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微型 Mindlin 板非线性动力学模型的建立*

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摘要　随着 MEMS 技术工艺的发展,微型结构在工程领域的应用越来越广泛. 对于微型结构,经典连续介质力学理论的本构关系中不包含任何特征长度尺度,不能反映结构在微米尺度下的尺寸效应. 本文基于 Von Karman 大变形理论和一阶剪切变形理论,把考虑尺寸效应的应变梯度理论推广至微型 Mindlin 板的非线性问题. 分别计算微结构的应变能,包括宏观变形应变能和微观变形应变能两部分,结合微型 Mindlin 板结构的动能及外力功,代入 Hamilton 原理,得到了微型 Mindlin 板在大变形情况下的非线性动力学方程及边界条件.

关键词　非线性,　Mindlin 板,　应变梯度,　尺度效应,　Hamilton 原理

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引言

近些年,微机电系统(MEMS)的迅速发展,使得整个业界都呈现出了微型化的特点,此类结构的尺寸都非常小,而实验结果表明:在微尺度下,结构会出现明显的尺度效应. 然而由于经典连续介质力学理论中并不包含特征长度参数,因此并不能预测在微尺度下板结构所表现出来的尺度效应.

从 Cosserat 兄弟提出偶应力理论开始,许多学者针对微结构的尺度效应提出了一系列的理论,1964 年,Mindlin[1] 给出了弹性全应变梯度理论,除了通常的两个拉梅常数以外,还引入五个新的材料常数;1992 年,Aifantis 和 Altan 在本构方程中引入了应变张量的二阶梯度,提出了一种梯度弹性理论. 1992 年,Aifantis[2] 构建了一种简单的应变梯度理论,采用了经典线性各向同性弹性理论所采用的拉梅常数;该理论中只使用了一个长度尺度参数. 2003 年,Lam[3] 等引入了相同的高阶平衡约束,提出了一种有三个额外材料参数的弹性应变梯度理论.

与此同时,许多学者利用应变梯度理论对微型结构进行了研究,2008 年,Papargyri-Beskou 和 Beskos[4] 运用应变梯度理论分析了弹性梯度弯曲 Kirchhoff 微型板结构的静态变形、稳定性和线性固有频率;2010 年,Papargyri-Beskou[5] 等用他们以前得到的基本方程分析了固定和简支圆形弹性梯度薄板的静态变形;2011 年,Wang[6] 等基于 Lam 等提出的弹性应变梯度理论提出了一种线性 Kirchhoff 微型板结构模型,并研究了该结构的静态变形、固有频率和屈曲;2012 年,Ramezani[7] 提出了基于标准弹性应变梯度理论的一阶剪切变形微型板结构模型,主要研究了结构的静态弯曲和线性固有频率;2014 年,Li[8] 对基于应变梯度理论的双层简支 Kirchhoff 微型板进行了研究,得出尺寸效应对板的偏转、轴向应力以及零应力面的位置有较大的影响.

以上各位学者针对微型结构进行的是线性研究,而在大变形非线性研究方面:2004 年,Lazopoulos[9] 基于 Aifantis 提出的弹性应变梯度理论,结合 von Karman 几何非线性建立了 Kirchhoff 微型板结构的方程;2011 年,Reddy[10] 研究了微结构 Euler-Bernoulli 和 Timoshenko 功能梯度材料梁的非线性振动问题;2012 年和 2013 年,Ramezani 和 Rajabi[11] 利用应变梯度理论研究了微型梁结构的非线性问题,并发现非线性是提升结构固有频率的主要原因,但在某些特殊的长厚比情况下,尺度效应也有明显提升固有频率的效果. 2013 年,Ramezani[12] 基于应变梯度理论对微型薄板结构进行了非线性振

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动的研究.

综上所述,针对微型结构的线性问题已进行了深入的研究,而应用应变梯度理论进行非线性研究工作目前还主要集中在梁结构和薄板结构,所以本文针对微型 Mindlin 板,综合考虑一阶剪切变形、Von Karman 大变形和尺寸效应的影响,把应变梯度理论推广到 Mindlin 板的非线性问题,分别计算微结构的势能,动能及外力功,代入 Hamilton 原理,得到了微型 Mindlin 板在大变形情况下的非线性动力学方程及边界条件,该方程及边界条件可应用于 MEMS 结构中微型 Mindlin 板结构,进行非线性动力学分析,同时也可应用于微型机械结构、微型太阳能帆板等微型 Mindlin 板结构的振动控制的研究.

