文章编号:1672-6553-2022-20(6)-106-08　　DOI:10.6052/1672-6553-2022-012

变质量力学系统的 Herglotz 型 Lagrange 方程与 Noether 对称性和守恒量*

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摘要　Herglotz 变分原理提供了非保守耗散问题的变分描述,同时变质量力学在自然界和工程领域有大量的应用,因此将 Herglotz 变分原理应用于变质量力学系统的 Lagrange 方程与守恒律研究,为研究变质量力学提供了一个新的途径. 文中建立了变质量力学系统的 Herglotz 型广义变分原理,导出了变质量系统的 Herglotz 型 Lagrange 方程. 定义了变质量力学系统的 Herglotz 型 Noether 对称性,建立并证明了 Herglotz 型 Noether 定理及其逆定理. 文末给出两个变质量非保守系统的具体例子以说明结果的应用.

关键词　变质量力学系统,　Herglotz 型广义变分原理,　Noether 定理,　守恒量

中图分类号:O316　　**文献标志码:**A

引言

众所周知,对于非保守这样一类自然界广泛存在的现象,人们无法将其纳入哈密顿变分原理的经典框架之中[1]. 德国数学家 Herglotz 提出的一类变分问题[2],不仅是对经典变分原理的推广,而且提供非保守和耗散过程的变分描述. Herglotz 变分问题与经典变分问题的不同之处在于其微分方程不仅取决于时间、曲线及其导数,还取决于积分泛函本身. Donchev[3] 和 Lazo 等[4,5] 对若干重要的非保守经典和量子系统的变分描述进行了研究,如:黏性力下振动弦、非保守电磁理论、非保守薛定谔方程等. Georgieva 和 Guenther 研究了 Herglotz 型 Noether 定理[6,7]. Santos 等[8,9] 将 Herglotz 变分问题推广到高阶微商情形. 近年来,张毅等将 Herglotz 变分问题引入力学系统[10],建立了非保守系统[11,12]、非完整系统[13] 和 Birkhoff 系统[14,15] 的 Herglotz 变分原理和 Noether 定理. 但是,到目前为止关于 Herglotz 变分原理的研究还仅限于常质量系统.

变质量系统力学研究质量变化的物体的运动与作用在其上的力之间的关系. 第一个系统地研究变质量力学的是 Meshchersky[16],通过引入质量以不为零的相对速度分离或并入物体时所产生的冲击力,他建立了变质量质点的动力学基本方程. 1989 年,杨来伍和梅凤翔[17] 在他们的专著中系统地介绍了变质量系统的牛顿力学和变质量系统的分析力学. 变质量系统在自然界和工程技术领域有大量应用实例[18-20],包括复杂系统,如火箭运动、移动机器人拾取或释放物体等,以及简单系统,如喷水系统或漏气的充气气球的运动等. 守恒律是变质量系统力学研究的一个重要方面. Cveticanin 利用 d'Alembert-Lagrange 原理构建了变质量系统的 Noether 守恒律[21]. 梅凤翔在其著作[1,22] 中研究了变质量力学系统的对称性与守恒量. 最近,姜文安等[23-25] 研究了一类变质量系统的 Noether 守恒律和 Mei 守恒律.

变质量力学系统的问题可分为两类[26]:一类

2022-01-04 收到第 1 稿, 2022-03-26 收到修改稿.
* 国家自然科学基金赞助项目(11972241,11572212),江苏省自然科学基金资助项目(BK20191454)
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是质量变化不改变运动学性质;另一类是质量的变化将引起系统运动学性质的改变. 本文研究第一类问题. 将 Herglotz 广义变分原理推广到变质量非保守力学系统,导出变质量系统的 Herglotz 型 Lagrange 方程,研究并证明其 Noether 定理及其逆定理.

