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一类具有记忆项的耦合非线性抽象方程组的整体解*

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摘要　运用 Galerkin 方法讨论了一类具有记忆项的耦合非线性抽象方程组的初值问题，根据方程组的特点，巧妙地对两个方程进行相加，并结合微积分的性质得到了所要的结果，然后研究收敛性，最后证明了方程组整体弱解的存在性.

关键词　记忆项，　耦合，　非线性，　抽象方程组，　整体解

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引言

2008年，Pedro Padro Durand Lazo[1]运用 Galerkin 方法证明了以下抽象方程

$$\ddot{u}+M(|A^{\frac{1}{2}}u|^2)Au+N(|A^{\alpha}u|)A^{\alpha}\dot{u}=f,$$

整体解的存在性，其中 $0<\alpha\leqslant 1$，$M(s)$，$N(s)\in C([0,+\infty);R)$.

2011年，张建文，丁霞霞[2]研究了如下抽象耦合非线性方程组

$$\begin{cases}\ddot{u}+M(|A^{\frac{1}{2}}u|^2+|A^{\frac{1}{2}}v|^2)Au+N(|A^{\alpha}u|^2)A^{\alpha}\dot{u}=f,\\ \ddot{v}+M(|A^{\frac{1}{2}}u|^2+|A^{\frac{1}{2}}v|^2)Av+N(|A^{\alpha}v|^2)A^{\alpha}\dot{v}=g.\end{cases}$$

在初始条件

$$\begin{cases}u(x,0)=u_0,\ v(x,0)=v_0,\ x\in\Omega,\\ \dot{u}(x,0)=u_1,\ \dot{v}(x,0)=v_1,\ x\in\Omega.\end{cases}$$

下的整体弱解的存在性，其中 $0<\alpha\leqslant 1$，$M(s)$，$N(s)\in C([0,+\infty);R)$.

2011年，Pedro Padro Durand Lazo[3]运用 Galerkin 方法证明了具有记忆项的非线性波动方程，特别地，当 $n=1$ 时，抽象方程

$$u''-M(|A^{\frac{1}{2}}u|^2)Au+\int_0^t h(t-\tau)Au(\tau)d\tau=0$$

的整体弱解的存在性.

2004年，陈树辉，黄建亮等[4]采用分离变量法分离时间变量和空间变量并利用 Galerkin 方法离散运动方程，再应用增量谐波平衡法进行非线性振动分析，建立横向运动微分方程

$$w_{,tt}+2vw_{,xt}+(v^2-1)w_{,xx}-\frac{3}{2}v_1^2w_{,x}^2w_{,xx}+v_f^2w_{,xxxx}=0$$

研究了运动辆横向非线性振动的内部共振.

在本文中，我们将在前人研究成果的基础上，讨论如下一类具有记忆项的耦合非线性抽象方程组

$$\begin{cases}\ddot{u}+M(|A^{\frac{1}{2}}u|^2+|A^{\frac{1}{2}}v|^2)Au+\int_0^t h(t-\tau)Au(\tau)d\tau=0,\\ \ddot{v}+M(|A^{\frac{1}{2}}u|^2+|A^{\frac{1}{2}}v|^2)Av+\int_0^t g(t-\tau)Av(\tau)d\tau=0.\end{cases}\tag{1}$$

在初始条件

$$\begin{cases}u(x,0)=u_0,\ v(x,0)=v_0,\ x\in\Omega,\\ \dot{u}(x,0)=u_1,\ \dot{v}(x,0)=v_1,\ x\in\Omega.\end{cases}\tag{2}$$

下整体弱解的存在性的问题，其中，算子 A 是定义在 Hilbert 空间 H 上的正定、自共轭算子，定义域 $D(A)$ 在 H 中稠密.

(A.1)　$M\in C^1([0,+\infty),R)$，$\forall\kappa,\beta,m_0>0$.

　(i)　$M\geqslant m_0>0$，

　(ii)　$\kappa M(\delta)\delta\leqslant\beta+\int_0^{\delta}M(s)ds$，$\forall\delta\geqslant 0$.