1　应变梯度理论

应变梯度理论在本构关系中引入了材料的特征长度尺寸,并考虑了应变梯度高阶张量对应变能密度函数的影响,能够描述和解释微构件力学性能的尺寸效应现象. 应变梯度理论建立了连续介质框架下考虑应变梯度影响的新的本构模型,成为刻划微米尺度效应的有力工具,也是联系经典力学与原子力学之间现实可行的桥梁,它们是一种广义连续介质力学理论.

应变梯度理论有三种形式:

Ⅰ型应变梯度 $\eta_{ijk}=u_{k,ij}$;

Ⅱ型应变梯度 $\eta_{ijk}=\varepsilon_{jk,i}$;

Ⅲ型应变梯度 $3\eta_{ijk}=u_{i,jk}+u_{j,ki}+u_{k,ij}$,

本文研究的是大变形问题,由于Ⅱ型应变梯度理论是由应变得到的,而Ⅰ型和Ⅲ型理论只与位移有关,所以只有由Ⅱ型理论得到的应变梯度是非线性表达式,能够反映大变形问题. 所以本文基于Ⅱ型表达式,把应变梯度理论推广到微型 Mindlin 板的非线性问题. 微型 Mindlin 板的应变能密度包括宏观应变能密度 $\overline{W}$ 和微观应变能密度 $\hat{W}$ 两部分,可以表示为:

$$W(\varepsilon,\eta)=\overline{W}+\hat{W} \tag{1}$$

其中:

$$\overline{W}=\sigma_{xx}\varepsilon_{xx}+\sigma_{yy}\varepsilon_{yy}+\tau_{xy}\gamma_{xy}+\tau_{xz}\gamma_{xz}+\tau_{yz}\gamma_{yz} \tag{2}$$

$$\hat{W}=a_1\eta_{iik}\eta_{kjj}+a_2\eta_{ijj}\eta_{ikk}+a_3\eta_{iik}\eta_{jjk}+a_4\eta_{ijk}\eta_{ijk}+a_5\eta_{iik}\eta_{kji} \tag{3}$$

$a_J(J=1,2,3,4,5)$为材料的尺度参数.

根据(3)式可以得到高阶应力的表达式:

$$\tau_{ijk}=\frac{\partial\hat{W}}{\partial\eta_{ijk}}=\frac{1}{2}a_1(\eta_{iip}\delta_{qr}+2\eta_{rii}\delta_{pq}+\eta_{iiq}\delta_{pr})+a_2(\eta_{pii}\delta_{qr}+\eta_{qii}\delta_{pr})+2a_3\eta_{iir}\delta_{pq}+2a_4\eta_{pqr}+a_5(\eta_{rqp}+\eta_{rpq}) \tag{4}$$

δ_{ij}为克罗内克函数.

2　模型的建立

针对微型 Mindlin 板,尺寸为 $a\times b\times h$,

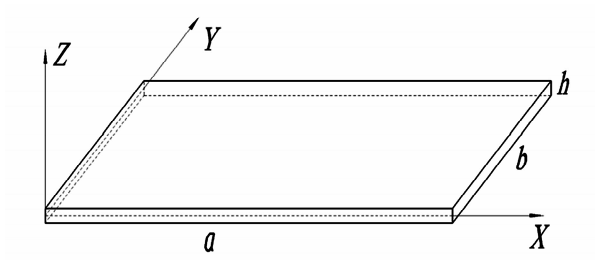

图 1　结构示意图

Fig. 1　The illustration of the Mindlin plate

综合考虑一阶剪切变形、Von Karman 大变形和尺度效应的影响,利用 Hamilton 原理建立板结构的非线性动力学模型.