1　变质量力学系统的 Herglotz 型 Lagrange 方程

假设变质量力学系统由 N 个质点组成. 在时刻 t,第 i 个质点的质量为 $m_i(i=1,2,\cdots,N)$;在时刻 $t+\mathrm{d}t$,由质点分离(或并入)的微粒质量为 $\mathrm{d}m_i$. 设系统所受完整约束中不含质点的质量,质点系的位形由 n 个广义坐标 $q_s(s=1,2,\cdots,n)$ 确定,并假设

$$m_i = m_i(t,q_s,\dot{q}_s) \tag{1}$$

变质量力学系统的 Herglotz 变分问题可表述如下:

确定函数 $q_s(t)$ $(t\in[t_1,t_2])$,使得泛函 $z(t_2)$ 取得极值,即 $z(t_2)\rightarrow\mathrm{extr.}$,其中 $z(t)$ 由微分方程

$$\dot{z}(t) = L(t,q_s(t),\dot{q}_s(t),z(t),m_i(t,q_s,\dot{q}_s)) \tag{2}$$

定义,且满足端点条件

$$q_s(t)|_{t=t_1}=q_{s_1},\ q_s(t)|_{t=t_2}=q_{s_2}\quad(s=1,2,\cdots,n) \tag{3}$$

和初始条件

$$z(t)|_{t=t_1}=z_1 \tag{4}$$

泛函 $z(t)$ 也称为 Herglotz 作用量.

令 $\frac{Д}{Дt},\frac{Д}{Дq_s}$ 以及 $\frac{Д}{Д\dot{q}_s}$ 分别表示将质量当作常数时对 t,q_s 以及 $\dot{q}_s$ 的偏导数,称为凝固偏导数;符号 $\frac{D}{\mathrm{D}t}$ 表示将质量当作常数时对时间的导数,称为凝固导数.

对方程(2)取等时变分,有

$$\delta\dot{z} = \frac{ДL}{Дq_s}\delta q_s + \frac{ДL}{Д\dot{q}_s}\delta\dot{q}_s + \frac{\partial L}{\partial m_i}\delta m_i + \frac{\partial L}{\partial z}\delta z \tag{5}$$

其中

$$\delta m_i = \frac{\partial m_i}{\partial q_s}\delta q_s + \frac{\partial m_i}{\partial\dot{q}_s}\delta\dot{q}_s \tag{6}$$

由交换关系

$$\delta\dot{z} = \frac{\mathrm{d}}{\mathrm{d}t}\delta z \tag{7}$$

则方程(5)可以表示为

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta z = A + \frac{\partial L}{\partial z}\delta z \tag{8}$$

其中

$$A = \frac{ДL}{Дq_s}\delta q_s + \frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial q_s}\delta q_s + \frac{ДL}{Д\dot{q}_s}\delta\dot{q}_s + \frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot{q}_s}\delta\dot{q}_s \tag{9}$$

方程(8)有解

$$\delta z(t)\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right) - \delta z(t_1) = \int_{t_1}^{t}A\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\mathrm{d}\tau \tag{10}$$

考虑到初始条件(4),且 $z(t_2)$ 取得极值,因此有

$$\delta z(t_1) = \delta z(t_2) = 0 \tag{11}$$

所以有

$$\int_{t_1}^{t_2}A\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\mathrm{d}\tau = 0 \tag{12}$$

将式(9)代入方程(12),并对其中含 $\delta\dot{q}_s$ 的项进行分部积分,得

$$\int_{t_1}^{t_2}\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left\{\frac{ДL}{Дq_s} + \frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial q_s} - \frac{D}{\mathrm{D}t}\frac{ДL}{Д\dot{q}_s} - \frac{\partial L}{\partial m_i}\frac{ДL}{Д\dot{q}_s}\dot{m}_i - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot{q}_s}\right) + \frac{\partial L}{\partial z}\frac{ДL}{Д\dot{q}_s} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot{q}_s}\right\}\delta q_s\mathrm{d}t = 0 \tag{13}$$

由于 δq_s 的独立性,根据变分学基本引理,可得

$$\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(-\frac{D}{\mathrm{D}t}\frac{ДL}{Д\dot{q}_s} + \frac{ДL}{Дq_s} + \frac{\partial L}{\partial z}\frac{ДL}{Д\dot{q}_s} + \psi_s\right) = 0\quad(s=1,2,\cdots,n) \tag{14}$$

其中

$$\psi_s = \frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial q_s} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot{q}_s} - \frac{\partial}{\partial m_i}\frac{ДL}{Д\dot{q}_s}\dot{m}_i - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot{q}_s}\right) \tag{15}$$

方程(14)称为变质量力学系统用凝固导数表示的 Herglotz 型 Lagrange 方程.