(A.2)　$h(t)=\exp(-\alpha t)t^{-v}$，$\kappa m_0\alpha>1$，$0<v<1$，

　$g(t)=\exp(-\gamma t)t^{-w}$，$\kappa m_0\gamma>1$，$0<w<1$.

$$\dot{h}(t)=-h(t)(\frac{\alpha t+v}{v})=-\exp(-\alpha t)(\frac{\alpha}{t^v}+\frac{v}{t^{1+v}}),$$

$$\dot{g}(t)=-g(t)(\frac{\gamma t+w}{w})=-\exp(-\gamma t)(\frac{\gamma}{t^w}+\frac{w}{t^{1+w}}).$$

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为研究问题方便，本文中我们记 $\Omega=(0,l)$，对 t 的一阶，二阶导数分别记为：$\dot u,\dot v,\dot g,\dot h,\ddot u,\ddot v,t\in[0,T](0<T<\infty)$，当我们重点考虑上述函数中的自变量 t 时，我们把函数记为 $u(x,t)=u(t),v(x,t)=v(t)$；我们记：$V=H_0^1(\Omega),H=L^2(\Omega),V'=H^{-1}(\Omega)$，$\nabla_{2s}=D(A^s)$，$\nabla_{2s}$，中的内积和范数定义如下 $\forall u,v\in\nabla_{2s},(u,v)_{2s}=(A^s u,A^s v)$，$\|u\|_{2s}^2=\{(u,u)_{2s}\}=|A^s u|^2$. $D(A^s)$ 是 Hilbert 空间，特别地，我们有：当 $s=0$ 时，记 $D(A^0)=H$；$s=\frac{1}{2}$，记 $D(A^{\frac{1}{2}})=V$，c,C 除特别声明外均表示不同的正常数.

1 主要定理

定理 1.1 设 A 是定义在 Hilbert 空间 H 上的正定、自共轭算子，(A.1)，(A.2)成立，若

$$\begin{cases}(u_0,u_1)\in V\times H,\\(v_0,v_1)\in V\times H.\end{cases}\tag{3}$$

则问题(1)~(2)存在一组解 $(u,v)=(u(x,t).v(x,t))$，对于 $\forall\phi\in V$，在 $D'(0,T)$ 中满足方程

$$\begin{cases}\frac{d}{dt}(\dot u,\phi)+M(|A^{\frac{1}{2}}u|^2+|A^{\frac{1}{2}}v|^2)(Au,\phi)+\\\quad(\int_0^t h(t-\tau)Au(\tau)d\tau,\phi)=0,\\\frac{d}{dt}(\dot v,\phi)+M(|A^{\frac{1}{2}}u|^2+|A^{\frac{1}{2}}v|^2)(Av,\phi)+\\\quad(\int_0^t g(t-\tau)Av(\tau)d\tau,\phi)=0.\end{cases}\tag{4}$$

及初始条件

$$\begin{aligned}&u(x,0)=u_0,\ v(x,0)=v_0,\\&\dot u(x,0)=u_1,\ \dot v(x,0)=v_1.\end{aligned}\tag{5}$$

其中

$$\begin{aligned}&u\in L^\infty(0,T;V)\cap H^{\frac{v}{2}}(0,T;V),\\&v\in L^\infty(0,T;V)\cap H^{\frac{w}{2}}(0,T;V),\\&\dot u\in L^\infty(0,T;H)\cap H^{1+\frac{v}{2}}(0,T;V'),\\&\dot v\in L^\infty(0,T;H)\cap H^{1+\frac{w}{2}}(0,T;V').\end{aligned}\tag{6}$$

2 近似解

选取 $\{\omega_k\}_{k=1}^\infty$ 为 V 的一组基，且为 H 的规范正交基，使得 $-A\omega_k=\lambda_k\omega_k(1\leqslant k<\infty)$，设 $V_m=\{\omega_1,\omega_2,\cdots,\Lambda\omega_m\}$ 为 V 的子空间，我们构造初值问题(1)~(2)的近似解序列 $\{u_m(x,t),v_m(x,t)\}$ 如下：

$$u_m(x,t)=\sum_{j=1}^m h_{jm}(t)\omega_j,v_m(x,t)=\sum_{j=1}^m g_{jm}(t)\omega_j,$$

其中 $h_{jm}(t)$ 和 $g_{jm}(t)$ 是待定函数，得到 Galerkin 方程组，即为关于 $h_{jm}(t)$ 和 $g_{jm}(t)$ 的常微分方程组的初值问题，其中 $(j=1,2,\cdots,\Lambda m)$，

$$\begin{cases}(\ddot u_m(t),\omega_j)+M(|A^{\frac{1}{2}}u_m(t)|^2+\\\quad|A^{\frac{1}{2}}v_m(t)|^2)(Au_m(t),\omega_j)+\\\quad(\int_0^t h(t-\tau)Au_m(\tau)d\tau,\omega_j)=0,\\(\ddot v_m(t),\omega_j)+M(|A^{\frac{1}{2}}u_m(t)|^2+\\\quad|A^{\frac{1}{2}}v_m(t)|^2)(Av_m(t),\omega_j)+\\\quad(\int_0^t g(t-\tau)Av_m(\tau)d\tau,\omega_j)=0.\end{cases}\tag{7}$$

