本文将介绍一种计算分数阶动力系统的有效数值方法, 并对其误差进行分析. 利用此方法对新

分数阶 jerk 混沌系统进行数值模拟. 并讨论了新分数阶 jerk 混沌系统的系统阶数 α 变化时, 系统动力学行为的变化. 该系统是印度学者 Sundarapandian^[19]根据分数阶 jerk 混沌系统扩展而来, 并利用最大 Lyapunov 指数证明了新分数阶 jerk 混沌系统比分数阶 jerk 混沌系统具有更大的复杂性. 新分数阶 jerk 混沌系统具有更加复杂的应用, 例如: 在图像加密方面, 新分数阶 jerk 混沌系统具有更加复杂的动力学行为和更加丰富的参数, 可以生成更加难以预测和破解的密钥, 可以提高加密的安全性. 新分数阶 jerk 混沌系统的形式如下所示, 其中 α 为系统阶数, a, b, c, d 为系统参数.

$$\begin{cases} D_t^\alpha x(t) = y & 0 < \alpha \leq 1 \\ D_t^\alpha y(t) = z \\ D_t^\alpha z(t) = -ay + bz + cxz + de^x + x^2 + 0.1 \end{cases} \quad (1)$$

1 数值算法

1.1 算法介绍

Caputo 分数阶积分的定义为

$$D_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t u(\tau)(t-\tau)^{\alpha-1} d\tau \quad (2)$$

考虑以下基于 Caputo 分数阶积分的分数阶微分方程

$$\begin{cases} D_t^\alpha y(t) = f[t, y(t)] \\ y(0) = y_0 \end{cases} \quad (3)$$

根据式(2), 可将式(3)化简为

$$y(t) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^t f[\tau, y(\tau)](t-\tau)^{\alpha-1} d\tau \quad (4)$$

令 $t = t_{k+1}, k = 0, 1, 2, \dots$, 式(4)可以表示为

$$\begin{aligned} & y(t_{k+1}) - y(0) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} f[\tau, y(\tau)](t_{k+1} - \tau)^{\alpha-1} d\tau \end{aligned} \quad (5)$$

移项得到

$$\begin{aligned} & y(t_{k+1}) \\ &= y(0) + \frac{1}{\Gamma(\alpha)} \sum_{m=0}^k \int_{t_m}^{t_{m+1}} f[\tau, y(\tau)](t_{k+1} - \tau)^{\alpha-1} d\tau \end{aligned} \quad (6)$$

为了简化式(6)右侧的积分, $f[\tau, y(\tau)]$ 由拉格朗日插值多项式近似得

$$f[\tau, y(\tau)] = f(t_m, y_m) \frac{\tau - t_{m-1}}{t_m - t_{m-1}} + f(t_{m-1}, y_{m-1}) \frac{\tau - t_m}{t_{m-1} - t_m} \quad (7)$$

令 $t_m - t_{m-1} = h$, 将式(7)代入式(6)得到

$$y_{k+1} = y(0) + \frac{1}{\Gamma(\alpha)} \sum_{m=0}^k \int_{t_m}^{t_{m+1}} f(t_m, y_m) \times \frac{\tau - t_{m-1}}{h} (t_{k+1} - \tau)^{\alpha-1} d\tau - \frac{1}{\Gamma(\alpha)} \times \sum_{m=0}^k \int_{t_m}^{t_{m+1}} f(t_{m-1}, y_{m-1}) \frac{\tau - t_m}{h} (t_{k+1} - \tau)^{\alpha-1} d\tau \quad (8)$$

令

$$\frac{1}{h} \int_{t_m}^{t_{m+1}} (\tau - t_{m-1}) \times (t_{k+1} - \tau)^{\alpha-1} d\tau = \frac{h^\alpha}{\alpha(\alpha+1)} [(k-m+1)^\alpha (k-m+2+\alpha) - (k-m)^\alpha (k-m+2+2\alpha)] \quad (9)$$