容易证明下述关系:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{ДL}{Д\dot{q}_s}=\frac{D}{Dt}\frac{ДL}{Д\dot{q}_s}+\frac{\partial}{\partial m_i}\frac{ДL}{Д\dot{q}_s}\dot{m}_i \tag{16}$$

将式(16)代入式(14)可得：

$$\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{ДL}{Д\dot{q}_s}+\frac{ДL}{Дq_s}+\frac{\partial L}{\partial z}\frac{ДL}{Д\dot{q}_s}+\varphi_s\right)=0\ (s=1,2,\cdots,n) \tag{17}$$

其中

$$\varphi_s=\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial q_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial \dot{q}_s}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial \dot{q}_s}\right) \tag{18}$$

方程(17)称为变质量力学系统用半凝固导数表示的 Herglotz 型 Lagrange 方程.

容易证明下述关系：

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{ДL}{Д\dot{q}_s}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial \dot{q}_s}\right) \tag{19}$$

$$\frac{\partial L}{\partial q_s}=\frac{ДL}{Дq_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial q_s} \tag{20}$$

将式(19)和式(20)代入式(17)可得

$$\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}+\frac{\partial L}{\partial q_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_s}\right)=0\ (s=1,2,\cdots,n) \tag{21}$$

方程(21)称为变质量力学系统用普通导数表示的 Herglotz 型 Lagrange 方程.

上述三类变质量系统的 Herglotz 型 Lagrange 方程是经典变质量完整力学系统广义坐标表示的 Lagrange 方程[17]的推广. 文献[17]指出：当质点质量依赖于广义坐标、广义速度和时间时，应用凝固导数表示的方程最简便，应用普通导数表示的方程最繁杂.

2 Herglotz 型 Noether 对称性

引入时间 t 和广义坐标 q_s 的 r 参数有限变换群 G_r 的无限小变换

$$\bar{t}=t+\Delta t,\ \bar{q}_s(t)=q_s(t)+\Delta q_s\ (s=1,2,\cdots,n) \tag{22}$$

或其展开式

$$\begin{aligned}&\bar{t}=t+\varepsilon_\alpha\xi_0^\alpha(t,q_s,\dot{q}_s,m_i,z),\\&\bar{q}_s(t)=q_s(t)+\varepsilon_\alpha\xi_s^\alpha(t,q_j,\dot{q}_j,m_i,z),\\&(s,j=1,2,\cdots,n;\alpha=1,2,\cdots,r;\\&i=1,2,\cdots,N)\end{aligned} \tag{23}$$

其中，ε_α 为无限小的参数，ξ_0^α 和 ξ_s^α 称为该变换的生成元. 在变换式(22)的作用下，Herglotz 作用量 z 变成

$$\bar{z}(\bar{t})=z(t)+\Delta z(t) \tag{24}$$

全变分 Δz 是 Herglotz 作用量 z 在无限小变换中相对于 ε_α 的主线性部分的前后之差. 对任意函数 $F(t)$，有如下关系

$$\Delta F=\delta F+\dot{F}\Delta t \tag{25}$$

对于完整约束系统，有交换关系

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta F=\delta\frac{\mathrm{d}}{\mathrm{d}t}F=\delta\dot{F} \tag{26}$$

因此得到

$$\Delta\dot{F}=\frac{\mathrm{d}}{\mathrm{d}t}\Delta F-\dot{F}\frac{\mathrm{d}}{\mathrm{d}t}\Delta t \tag{27}$$

对式(2)进行非等时变分：

$$\Delta\dot{z}=\frac{ДL}{Дt}\Delta t+\frac{ДL}{Дq_s}\Delta q_s+\frac{ДL}{Д\dot{q}_s}\Delta\dot{q}_s+\frac{\partial L}{\partial z}\Delta z+\frac{\partial L}{\partial m_i}\Delta m_i \tag{28}$$

由于

$$\Delta m_i=\frac{\partial m_i}{\partial t}\Delta t+\frac{\partial m_i}{\partial q_s}\Delta q_s+\frac{\partial m_i}{\partial \dot{q}_s}\Delta\dot{q}_s \tag{29}$$