对于 Mindlin 板,板结构内任意一点沿 X、Y、Z 方向的位移分别为:

$$\begin{aligned}u(x,y,z,t)&=u_0(x,y,t)+z\varphi_x(x,y,t)\\v(x,y,z,t)&=v_0(x,y,t)+z\varphi_y(x,y,t)\\w(x,y,z,t)&=w_0(x,y,t)\end{aligned} \tag{5}$$

其中 u_0,v_0,w_0 为中面上任意点在 X、Y、Z 方向的位移,φ_x,φ_y 分别为绕 Y 轴和 X 轴的转角.

由 Von Karman 大变形理论,可以得到非线性应变表达式:

$$\begin{aligned}\varepsilon_{xx}&=\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2+z\frac{\partial\varphi_x}{\partial x}=\varepsilon_{xx}^{(0)}+z\varepsilon_{xx}^{(1)}\\\varepsilon_{yy}&=\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2+z\frac{\partial\varphi_y}{\partial y}=\varepsilon_{yy}^{(0)}+z\varepsilon_{yy}^{(1)}\\\gamma_{xy}&=\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)+z\left(\frac{\partial\varphi_x}{\partial y}+\frac{\partial\varphi_y}{\partial x}\right)=\gamma_{xy}^{(0)}+z\gamma_{xy}^{(1)}\\\gamma_{xz}&=\varphi_x+\frac{\partial w}{\partial x}\\\gamma_{yz}&=\varphi_y+\frac{\partial w}{\partial y}\end{aligned} \tag{6}$$

由本构关系得到应力表达式为:

$$\sigma_{xx}=\frac{E}{1-\upsilon^2}(\varepsilon_{xx}+\upsilon\varepsilon_{yy})$$

$$\sigma_{yy}=\frac{E}{1-\upsilon^2}(\varepsilon_{yy}+\upsilon\varepsilon_{xx})$$

$$\tau_{xy}=\frac{E}{2(1+\upsilon)}\gamma_{xy}$$

$$\tau_{xz}=\frac{E}{2(1+\upsilon)}\gamma_{xz}$$

$$\tau_{yz}=\frac{E}{2(1+\upsilon)}\gamma_{yz} \tag{7}$$

E,υ 分别为材料的弹性模量和泊松比.

设:

$$\begin{Bmatrix}\overline{Q}_x\\ \overline{Q}_y\end{Bmatrix}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\begin{Bmatrix}\tau_{xz}\\ \tau_{yz}\end{Bmatrix}dz$$

$$\begin{Bmatrix}\overline{N}_x\\ \overline{N}_y\\ \overline{N}_{xy}\end{Bmatrix}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}dz \tag{8}$$

$$\begin{Bmatrix}\overline{M}_x\\ \overline{M}_y\\ \overline{M}_{xy}\end{Bmatrix}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{Bmatrix}zdz$$

其中:h 为板结构厚度.

基于经典连续介质力学,微型板结构宏观变形的应变能表达式为:

$$\overline{U}=\frac{1}{2}\int_{\Omega}\overline{N}_{xx}\varepsilon_{xx}^{(0)}+\overline{N}_{yy}\varepsilon_{yy}^{(0)}+\overline{N}_{xy}\gamma_{xy}^{(0)}+\overline{M}_x\varepsilon_{xx}^{(1)}+\overline{M}_y\varepsilon_{yy}^{(1)}+\overline{M}_{xy}\varepsilon_{xy}^{(1)}+\overline{Q}_{xz}\gamma_{xz}+\overline{Q}_{yz}\gamma_{yz})\mathrm{d}x\mathrm{d}y \tag{9}$$

把由 Von Karman 大变形理论得到的非线性应变表达式(6)代入Ⅱ型应变梯度理论表达式 $\eta_{ijk}=\eta_{ikj}=\varepsilon_{jk,i}=\varepsilon_{kj,i}(i,j,k=x,y,z)$,得到非线性应变梯度表达式如下:

$$\eta_{xxx}=\frac{\partial^2}{\partial x^2}u_0+(\frac{\partial}{\partial x}w)(\frac{\partial^2}{\partial x^2}w)+z(\frac{\partial^2}{\partial x^2}\varphi_x)=\eta_{xxx}^{(0)}+z\eta_{xxx}^{(1)}$$