以及

$$\frac{1}{h} \int_{t_m}^{t_{m+1}} (\tau - t_m) \times (t_{k+1} - \tau)^{\alpha-1} d\tau = \frac{h^\alpha}{\alpha(\alpha+1)} [(k-m+1)^{\alpha+1} - (k-m)^\alpha (k-m+1+\alpha)] \quad (10)$$

将式(9)和式(10)代入到式(7)得

$$y_{k+1} = y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{m=0}^k f(t_m, y_m) \times [(k-m+1)^\alpha (k-m+2+\alpha) - (k-m)^\alpha (k-m+2+2\alpha)] - \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{m=0}^k f(t_{m-1}, y_{m-1}) [(k-m+1)^{\alpha+1} - (k-m)^\alpha (k-m+1+\alpha)] \quad (11)$$

1.2 误差分析

将式(5)与 α 阶分数阶积分结合起来, 利用拉格朗日多项式插值, 可以得到

$$y(t_{k+1}) - y(0) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} f[\tau, y(\tau)] (t_{k+1} - \tau)^{\alpha-1} d\tau = \frac{1}{\Gamma(\alpha)} \sum_{m=0}^k \int_{t_m}^{t_{m+1}} f[\tau, y(\tau)] (t_{k+1} - \tau)^{\alpha-1} d\tau = \frac{1}{\Gamma(\alpha)} \left[\frac{f(t_m, y_m)}{h} \int_{t_m}^{t_{m+1}} (\tau - t_{m-1}) \times (t_{k+1} - \tau)^{\alpha-1} d\tau - \frac{f(t_{m-1}, y_{m-1})}{h} \int_{t_m}^{t_{m+1}} (\tau - t_m) \times (t_{k+1} - \tau)^{\alpha-1} d\tau \right] + E_k^\alpha \quad (12)$$

又

$$E_k^\alpha = \frac{1}{\Gamma(\alpha)} \sum_{m=0}^k \int_{t_m}^{t_{m+1}} \frac{(\tau - t_m)(\tau - t_{m-1})}{2!} \times \frac{\partial^2}{\partial \tau^2} \{f[\tau, y(\tau)]\}_{\tau=\gamma_m} \times (t_{k+1} - \tau)^{\alpha-1} d\tau \quad (13)$$

由于 τ 在区间 $[t_m, t_{m+1}]$ 内为正, 则可以找到 $\gamma_m \in [t_m, t_{m+1}]$, 得到

$$E_k^\alpha = \frac{1}{\Gamma(\alpha)} \sum_{m=0}^k \frac{\partial^2}{\partial \tau^2} \{f[\tau, y(\tau)]\}_{\tau=\gamma_m} \frac{(\gamma_m - t_m)}{2} \times \int_{t_m}^{t_{m+1}} (\tau - t_{m-1})(t_{k+1} - \tau)^{\alpha-1} d\tau = \frac{1}{\Gamma(\alpha)} \sum_{m=0}^k \frac{\partial^2}{\partial \tau^2} \{f[\tau, y(\tau)]\}_{\tau=\gamma_m} \frac{(\gamma_m - t_m)}{2} h^{\alpha+1} \times \frac{1}{\alpha(\alpha+1)} [(k-m+1)^\alpha (k-m+2+\alpha) - (k-m)^\alpha (k-m+2+\alpha)] \quad (14)$$

取式(14)的绝对值, 可以得到

$$|E_k^\alpha| \leq \frac{h^{\alpha+2}}{2\Gamma(\alpha+2)} \sup_{\tau \in [0, t_{k+1}]} \left| \frac{\partial^2}{\partial \tau^2} f[\tau, y(\tau)] \right| \times \sum_{m=0}^k |[(k-m+1)^\alpha (k-m+2+\alpha) - (k-m)^\alpha (k-m+2+\alpha)]| \quad (15)$$

又有

$$\sum_{m=0}^k |[(k-m+1)^\alpha (k-m+2+\alpha) - (k-m)^\alpha (k-m+2+\alpha)]| = [(k+1)^\alpha - \alpha k^\alpha] \times \frac{k(k+4+2\alpha)}{2} \quad (16)$$