将式(29)代入式(28)，并利用关系式(27)，得

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Delta z=&\frac{ДL}{Дt}\Delta t+\frac{ДL}{Дq_s}\Delta q_s+\frac{ДL}{Д\dot{q}_s}\Delta\dot{q}_s+\frac{\partial L}{\partial z}\Delta z+\\&\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial t}\Delta t+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial q_s}\Delta q_s+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial \dot{q}_s}\Delta\dot{q}_s+\\&L\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\end{aligned} \tag{30}$$

方程(30)的解为

$$\begin{aligned}&\Delta z(t)\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)-\Delta z(t_1)=\\&\int_{t_1}^{t}\left\{\exp\left(-\int_{t_1}^{\tau}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{ДL}{Дt}\Delta t+\right.\right.\\&\frac{ДL}{Дq_s}\Delta q_s+\frac{ДL}{Д\dot{q}_s}\Delta\dot{q}_s+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial t}\Delta t+\\&\left.\left.\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial q_s}\Delta q_s+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial \dot{q}_s}\Delta\dot{q}_s+L\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right)\right\}\mathrm{d}\tau\end{aligned} \tag{31}$$

显然 $\Delta z(t_1)=0$，所以方程(31)还可以写为

$$\begin{aligned}&\Delta z(t)\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=\\&\int_{t_1}^{t}\left\{\frac{\mathrm{d}}{\mathrm{d}\tau}\left\{\left[L\Delta t+\left(\frac{ДL}{Д\dot{q}_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial \dot{q}_s}\right)(\Delta q_s-\dot{q}_s\Delta t)\right]\right.\right.\end{aligned}$$

$$\exp\left(-\int_t^\tau\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\Big\}+$$
$$\exp\left(-\int_{t_1}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(-\frac{D}{Dt}\frac{\text{Д}L}{\text{Д}\dot q_s}+\frac{\text{Д}L}{\text{Д}q_s}+\frac{\partial L}{\partial z}\frac{\text{Д}L}{\text{Д}\dot q_s}+\psi_s\right)(\Delta q_s-\dot q_s\Delta t)\Big\}\mathrm{d}\tau=0 \qquad (32)$$

由于

$$\Delta t=\varepsilon_\alpha\xi_0^\alpha,\Delta q_s=\varepsilon_\alpha\xi_s^\alpha \qquad (33)$$

将式(33)代入式(31)可得

$$\Delta z(t)\exp\left(-\int_{t_1}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=$$
$$\int_{t_1}^t\Big\{\exp\left(-\int_{t_1}^\tau\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\Big[\left(\frac{\text{Д}L}{\text{Д}\tau}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\tau}\right)\xi_0^\alpha+\left(\frac{\text{Д}L}{\text{Д}q_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial q_s}\right)\xi_s^\alpha+\left(\frac{\text{Д}L}{\text{Д}\dot q_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot q_s}\right)\dot\xi_s^\alpha+$$
$$\left(L-\frac{\text{Д}L}{\text{Д}\dot q_s}\dot q_s-\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot q_s}\dot q_s\right)\dot\xi_0^\alpha\Big]\Big\}\varepsilon_\alpha\mathrm{d}\tau \qquad (34)$$

将式(33)代入式(32)可得

$$\Delta z(t)\exp\left(-\int_{t_1}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=$$
$$\int_{t_1}^t\Big\{\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\{\Big[L\xi_0^\alpha+\left(\frac{\text{Д}L}{\text{Д}\dot q_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot q_s}\right)\bar\xi_s^\alpha\Big]\exp\left(-\int_{t_1}^\tau\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\Big\}+$$
$$\exp\left(-\int_{t_1}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(-\frac{D}{\mathrm{D}t}\frac{\text{Д}L}{\text{Д}\dot q_s}+\frac{\text{Д}L}{\text{Д}q_s}+\frac{\partial L}{\partial z}\frac{\text{Д}L}{\text{Д}\dot q_s}+\psi_s\right)\bar\xi_s^\alpha\Big\}\varepsilon_\alpha\mathrm{d}\tau=0 \qquad (35)$$

其中

$$\bar\xi_s^\alpha=\xi_s^\alpha-\dot q_s\xi_0^\alpha \qquad (36)$$

定义 1　对于变质量力学系统(14),如果对于无限小变换(22)的每一个变换,等式

$$\Delta z(t_2)=0 \qquad (37)$$

成立,则称变换是该系统的 Noether 对称变换.