$$\eta_{xxy}=\eta_{xyx}=\frac{\partial^2}{\partial x\partial y}u_0+\frac{\partial^2}{\partial x^2}v_0+(\frac{\partial}{\partial x}w)(\frac{\partial^2}{\partial x\partial y}w)+(\frac{\partial}{\partial y}w)(\frac{\partial^2}{\partial x^2}w)+z(\frac{\partial^2}{\partial x\partial y}\varphi_x+\frac{\partial^2}{\partial x^2}\varphi_y)=\eta_{xxy}^{(0)}+z\eta_{xxy}^{(1)}=\eta_{xyx}^{(0)}+z\eta_{xyx}^{(1)}$$

$$\eta_{xyy}=\frac{\partial^2}{\partial x\partial y}v_0+(\frac{\partial}{\partial y}w)(\frac{\partial^2}{\partial x\partial y}w)+z(\frac{\partial^2}{\partial x\partial y}\varphi_y)=\eta_{xyy}^{(0)}+z\eta_{xyy}^{(1)}$$

$$\eta_{xxz}=\eta_{xzx}=\frac{\partial}{\partial x}\varphi_x+\frac{\partial^2}{\partial x^2}w=\eta_{xxz}^{(2)}=\eta_{xzx}^{(2)}$$

$$\eta_{xyz}=\eta_{xzy}=\frac{\partial}{\partial x}\varphi_y+\frac{\partial^2}{\partial x\partial y}w=\eta_{xyz}^{(2)}=\eta_{xzy}^{(2)}$$

$$\eta_{yxx}=\frac{\partial^2}{\partial x\partial y}u_0+(\frac{\partial}{\partial x}w)(\frac{\partial^2}{\partial x\partial y}w)+z(\frac{\partial^2}{\partial x\partial y}\varphi_x)=\eta_{yxx}^{(0)}+z\eta_{yxx}^{(1)}$$

$$\eta_{yxy}=\eta_{yyx}=\frac{\partial^2}{\partial y^2}u_0+\frac{\partial^2}{\partial x\partial y}v_0+(\frac{\partial}{\partial y}w)(\frac{\partial^2}{\partial x\partial y}w)+(\frac{\partial}{\partial x}w)(\frac{\partial^2}{\partial y^2}w)+z(\frac{\partial^2}{\partial y^2}\varphi_x+\frac{\partial^2}{\partial x\partial y}\varphi_y)=\eta_{yxy}^{(0)}+z\eta_{yxy}^{(1)}=\eta_{yyx}^{(0)}+z\eta_{yyx}^{(1)}$$

$$\eta_{yyy}=\frac{\partial^2}{\partial y^2}v_0+(\frac{\partial}{\partial y}w)(\frac{\partial^2}{\partial y^2}w)+z(\frac{\partial^2}{\partial y^2}\varphi_y)=\eta_{yyy}^{(0)}+z\eta_{yyy}^{(1)}$$

$$\eta_{yxz}=\eta_{yzx}=\frac{\partial}{\partial y}\varphi_x+\frac{\partial^2}{\partial x\partial y}w=\eta_{yxz}^{(2)}=\eta_{yzx}^{(2)}$$

$$\eta_{yyz}=\eta_{yzy}=\frac{\partial}{\partial y}\varphi_y+\frac{\partial^2}{\partial y^2}w=\eta_{yyz}^{(2)}=\eta_{yzy}^{(2)}$$

$$\eta_{zxx}=\frac{\partial}{\partial x}\varphi_x=\eta_{zxx}^{(2)}$$

$$\eta_{zxy}=\eta_{zyx}=\frac{\partial}{\partial y}\varphi_x+\frac{\partial}{\partial x}\varphi_y=\eta_{zxy}^{(2)}=\eta_{zyx}^{(2)}$$

$$\eta_{zyy}=\frac{\partial}{\partial y}\varphi_y=\eta_{zyy}^{(2)}$$

$$\eta_{xzz}=\eta_{yzz}=\eta_{zzz}=\eta_{zxz}=\eta_{zzx}=\eta_{zyz}=\eta_{zzy}=0 \tag{10}$$