由此可得, 误差为

$$|E_k^\alpha| \leq \frac{h^{\alpha+2}}{2\Gamma(\alpha+2)} \sup_{\tau \in [0, t_{k+1}]} \left| \frac{\partial^2}{\partial \tau^2} f[\tau, y(\tau)] \right| \times [(k+1)^\alpha - \alpha k^\alpha] \times \frac{k(k+4+2\alpha)}{2} \quad (17)$$

2 研究结果与讨论

首先根据式(11)将新分数阶 jerk 混沌系统离散化处理, 得到离散格式为

$$x_{k+1} = x_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{m=0}^k y_m [(k-m+1)^\alpha (k-m+2+\alpha) - (k-m)^\alpha (k-m+2+2\alpha)] - \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{m=0}^k y_{m-1} [(k-m+1)^{\alpha+1} - (k-m)^\alpha (k-m+1+\alpha)], y_{k+1} = y_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{m=0}^k z_m [(k-m+1)^\alpha (k-m$$

$$\begin{aligned}
 & m + 2 + \alpha) - (k - m)^\alpha (k - m + 2 + 2\alpha)] - \\
 & \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{m=0}^k z_{m-1} [(k - m + 1)^{\alpha+1} - \\
 & (k - m)^\alpha (k - m + 1 + \alpha)], \\
 z_{k+1} = & z_0 + \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{m=0}^k [-ay_m + bz_m + \\
 & cx_m z_m + de^{x_m} + (x_m)^2 + 0.1] \times \\
 & [(k - m + 1)^\alpha (k - m + 2 + \alpha) - (k - m)^\alpha \times \\
 & (k - m + 2 + 2\alpha)] - \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{m=0}^k [-ay_{m-1} + \\
 & bz_{m-1} + cx_{m-1} z_{m-1} + de^{x_{m-1}} + (x_{m-1})^2 + 0.1] \times \\
 & [(k - m + 1)^{\alpha+1} - (k - m)^\alpha (k - m + 1 + \alpha)]
 \end{aligned}
 \tag{18}$$

为了验证离散格式的有效性,在式(18)中,设置初始条件 $[x_0 \ y_0 \ z_0] = [-1.6 \ 1.5 \ 0.1]$,同时 $a = 9, b = 6, c = 1.7, d = -7.0, h = 0.01, \alpha = 0.97, T = 1\ 000$ 时,图 1 为新分数阶 jerk 混沌系统的时间历程图(注:实线是提出方法所得解,虚线是预测校正法所得解,下同),图 2 为新分数阶 jerk 混沌系统的相图,由图 1 和图 2 可以观察到,时间历程图和相图的吻合效果良好,说明该数值方法是合理、有效的。

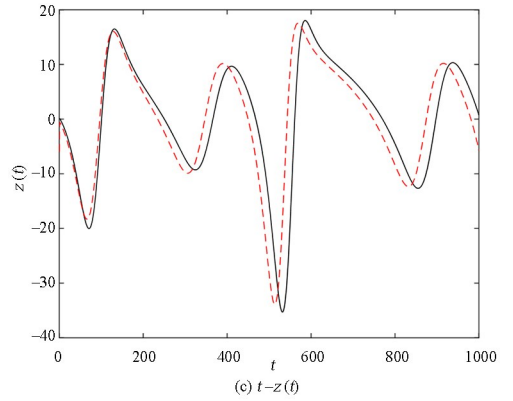
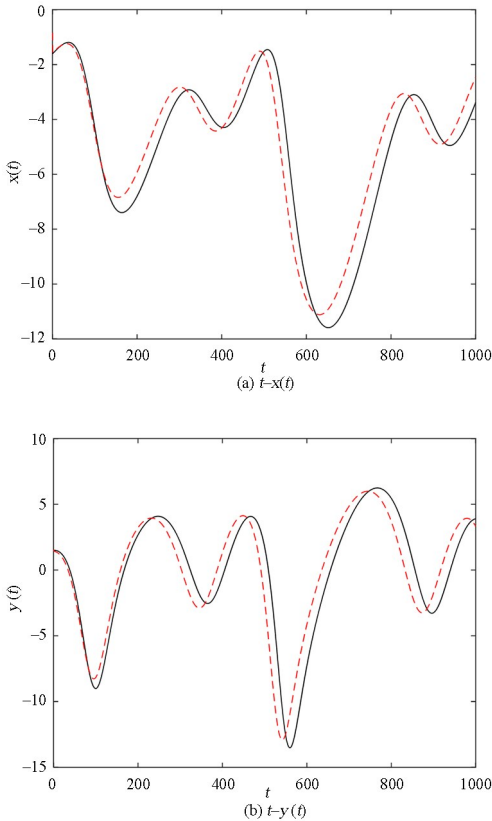