根据定义 1 以及方程(34),可以得到如下判据.

判据 1　在变质量力学系统(14)中,对于无限小变换(23),假设生成元 $\xi_0^\alpha,\xi_s^\alpha$ 符合下列条件

$$\left(\frac{\text{Д}L}{\text{Д}t}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial t}\right)\xi_0^\alpha+\left(\frac{\text{Д}L}{\text{Д}q_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial q_s}\right)\xi_s^\alpha+\left(\frac{\text{Д}L}{\text{Д}\dot q_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot q_s}\right)\dot\xi_s^\alpha+\left(L-\frac{\text{Д}L}{\text{Д}\dot q_s}\dot q_s-\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot q_s}\dot q_s\right)\dot\xi_0^\alpha=0(\alpha=1,2,\cdots,r) \qquad (38)$$

则称变换是该系统的 Noether 对称变换.

当 $r=1$ 时,式(38)给出

$$\left(\frac{\text{Д}L}{\text{Д}t}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial t}\right)\xi_0+\left(\frac{\text{Д}L}{\text{Д}q_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial q_s}\right)\xi_s+\left(\frac{\text{Д}L}{\text{Д}\dot q_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot q_s}\right)\dot\xi_s+\left(L-\frac{\text{Д}L}{\text{Д}\dot q_s}\dot q_s-\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot q_s}\dot q_s\right)\dot\xi_0=0 \qquad (39)$$

式(39)为变质量力学系统(14)的 Noether 等式.

3　Herglotz 型 Noether 定理

定理 1　对于变质量力学系统(14),若给定的有限变换群 G_r 的无限小变换(23)是 Herglotz 型 Noether 对称变换,则系统存在 r 个线性独立的守恒量

$$I_{\mathrm{N}}^\alpha=\Big[L\xi_0^\alpha+\left(\frac{\text{Д}L}{\text{Д}\dot q_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot q_s}\right)\bar\xi_s^\alpha\Big]\exp\left(-\int_{t_1}^\tau\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=c_\alpha \qquad (40)$$

证明　由于无限小变换(23)是变质量力学系统(14)的 Herglotz 型 Noether 对称变换,根据定义 1,有 $\Delta z(t_2)=0$. 利用公式(35),得到

$$\int_{t_1}^{t_2}\Big\{\frac{\mathrm{d}}{\mathrm{d}\tau}\Big\{\Big[L\xi_0^\alpha+\left(\frac{\text{Д}L}{\text{Д}\dot q_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot q_s}\right)\bar\xi_s^\alpha\Big]\exp\left(-\int_{t_1}^\tau\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\Big\}+\exp\left(-\int_{t_1}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(-\frac{D}{\mathrm{D}t}\frac{\text{Д}L}{\text{Д}\dot q_s}+\frac{\text{Д}L}{\text{Д}q_s}+\frac{\partial L}{\partial z}\frac{\text{Д}L}{\text{Д}\dot q_s}+\psi_s\right)\bar\xi_s^\alpha\Big\}\varepsilon_\alpha\mathrm{d}\tau=0$$

(41)

由于区间 $[t_1,t_2]$ 是任意的, ε_α 是相互独立的,根据变分法的基本引理,得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\Big[L\xi_0^\alpha+\left(\frac{\text{Д}L}{\text{Д}\dot q_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot q_s}\right)\bar\xi_s^\alpha\Big]\exp\left(-\int_{t_1}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\Big\}+\exp\left(-\int_{t_1}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(-\frac{D}{Dt}\frac{\text{Д}L}{\text{Д}\dot q_s}+\frac{\text{Д}L}{\text{Д}q_s}+\frac{\partial L}{\partial z}\frac{\text{Д}L}{\text{Д}\dot q_s}+\psi_s\right)\bar\xi_s^\alpha=0$$

(42)

对于变质量力学系统有 Herglotz 型 Lagrange 方程(14),因此得

$$\frac{d}{\mathrm{d}t}\Big\{\Big[L\xi_0^\alpha+\left(\frac{\text{Д}L}{\text{Д}\dot q_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial\dot q_s}\right)\bar\xi_s^\alpha\Big]\exp$$

$$\left(-\int_{t}^{\tau}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\Big\}=0 \tag{43}$$

积分便可得到守恒量(40).