及初始条件

$$\begin{cases}u_m(x,0)=u_{0m}=\sum_{j=1}^m h_{jm}(0)\omega_j=\\\quad\sum_{j=1}^m((u_0,\omega_j))\omega_j\to u_0,\\v_m(x,0)=v_{0m}=\sum_{j=1}^m g_{jm}(0)\omega_j=\\\quad\sum_{j=1}^m((v_0,\omega_j))\omega_j\to v_0.\end{cases}\quad 在\ V\ 中$$

$$\begin{cases}\dot u_m(x,0)=u_{1m}=\sum_{j=1}^m \dot h_{jm}(0)\omega_j=\\\quad\sum_{j=1}^m(u_1,\omega_j)\to u_1,\\\dot v_m(x,0)=v_{1m}=\sum_{j=1}^m \dot g_{jm}(0)\omega_j=\\\quad\sum_{j=1}^m(v_1,\omega_j)\to v_1.\end{cases}\quad 在\ H\ 中$$

由常微分方程理论知，存在 $t_j>0$，满足方程组(7)在相应的初始条件下在 $[0,T]$（取 $T=\min_{1\leqslant j\leqslant m}\{t_j\}$）上存在解 $\{h_{jm}(t),g_{jm}(t)\}$，从而得到相应的近似解 $\{u_m(x,t),v_m(x,t)\}$.

3 先验积分估计

估计一 式(7)第一个式子两端同乘 $\dot h_{jm}(t)$，(7)式第二个式子两端同乘 $\dot g_{jm}(t)$，并对 j 从 1 到 m 求和后，分别从 0 到 $t(t\in[0,T])$ 积分，所得两式相加，得

$$|\dot u_m(t)|^2+|\dot v_m(t)|^2+\int_0^{|A^{\frac{1}{2}}u_m(t)|^2+|A^{\frac{1}{2}}v_m(t)|^2}M(s)ds+$$

$$2\int_0^t\int_0^s h(s-\tau)(A^{\frac{1}{2}}u_m(\tau),A^{\frac{1}{2}}\dot u_m(s))d\tau ds+$$

$$2\int_0^t\int_0^s g(s-\tau)(A^{\frac{1}{2}}v_m(\tau),A^{\frac{1}{2}}\dot{v}_m(s)d\tau ds =$$

$$|u_{1m}|^2+|v_{1m}|^2+\int_0^{|A^{\frac{1}{2}}u_{0m}|^2+|A^{\frac{1}{2}}v_{0m}|^2}M(s)ds.$$

令

$$\Psi_m(t) = |\dot{u}_m(t)|^2+|\dot{v}_m(t)|^2+\int_0^{|A^{\frac{1}{2}}u_m(t)|^2+|A^{\frac{1}{2}}v_m(t)|^2}M(s)ds,$$

$$\Phi(t) = 2\int_0^t\int_0^s h(s-\tau)(A^{\frac{1}{2}}u_m(\tau),A^{\frac{1}{2}}\dot{u}_m(s))d\tau ds + 2\int_0^t\int_0^s g(s-\tau)(A^{\frac{1}{2}}v_m(\tau),A^{\frac{1}{2}}\dot{v}_m(s))d\tau ds,$$

$$\Psi_m(0) = |u_{1m}|^2+|v_{1m}|^2+\int_0^{|A^{\frac{1}{2}}u_{0m}|^2+|A^{\frac{1}{2}}v_{0m}|^2}M(s)ds,$$

我们有　$\Psi_m(t)+\Phi(t)=\Psi_m(0)\leqslant\Psi(0).$

为了方便地计算,记

$$\Theta_1(s,\tau)=h(s-\tau)(A^{\frac{1}{2}}u_m(\tau),A^{\frac{1}{2}}\dot{u}_m(s)),$$

$$\Theta_2(s,\tau)=g(s-\tau)(A^{\frac{1}{2}}v_m(\tau),A^{\frac{1}{2}}\dot{v}_m(s)),$$

$$\Theta(s,\tau)=\Theta_1(s,\tau)+\Theta_2(s,\tau).$$

则

$$\Phi(t)=2\int_0^t\int_0^s\Theta(s,\tau)d\tau ds=\Phi_1(t)+\Phi_2(t)+\Phi_3(t),$$