代入(4)式,得到非线性高阶应力表达式:

$$\tau_{xxx}=2(a_1+a_2+a_3+a_4+a_5)\eta_{xxx}+(a_1+2a_2)\eta_{xyy}+(a_1+2a_3)\eta_{yyx}$$

$$\tau_{xxy}=(a_1+2a_5)\eta_{yxx}+(a_1+2a_3)\eta_{yyy}+2(a_3+a_4)\eta_{xxy}$$

$$\tau_{xyx}=2(a_4+a_5)\eta_{xxy}$$

$$\tau_{xyy}=(a_1+2a_2)\eta_{xxx}+(a_1+2a_5)\eta_{yyx}+2(a_2+a_4)\eta_{xyy}$$

$$\tau_{xxz}=(a_1+2a_5)\eta_{zxx}+2(a_3+a_4)\eta_{xxz}+a_1\eta_{zyy}+2a_3\eta_{yyz}$$

$$\tau_{xzx}=2(a_4+a_5)\eta_{xxz}$$

$$\tau_{xyz}=2a_4\eta_{xyz}+2a_5\eta_{zxy}$$

$$\tau_{xzy}=2a_4\eta_{xyz}+2a_5\eta_{yxz}$$

$$\tau_{xzz}=(a_1+2a_2)\eta_{xxx}+a_1\eta_{yyx}+2a_2\eta_{xyy}$$

$$\tau_{yxx}=(a_1+2a_5)\eta_{xxy}+(a_1+2a_2)\eta_{yyy}+(2a_2+2a_4)\eta_{yxx}$$

$$\tau_{yzy}=(2a_4+2a_5)\eta_{yxy}$$

$$\tau_{yyx}=(a_1+2a_3)\eta_{xxx}+(a_1+2a_5)\eta_{xyy}+(2a_3+2a_4)\eta_{yyx}$$

$$\tau_{yyy}=(a_1+2a_3)\eta_{xxy}+(a_1+2a_2)\eta_{yxx}+(2a_1+2a_2+2a_3+2a_4+2a_5)\eta_{yyy}$$

$$\tau_{yxz}=2a_4\eta_{yxz}+2a_5\eta_{zxy}$$

$$\tau_{yzx}=2a_4\eta_{yxz}+2a_5\eta_{xyz}$$

$$\tau_{yyz}=(a_1+2a_5)\eta_{zyy}+(2a_3+2a_4)\eta_{yyz}+$$

$$a_1\eta_{zxx}+2a_3\eta_{xxz}$$
$$\tau_{yzy}=(2a_4+2a_5)\eta_{yzy}$$
$$\tau_{yzz}=(a_1+2a_2)\eta_{yyy}+a_1\eta_{xxy}+2a_2\eta_{yxx}$$
$$\tau_{zxx}=(a_1+2a_5)\eta_{xxz}+(2a_2+2a_4)\eta_{zxx}+a_1\eta_{yyz}+2a_2\eta_{zyy}$$
$$\tau_{zxy}=2a_4\eta_{zxy}+2a_5\eta_{yxz}$$
$$\tau_{zyx}=2a_4\eta_{zxy}+2a_5\eta_{xyz}$$
$$\tau_{zyy}=(a_1+2a_5)\eta_{yyz}+(2a_2+2a_4)\eta_{zyy}+a_1\eta_{xxz}+2a_2\eta_{zxx}$$
$$\tau_{zzx}=(a_1+2a_3)\eta_{xxx}+a_1\eta_{xyy}+2a_3\eta_{yyx}$$
$$\tau_{zzy}=(a_1+2a_3)\eta_{yyy}+a_1\eta_{yxx}+2a_3\eta_{xxy}$$
$$\tau_{zzz}=(a_1+2a_3)\eta_{xxz}+(a_1+2a_3)\eta_{yyz}+(a_1+2a_2)\eta_{zxx}+(a_1+2a_2)\eta_{zyy}$$
$$\tau_{zxz}=\tau_{zyz}=0 \tag{11}$$