4 Herglotz 型 Noether 逆定理

已知变质量力学系统(14)有 r 个线性独立的守恒量

$$I^{\alpha}=I^{\alpha}(t,q_s,\dot{q}_s,m_i,z)=c_{\alpha}(\alpha=1,2,\cdots,r) \tag{44}$$

求满足 Herglotz 型 Noether 对称性的生成函数 ξ_0^{α} 和 ξ_s^{α}.

将式(14)乘 $\bar{\xi}_s^{\alpha}$,并对 s 求和,得到

$$\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(-\frac{D}{Dt}\frac{\partial L}{\partial \dot{q}_s}+\frac{\partial L}{\partial q_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_s}+\psi_s\right)\bar{\xi}_s^{\alpha}=0 \tag{45}$$

将式(44)对 t 求导数后与式(45)相加,展开得

$$\frac{\partial I^{\alpha}}{\partial t}+\frac{\partial I^{\alpha}}{\partial q_s}\dot{q}_s+\frac{\partial I^{\alpha}}{\partial \dot{q}_s}\ddot{q}_s+\frac{\partial I^{\alpha}}{\partial m_i}\dot{m}_s+\frac{\partial I^{\alpha}}{\partial z}L+\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(-\frac{D}{\mathrm{D}t}\frac{\partial L}{\partial \dot{q}_s}+\frac{\partial L}{\partial q_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_s}+\psi_s\right)\bar{\xi}_s^{\alpha}=0 \tag{46}$$

令含 $\ddot{q}_k$ 项的系数为0,可得

$$\frac{\partial I^{\alpha}}{\partial \dot{q}_s}+\frac{\partial I^{\alpha}}{\partial m_i}\frac{\partial m_i}{\partial \dot{q}_s}-\bar{\xi}_k^{\alpha}\exp\left(-\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k}+\frac{\partial}{\partial m_i}\frac{\partial L}{\partial \dot{q}_k}\frac{\partial m_i}{\partial \dot{q}_s}+\frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial \dot{q}_k}\right)\right]=0\ (s=1,2,\cdots n;\alpha=1,2,\cdots,r) \tag{47}$$

再令式(44)等于式(40),可得

$$\xi_0^{\alpha}=\frac{1}{L}\left[\exp\left(\int_{t_1}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)I^{\alpha}-\left(\frac{\partial L}{\partial \dot{q}_s}+\frac{\partial L}{\partial m_i}\frac{\partial m_i}{\partial \dot{q}_s}\right)\bar{\xi}_s^{\alpha}\right](\alpha=1,2,\cdots,r) \tag{48}$$

由式(47)和式(48)可以找到无限小生成函数,它们对应变质量力学系统(14)的 Noether 对称变换. 于是有:

定理2 对于变质量力学系统(14),如果已知 r 个线性独立的守恒量(44),那么生成函数由式(47)和式(48)确定的无限小变换相应于系统的 Herglotz 型 Noether 对称性.

定理2称为变质量力学系统的 Herglotz 型 Noether 逆定理.

5 算例

例1 假设一变质量质点在铅垂平面内运动,其质量为

$$m(t)=m_0\exp(-\alpha t)\ (m_0=\text{const.};\ \alpha=\text{const.},\alpha>0) \tag{49}$$

设质点的运动受黏性阻尼的作用,阻尼系数为 $c=$ const. ,微粒分离的绝对速度为零. 试研究该系统的 Noether 对称性和守恒量.

选取广义坐标 $q_1=x,q_2=y$,则系统的 Herglotz 型 Lagrange 函数为

$$L=\frac{1}{2}m(t)(\dot{q}_1^2+\dot{q}_2^2)-m(t)gq_2-\left[\frac{c}{m(t)}+\alpha\right]z \tag{50}$$

其中,泛函 z 满足

$$\dot{z}=\frac{1}{2}m(t)(\dot{q}_1^2+\dot{q}_2^2)-m(t)gq_2-\left[\frac{c}{m(t)}+\alpha\right]z \tag{51}$$

根据式(14)给出该系统的运动方程为

$$\exp[\lambda(t)][-m(t)\ddot{q}_1-c\dot{q}_1]=0,\quad \exp[\lambda(t)][-m(t)\ddot{q}_2-m(t)g-c\dot{q}_2]=0 \tag{52}$$

其中, $\lambda(t)=\dfrac{c(\mathrm{e}^{\alpha t}-\mathrm{e}^{\alpha t_1})}{\alpha m_0}+\alpha(t-t_1)$.