图 1 新分数阶 jerk 混沌系统的时间历程图
Fig. 1 Time course diagram of the new fractional-order jerk chaotic system

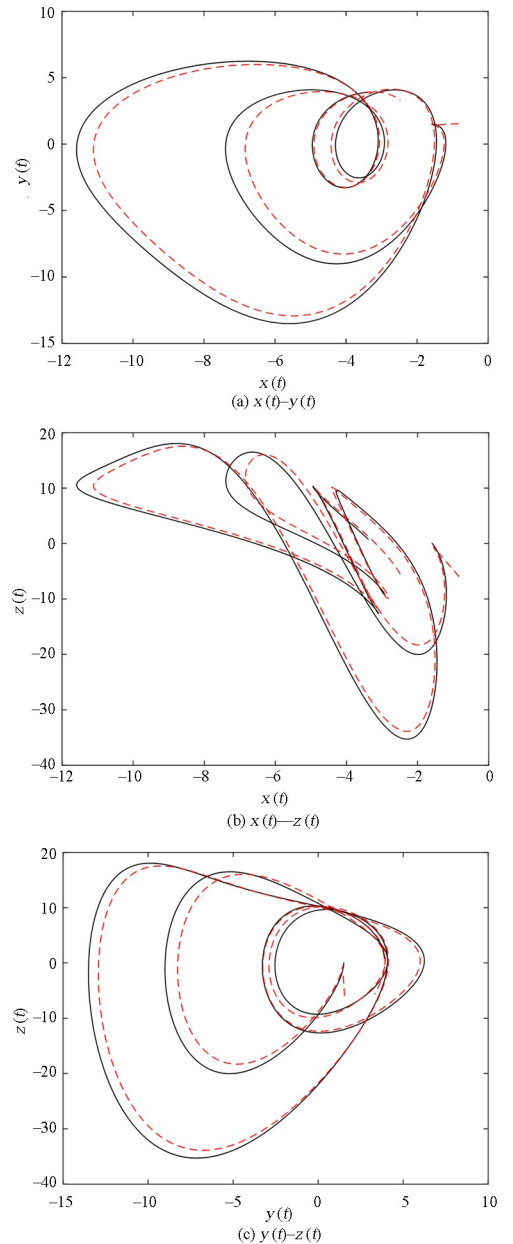


图 2 新分数阶 jerk 混沌系统的相图
Fig. 2 Phase diagram of the new fractional-order chaotic jerk system

为了考察系统阶数 α 对系统的影响,同样保持式(18)中的其它数值不变,只改变系统阶数,将其它参数值固定为 $a=9, b=6, c=1.7, d=-7.0$. 关于系统阶数 α 的最大 Lyapunov 指数图如图 3 所示,其中 α 从 0.5 到 1 变化,步长 $h=0.01$. 由最大 Lyapunov 指数图可知,当 α 分别取 0.85、0.95 时,系统的最大 Lyapunov 指数大于 0,系统呈现混沌状态.