其中

$$\Phi_1(t) = 2\int_\delta^t\int_0^{s-\delta}\Theta(s,\tau)d\tau ds,$$

$$\Phi_2(t) = 2\int_\delta^t\int_{s-\delta}^s\Theta(s,\tau)d\tau ds,$$

$$\Phi_3(t) = 2\int_\delta^t\int_0^s\Theta(s,\tau)d\tau ds.$$

$$\begin{aligned}\Phi_1(t) ={}& 2\int_0^{t-\delta}(h(t-\tau)A^{\frac{1}{2}}u_m(t),A^{\frac{1}{2}}u_m(\tau))d\tau - \\ & 2\int_0^{t-\delta}(h(\delta)A^{\frac{1}{2}}u_m(\tau+\delta),A^{\frac{1}{2}}u_m(\tau))d\tau + \\ & 2\alpha\int_0^{t-\delta}(\int_{\tau+\delta}^t h(s-\tau)A^{\frac{1}{2}}u_m(s),A^{\frac{1}{2}}u_m(\tau))d\tau + \\ & 2v\int_0^{t-\delta}(\int_{\tau+\delta}^t\frac{h(s-\tau)}{s-\tau}A^{\frac{1}{2}}u_m(s)ds,A^{\frac{1}{2}}u_m(\tau))d\tau + \\ & 2\int_0^{t-\delta}(g(t-\tau)A^{\frac{1}{2}}v_m(\tau),A^{\frac{1}{2}}v_m(\tau))d\tau - \\ & 2\int_0^{t-\delta}(g(\delta)A^{\frac{1}{2}}v_m(\tau+\delta),A^{\frac{1}{2}}v_m(\tau))d\tau + \\ & 2\gamma\int_0^{t-\delta}(\int_{\tau+\delta}^t g(s-\tau)A^{\frac{1}{2}}v_m(s)ds,A^{\frac{1}{2}}v_m(\tau))d\tau + \\ & 2w\int_0^{t-\delta}(\int_{\tau+\delta}^t\frac{g(s-\tau)}{s-\tau}A^{\frac{1}{2}}v_m(s)ds,A^{\frac{1}{2}}v_m(\tau))d\tau = \\ & \phi_1(t)+J_1(t)+\phi_2(t)+J_2(t),\end{aligned}$$

其中 $J_1(t) = \frac{v}{2}\int_\delta^t\int_0^{s-\delta}\exp(-\alpha(s-\tau))$

$$\left[\frac{|A^{\frac{1}{2}}u_m(s)+A^{\frac{1}{2}}u_m(\tau)|^2}{(s-\tau)^{1+v}}-\frac{|A^{\frac{1}{2}}u_m(s)-A^{\frac{1}{2}}u_m(\tau)|^2}{(s-\tau)^{1+v}}\right]d\tau ds,$$

同理可得

$$J_2(t) = \frac{w}{2}\int_\delta^t\int_0^{s-\delta}\exp(-\gamma(s-\tau))\left[\frac{|A^{\frac{1}{2}}v_m(s)+A^{\frac{1}{2}}v_m(\tau)|^2}{(s-\tau)^{1+w}}-\frac{|A^{\frac{1}{2}}v_m(s)-A^{\frac{1}{2}}v_m(\tau)|^2}{(s-\tau)^{1+w}}\right]d\tau ds.$$

因为

$$0<\exp(-\alpha(s-\tau))\leqslant 1,\ 0<\exp(-\gamma(s-\tau))\leqslant 1,$$

所以

$$\exp(-\alpha(s-\tau))\frac{|A^{\frac{1}{2}}u_m(s)-A^{\frac{1}{2}}u_m(\tau)|^2}{(s-\tau)^{1+v}}\leqslant\frac{|A^{\frac{1}{2}}u_m(s)-A^{\frac{1}{2}}u_m(\tau)|^2}{(s-\tau)^{1+v}},$$

$$\exp(-\gamma(s-\tau))\frac{|A^{\frac{1}{2}}v_m(s)-A^{\frac{1}{2}}v_m(\tau)|^2}{(s-\tau)^{1+w}}\leqslant\frac{|A^{\frac{1}{2}}v_m(s)-A^{\frac{1}{2}}v_m(\tau)|^2}{(s-\tau)^{1+w}}.$$