同时设：

$$\{\hat{Q}_{zjk}\}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\{\tau_{zjk}\}\mathrm{d}z\ (j,k=x,y)$$
$$\{\hat{Q}_{izk}\}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\{\tau_{izk}\}\mathrm{d}z\ (i,k=x,y)$$
$$\{\hat{Q}_{ijz}\}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\{\tau_{ijz}\}\mathrm{d}z\ (i,j=x,y)$$
$$\{\hat{N}_{ijk}\}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\{\tau_{ijk}\}\mathrm{d}z\ (i,j,k=x,y)$$
$$\{\hat{M}_{ijk}\}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\{\tau_{ijk}\}z\mathrm{d}z\ (i,j,k=x,y) \tag{12}$$

则微观变形部分的应变能为：

$$\hat{U}=\frac{1}{2}\int_\Omega(\hat{N}_{ijk}\eta_{ijk}^{(0)}+\hat{M}_{ijk}\eta_{ijk}^{(1)}+\hat{Q}_{zjk}\eta_{zjk}^{(2)}+\hat{Q}_{izk}\eta_{izk}^{(2)}+\hat{Q}_{ijz}\eta_{ijz}^{(2)})\mathrm{d}x\mathrm{d}y \tag{13}$$

由此考虑尺度效应微型板结构的总应变能为：

$$U=\overline{U}+\hat{U} \tag{14}$$

结构的动能和外力功分别为：

$$T=\frac{1}{2}\int_\Omega\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho(\dot{u}^2+\dot{v}^2+\dot{w}^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{15}$$

$$W_E=\int_\Omega(qw)\mathrm{d}x\mathrm{d}y \tag{16}$$

其中：τ_{ijk}为高阶应力，η_{ijk}为应变梯度，ρ 为材料密度，$q(x,y)$为结构所受横向分布力.

将(14)、(15)、(16)三式代入 Hamilton 原理：

$$\int_{t_1}^{t2}(\delta T-\delta U+\delta W_E)\mathrm{d}t=0 \tag{17}$$

同时设：$\begin{Bmatrix}I_0\\I_1\\I_2\end{Bmatrix}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\begin{Bmatrix}1\\z\\z^2\end{Bmatrix}\mathrm{d}z$

整理得到微型 Mindlin 板的非线性动力学方程：

$$\frac{\partial}{\partial x}\overline{N}_x+\frac{\partial}{\partial y}\overline{N}_{xy}-\frac{\partial^2}{\partial x^2}\hat{N}_{xxx}-\frac{\partial^2}{\partial x\partial y}\hat{N}_{xxy}-\frac{\partial^2}{\partial x\partial y}\hat{N}_{yxx}-\frac{\partial^2}{\partial y^2}\hat{N}_{yxy}=I_0\frac{\partial^2u_0}{\partial t^2}+I_1\frac{\partial^2\varphi_x}{\partial t^2} \tag{18}$$

$$\frac{\partial}{\partial x}\overline{N}_{xy}+\frac{\partial}{\partial y}\overline{N}_y-\frac{\partial^2}{\partial y^2}\hat{N}_{yyy}-\frac{\partial^2}{\partial x^2}\hat{N}_{xxy}-\frac{\partial^2}{\partial x\partial y}\hat{N}_{xyy}-\frac{\partial^2}{\partial x\partial y}\hat{N}_{yxy}=I_0\frac{\partial^2v_0}{\partial t^2}+I_1\frac{\partial^2\varphi_y}{\partial t^2} \tag{19}$$