Noether 等式(39)给出

$$\left[-\frac{1}{2}\alpha m(t)(\dot{q}_1^2+\dot{q}_2^2)+\alpha m(t)gq_2-\frac{\alpha c}{m(t)}z\right]\xi_0-m(t)g\xi_2+m(t)\dot{q}_1\dot{\xi}_1+m(t)\dot{q}_2\dot{\xi}_2-\left\{\frac{1}{2}m(t)(\dot{q}_1^2+\dot{q}_2^2)+m(t)gq_2+\left[\frac{c}{m(t)}+\alpha\right]z\right\}\dot{\xi}_0=0 \tag{53}$$

方程(53)有解

$$\xi_0=0,\xi_1=1,\xi_2=0 \tag{54}$$

$$\xi_0=0,\xi_1=-\frac{c}{\alpha m(t)}-\ln\dot{q}_1,\xi_2=0 \tag{55}$$

由定理1,得到

$$I_N=\exp[\lambda(t)]m(t)\dot{q}_1=\text{const.} \tag{56}$$

$$I_N=\exp[\lambda(t)]m(t)\dot{q}_1\left[-\frac{c}{\alpha m(t)}-\ln\dot{q}_1\right]=$$

const. (57)

式(56)和式(57)分别为生成函数(54)、函数(55)相应的 Noether 守恒量.

根据 Herglotz 型 Noether 逆定理,由已知的守恒量求 Noether 对称性. 假设系统有守恒量(56),则由式(47)和式(48),可以得到

$$\exp[\lambda(t)]m(t)(1-\bar{\xi}_1)=0,$$
$$-\exp[\lambda(t)]m(t)\bar{\xi}_2=0 \tag{58}$$

以及

$$\xi_0=\frac{m\dot{q}_1-m\dot{q}_1\bar{\xi}_1-m\dot{q}_2\bar{\xi}_2}{L} \tag{59}$$

由方程(58)和方程(59)解得

$$\xi_0=0,\xi_1=1,\xi_2=0 \tag{60}$$

例 2　球形的雨滴在大气中下落,沿途受到水汽的充实. 设雨滴的初始半径为 r_0,由于凝聚,雨滴的质量以正比于其表面积的速率增加[22],比例系数为 α. 假设雨滴下落过程中受到与速度成正比的阻力作用,阻力系数为 μ. 试研究该系统的 Noether 对称性与守恒量.

因为

$$\frac{\mathrm{d}m}{\mathrm{d}t}=4\pi r^2\alpha,r=r_0+\alpha t \tag{61}$$

故有

$$m(t)=\frac{4}{3}\pi r^3 \tag{62}$$

系统的 Herglotz 型 Lagrange 函数为

$$L=\frac{1}{2}m(t)(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)-mgq_3-(\frac{\mu}{m}-\frac{3\alpha}{r})z \tag{63}$$

式中的 Herglotz 作用量 $z(t)$ 满足方程

$$\dot{z}=\frac{1}{2}m(t)(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)-mgq_3-(\frac{\mu}{m}-\frac{3\alpha}{r})z \tag{64}$$

根据式(14),该系统的运动微分方程为

$$\exp[\lambda(t)](-m\ddot{q}_1-\mu\dot{q}_1)=0,$$
$$\exp[\lambda(t)](-m\ddot{q}_2-\mu\dot{q}_2)=0,$$
$$\exp[\lambda(t)](-m\ddot{q}_3-mg-\mu\dot{q}_3)=0 \tag{65}$$

其中

$$\lambda(t)=3\ln(\frac{r_0+\alpha t_1}{r})+\frac{3\mu}{8\pi\alpha(r_0+\alpha t_1)^2}-\frac{\mu r}{2\alpha m} \tag{66}$$