当 $\alpha=0.85$ 时,设置初始条件 $[x_0 \ y_0 \ z_0] = [-1.6 \ 1.5 \ 0.1]$, $a=9, b=6, c=1.7, d=-7.0$, $h=0.01, T=1\ 000$,得到系统相图,如图 4 所示.

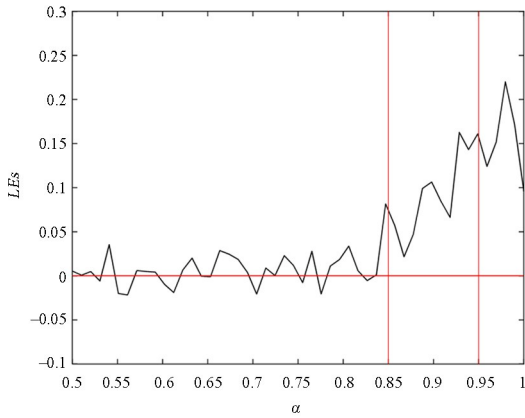
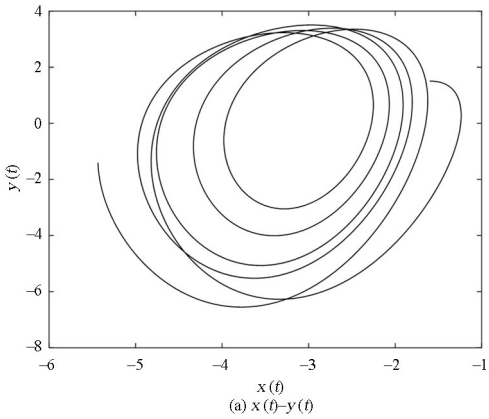
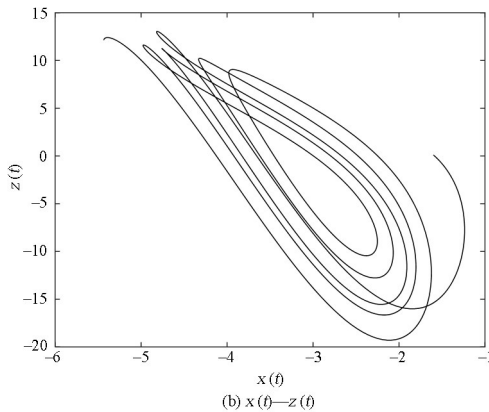


图 3 新分数阶 jerk 混沌系统关于系统阶数的最大 Lyapunov 指数图

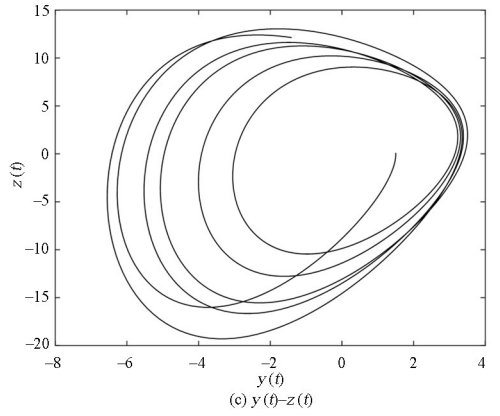
Fig. 3 Maximum Lyapunov exponential map of fractional-order jerk chaotic systems with respect to system order