$$\begin{aligned}&\frac{\partial}{\partial y}\overline{Q}_y+\frac{\partial}{\partial x}\overline{Q}_x-\frac{\partial^2}{\partial x^2}\hat{Q}_{xxz}-\frac{\partial^2}{\partial x\partial y}\hat{Q}_{yxz}-\frac{\partial^2}{\partial x\partial y}\hat{Q}_{xyz}-\frac{\partial^2}{\partial y^2}\hat{Q}_{yyz}+2\overline{N}_{xy}\left(\frac{\partial^2w}{\partial x\partial y}\right)+\\&\overline{N}_x\left(\frac{\partial^2w}{\partial x^2}\right)+\overline{N}_y\left(\frac{\partial^2w}{\partial y^2}\right)+\left(\frac{\partial\overline{N}_{xy}}{\partial y}\right)\left(\frac{\partial w}{\partial x}\right)+\\&\left(\frac{\partial\overline{N}_y}{\partial y}\right)\left(\frac{\partial w}{\partial y}\right)+\left(\frac{\partial\overline{N}_x}{\partial x}\right)\left(\frac{\partial w}{\partial x}\right)+\\&\left(\frac{\partial\overline{N}_{xy}}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial^2}{\partial x\partial y}\hat{N}_{xxy}\right)-\\&\left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial^2}{\partial x\partial y}\hat{N}_{yxy}\right)-\left(\frac{\partial^2w}{\partial y^2}\right)\left(\frac{\partial}{\partial x}\hat{N}_{xyy}\right)-\\&\left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial^2}{\partial x\partial y}\hat{N}_{xyy}\right)-\left(\frac{\partial^2w}{\partial x^2}\right)\left(\frac{\partial}{\partial y}\hat{N}_{yxx}\right)-\\&\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial^2}{\partial x\partial y}\hat{N}_{yxx}\right)-2\left(\frac{\partial^2w}{\partial x\partial y}\right)\left(\frac{\partial}{\partial x}\hat{N}_{xxy}\right)-\\&\left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial^2}{\partial x^2}\hat{N}_{xxy}\right)-\left(\frac{\partial^2w}{\partial x^2}\right)\left(\frac{\partial}{\partial x}\hat{N}_{xxx}\right)-\\&\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2}\hat{N}_{xxx}\right)-2\left(\frac{\partial^2w}{\partial x\partial y}\right)\left(\frac{\partial}{\partial y}\hat{N}_{yxy}\right)-\\&\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial^2}{\partial y^2}\hat{N}_{yxy}\right)-\left(\frac{\partial^2w}{\partial y^2}\right)\left(\frac{\partial}{\partial y}\hat{N}_{yyy}\right)-\\&\left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial^2}{\partial y^2}\hat{N}_{yyy}\right)+q=I_0\left(\frac{\partial^2w}{\partial t^2}\right)\end{aligned} \tag{20}$$

$$\frac{\partial}{\partial x}\overline{M}_x+\frac{\partial}{\partial y}\overline{M}_{xy}-\overline{Q}_x-\frac{\partial^2}{\partial x^2}\hat{M}_{xxx}-\frac{\partial^2}{\partial x\partial y}\hat{M}_{xxy}-\frac{\partial^2}{\partial x\partial y}\hat{M}_{yxx}-\frac{\partial^2}{\partial y^2}\hat{M}_{yxy}+\frac{\partial}{\partial x}\hat{Q}_{xxz}+\frac{\partial}{\partial y}\hat{Q}_{yxz}=I_1\frac{\partial^2u_0}{\partial t^2}+I_2\frac{\partial^2\varphi_x}{\partial t^2} \tag{21}$$

$$\frac{\partial}{\partial x}\overline{M}_{xy}+\frac{\partial}{\partial y}\overline{M}_y-\overline{Q}_y-\frac{\partial^2}{\partial x^2}\hat{M}_{xxy}-\frac{\partial^2}{\partial x\partial y}\hat{M}_{xyy}-\frac{\partial^2}{\partial x\partial y}\hat{M}_{yxy}-\frac{\partial^2}{\partial y^2}\hat{M}_{yyy}+\frac{\partial}{\partial x}\hat{Q}_{xyz}+\frac{\partial}{\partial y}\hat{Q}_{yyz}=I_1\frac{\partial^2v_0}{\partial t^2}+I_2\frac{\partial^2\varphi_y}{\partial t^2} \tag{22}$$

同时得到了微型 Mindlin 板的边界条件:见表 1、表 2:

表 1　$x=0,\ a$ 边界条件

Table 1　Boundary conditions for $x=0,\ a$

force boundary conditions	displacement boundary conditions
$\overline{N}_x-\frac{\partial}{\partial x}\hat{N}_{xxx}=0$	$u_0=0$
$\overline{N}_{xy}-\frac{\partial}{\partial x}\hat{N}_{xxy}=0$	$v_0=0$
$\overline{Q}_x-\frac{\partial}{\partial x}\hat{N}_{xxz}=0$	$w=0$
$\overline{M}_x+\hat{Q}_{zxx}+\hat{Q}_{xxz}-\frac{\partial}{\partial x}\hat{M}_{xxx}=0$	$\varphi_x=0$
$\overline{M}_{xy}+\hat{Q}_{zxy}+\hat{Q}_{xyz}-\frac{\partial}{\partial x}\hat{M}_{xxy}=0$	$\varphi_y=0$
$\hat{N}_{xxx}=0$	$\frac{\partial u_0}{\partial x}=0$
$\hat{N}_{xxy}=0$	$\frac{\partial v_0}{\partial x}=0$
$\hat{Q}_{xxz}+\left(\frac{\partial w}{\partial x}\right)\hat{N}_{xxx}=0$	$\frac{\partial w}{\partial x}=0$
$\hat{M}_{xxx}=0$	$\frac{\partial \varphi_x}{\partial x}=0$
$\hat{M}_{xxy}=0$	$\frac{\partial \varphi_y}{\partial x}=0$

表 2　$y=0,\ b$ 边界条件

Table 2　Boundary conditions for $y=0,\ b$

force boundary conditions	displacement boundary conditions
$\overline{N}_{xy}-\frac{\partial}{\partial x}\hat{N}_{yxy}=0$	$u_0=0$
$\overline{N}_y-\frac{\partial}{\partial x}\hat{N}_{yyy}=0$	$v_0=0$
$\overline{Q}_y-\frac{\partial}{\partial y}\hat{N}_{yyz}=0$	$w=0$
$\overline{M}_{xy}+\hat{Q}_{zxy}+\hat{Q}_{yxz}-\frac{\partial}{\partial y}\hat{M}_{yxy}=0$	$\varphi_x=0$
$\overline{M}_y+\hat{Q}_{zyy}+\hat{Q}_{yyz}-\frac{\partial}{\partial y}\hat{M}_{yyy}=0$	$\varphi_y=0$
$\hat{N}_{yxy}=0$	$\frac{\partial u_0}{\partial y}=0$
$\hat{N}_{yyy}=0$	$\frac{\partial v_0}{\partial y}=0$
$\hat{Q}_{yyz}+\left(\frac{\partial w}{\partial y}\right)\hat{N}_{yyy}=0$	$\frac{\partial w}{\partial y}=0$
$\hat{M}_{yxy}=0$	$\frac{\partial \varphi_x}{\partial y}=0$
$\hat{M}_{yyy}=0$	$\frac{\partial \varphi_y}{\partial y}=0$

分别把(7)、(11)代入(8)和(12)式,再代入到动力学方程(18)~(22)和边界条件中,得到由位移表示的非线性动力学方程和边界条件. 限于篇幅有限,本文并未列出.

3　结论

本文基于 Von Karman 大变形理论和一阶剪切变形理论,将应变梯度理论推广到 Mindlin 板的非线性问题,采用能够反映大变形的Ⅱ型应变梯度理论,分别计算微结构宏观变形的应变能和微观变形的应变能,结合微型 Mindlin 板结构的动能及外力功,基于 Hamilton 原理,得到了微型 Mindlin 板大变形情况下的非线性动力学方程及边界条件,方程和边界条件中分别包含宏观变形项和微观变形项,为 MEMS 结构中微型 Mindlin 板结构的非线性动力学分析提供了理论基础,同时方程可应用于微型机械结构、微型太阳能帆板等微型 Mindlin 板结构的振动控制的研究.

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NONLINEAR DYNAMICS MODEL OF THE MICRO-MINDLIN PLATE BASED ON THE STRAIN GRADIENT THEORY *

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Abstract With the development of MEMS fabricating technology, MEMS structure now has a wide application in manufacturing industry. Meanwhile, because of no any characteristic length scale parameters in constitutive relationship, the classical theory of continuum mechanics is unlikely to predict the size effect under micro scale. This paper investigates the nonlinear problems of micro Mindlin plates based on the first-order shear deformation theory and von Karman non-linearity, and the strain gradient theory is also utilized to take size effect into consideration. Nonlinear dynamic model of the micro Mindlin plate along with its boundary conditions are obtained by taking strain energy, including strain energy of both macro and micro deformation, kinetic energy and external work into Hamilton's principle.

Key words nonlinearity, Mindlin plate, strain gradient theory, size effect, Hamilton principle

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