Noether 等式(39)给出

$$\left[2\pi r^2\alpha(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)-4\pi r^2\alpha gq_3-\frac{3\alpha\mu}{mr}z\right]\xi_0-m(t)g\xi_3+m(t)\dot{q}_1\dot{\xi}_1+m(t)\dot{q}_2\dot{\xi}_2+m(t)\dot{q}_3\dot{\xi}_3-\left[\frac{1}{2}m(t)(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2)+m(t)gq_3+(\frac{\mu}{m}-\frac{3\alpha}{r})z\right]\dot{\xi}_0=0 \tag{67}$$

方程(67)有解

$$\xi_0=0,\xi_1=1,\xi_2=1,\xi_3=0 \tag{68}$$

$$\xi_0=0,\xi_1=\frac{3\mu}{8\pi\alpha r^2}-\ln\dot{q}_1,\xi_2=1,\xi_3=0 \tag{69}$$

由定理 1,得到

$$I_N=\exp[\lambda(t)][m(t)\dot{q}_1+m(t)\dot{q}_2]=\text{const.} \tag{70}$$

$$I_N=\exp[\lambda(t)][m\dot{q}_2+m\dot{q}_1(\frac{3\mu}{8\pi\alpha r^2}-\ln\dot{q}_1)]=\text{const.} \tag{71}$$

式(70)、式(71)分别为生成函数(68)、函数(69)所导致的 Noether 守恒量.

根据 Herglotz 型 Noether 逆定理,由已知守恒量求 Noether 对称性. 假设系统有守恒量(70),则由式(47)和式(48),可以得到

$$\exp[\lambda(t)]m(t)(1-\bar{\xi}_1)=0,$$
$$\exp[\lambda(t)]m(t)(1-\bar{\xi}_2)=0,$$
$$-\exp[\lambda(t)]m(t)\bar{\xi}_3=0 \tag{72}$$

以及

$$\xi_0=\frac{m(t)(\dot{q}_1+\dot{q}_2-\dot{q}_1\bar{\xi}_1-\dot{q}_2\bar{\xi}_2-\dot{q}_3\bar{\xi}_3)}{L} \tag{73}$$

由方程(72)、方程(73)解得

$$\xi_0=0,\xi_1=1,\xi_2=1,\xi_3=0 \tag{74}$$

6　结论

变质量力学系统研究的是物体质量在不断变化的状态下,物体运动状态与作用在其上力的关系. 由于变质量系统在自然界和工程领域普遍存在,因此变质量力学的研究具有重要意义. 本文研究了变质量力学系统的 Herglotz 型 Lagrange 方程与 Noether 对称性及其守恒量,主要工作如下:提出了变质量力学系统中的 Herglotz 型广义变分原理,并基于该原理建立了分别用凝固导数、半凝固导数、普通导数表示的变质量力学系统的 Herglotz 型

Lagrange 方程(14)、式(17)和式(21);提出了变质量力学系统 Herglotz 型 Noether 对称性的定义及其判据方程(39);建立并证明了变质量力学系统 Herglotz 型 Noether 定理(定理 1)及其逆定理(定理 2).

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HERGLOTZ TYPE LAGRANGE EQUATIONS AND NOETHER SYMMETRY AND CONSERVED QUANTITY FOR MECHANICAL SYSTEMS WITH VARIABLE MASS*

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Abstract Herglotz's variational principle provides a variational description of non-conservative dissipation problems, and variable mass mechanics is widely used in nature and engineering. Therefore, it provides a new way to study variable mass mechanics by applying Herglotz's variational principle to Lagrange equations and conservation laws of variable mass mechanics systems. In this paper, the Herglotz type generalized variational principle of mechanical systems with variable mass is established and the Herglotz type Lagrange equations of mechanical systems with variable mass are derived. Herglotz type Noether symmetry of variable mass mechanical systems is defined, and the Herglotz Noether theorem and its inverse theorem are established and proved. At the end of this paper, two concrete examples of non-conservative systems with variable mass are given to illustrate the application of the results.

Key words variable mass mechanical system, Herglotz type generalized variational principle, Noether theorem, conserved quantity

Received 4 January 2022, revised 26 March 2022.

* The project supported by the National Natural Science Foundation of China(11972241,11572212), and the Natural Science Foundation of Jiangsu Province of China(BK20191454)

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