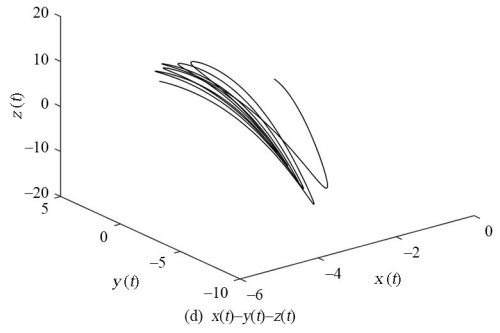
(a) $x(t)-y(t)$



(b) $x(t)-z(t)$



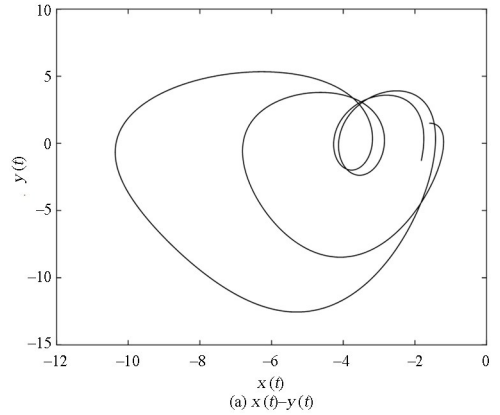
(c) $y(t)-z(t)$



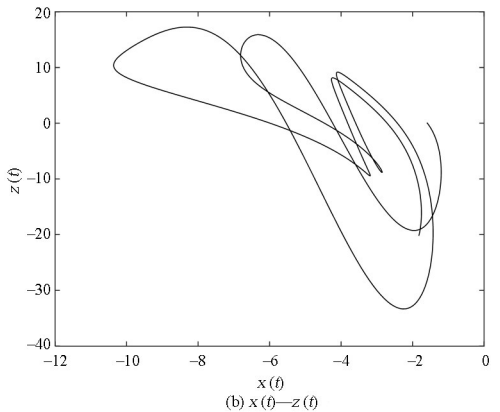
(d) $x(t)-y(t)-z(t)$

图 4 当 $\alpha=0.85$ 时,新分数阶 jerk 混沌系统的相图
Fig. 4 Phase diagram of the new fractional-order chaotic jerk system when $\alpha=0.85$

当 $\alpha=0.95$ 时,保持其它参数值不变,得到系统相图,如图 5 所示.



(a) $x(t)-y(t)$



(b) $x(t)-z(t)$

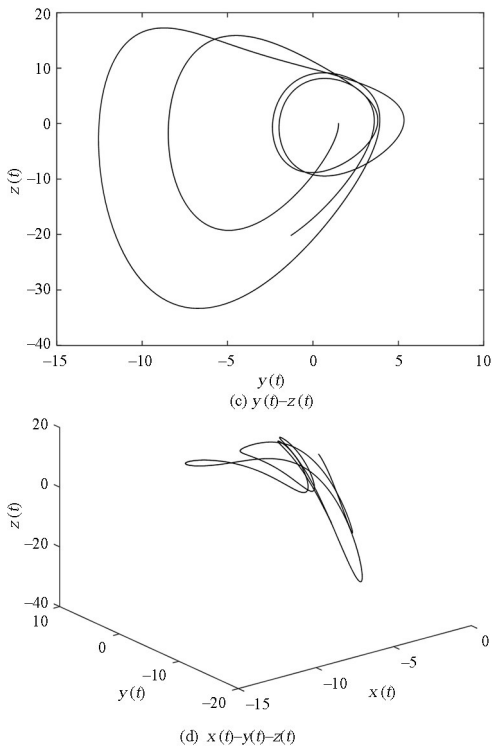


图5 当 $\alpha=0.95$ 时,新分数阶 jerk 混沌系统的相图
Fig. 5 Phase diagram of the new fractional-order chaotic jerk system when $\alpha=0.95$

通过观察图4和图5,得知当导数阶数 α 的取值不同,系统的混沌程度也不同.当导数阶数 α 发生改变时,新分数阶 jerk 系统的动力学行为也会发生改变.

3 结论

本文详细阐述了一种求解分数阶动力系统的数值方法,并利用此方法对新分数阶 jerk 混沌系统进行有效数值模拟.数值结果的研究表明,该方法是可行、有效的.该方法的主要优点是能够有效地捕捉并描述新分数阶 jerk 混沌系统的动力学行为,并为新分数阶 jerk 混沌系统的研究提供了新的视角和理论基础.此外,通过探讨系统在不同系统阶数 α 下的相图,可以发现系统阶数 α 对新分数阶 jerk 混沌系统的动力学行为有显著影响.

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