因此,有 $\forall\rho>0$, $\exists M(s,\delta)=\{(\tau,s):0<\tau\leqslant s-\delta,\ \delta\leqslant s\leqslant t,\delta>0\}$,使得

$$\int_{M(s,\delta)}\exp(-\alpha(s-\tau))\frac{|A^{\frac{1}{2}}u_m(s)-A^{\frac{1}{2}}u_m(\tau)|^2}{(s-\tau)^{1+v}}d\tau ds = \int_{M(s,\delta)}\frac{|A^{\frac{1}{2}}u_m(s)-A^{\frac{1}{2}}u_m(\tau)|^2}{(s-\tau)^{1+v}}d\tau ds-\rho,$$

$$\int_{M(s,\delta)}\exp(-\gamma(s-\tau))\frac{|A^{\frac{1}{2}}v_m(s)-A^{\frac{1}{2}}v_m(\tau)|^2}{(s-\tau)^{1+w}}d\tau ds = \int_{M(s,\delta)}\frac{|A^{\frac{1}{2}}v_m(s)-A^{\frac{1}{2}}v_m(\tau)|^2}{(s-\tau)^{1+w}}d\tau ds-\rho.$$

所以

$$c\left(\int_{M(s,\delta)}\frac{|A^{\frac{1}{2}}u_m(s)-A^{\frac{1}{2}}u_m(\tau)|^2}{(s-\tau)^{1+v}}d\tau ds+\int_{M(s,\delta)}\frac{|A^{\frac{1}{2}}v_m(s)-A^{\frac{1}{2}}v_m(\tau)|^2}{(s-\tau)^{1+w}}\right)\leqslant\Psi(0).$$

$$\Phi(t) = \Psi_1(t)+\Psi_2(t)+\Psi_3(t)+\Psi_4(t),$$

其中

$$\Psi_1(t)=2\int_0^t\int_0^s h(s-\tau)(s-\tau)(\frac{A^{\frac{1}{2}}u_m(\tau)-A^{\frac{1}{2}}u_m(s)}{s-\tau},A^{\frac{1}{2}}\dot{u}_m(s))d\tau ds,$$

$$\Psi_2(t)=2\int_0^t\int_0^s h(s-\tau)(A^{\frac{1}{2}}u_m(s),A^{\frac{1}{2}}\dot{u}_m(s))d\tau ds,$$

$$\Psi_3(t)=2\int_0^t\int_0^s g(s-\tau)(s-\tau)(\frac{A^{\frac{1}{2}}v_m(\tau)-A^{\frac{1}{2}}v_m(s)}{s-\tau},A^{\frac{1}{2}}\dot{v}_m(s))d\tau ds,$$

$$\Psi_4(t)=2\int_0^t\int_0^s g(s-\tau)(A^{\frac{1}{2}}v_m(s),A^{\frac{1}{2}}\dot{v}_m(s))d\tau ds,$$

$$\Psi_1(t)\leqslant\frac{1}{2}\int_0^t\int_0^s h(s-\tau)(s-\tau)\left|\frac{A^{\frac{1}{2}}u_m(s)-A^{\frac{1}{2}}u_m(\tau)}{s-\tau}-A^{\frac{1}{2}}\dot{u}_m(s)\right|^2 d\tau ds,$$

根据导数的连续性知：对于

$$\forall\varepsilon=\sqrt{\frac{h(\tau)h(s)}{(s-\tau)h(s-\tau)}}>0,\ \exists\delta=\delta(\varepsilon),$$

当$|s-\tau|<\delta$时，有

$$\Psi_1(t)\leqslant\frac{1}{2}\int_0^t(\int_0^s h(s-\tau)(s-\tau)\frac{h(\tau)h(s)}{h(s-\tau)(s-\tau)}d\tau)ds=\frac{1}{2}\int_0^t(\int_0^s h(\tau)h(s)d\tau)ds,$$

同理有

$$\Psi_3(t)\leqslant\frac{1}{2}\int_0^t(\int_0^s g(s-\tau)(s-\tau)\frac{g(\tau)g(s)}{g(s-\tau)(s-\tau)}d\tau)ds=\frac{1}{2}\int_0^t(\int_0^s g(\tau)g(s)d\tau)ds.$$

因为

$$\int_0^\infty h(\delta)d\delta\leqslant\frac{1}{\alpha},\quad\int_0^\infty g(\delta)d\delta\leqslant\frac{1}{\gamma}.$$

所以

$$\Psi_2(t)\leqslant\frac{1}{\alpha}|A^{\frac{1}{2}}u_m(t)|^2+\frac{1}{\alpha}|A^{\frac{1}{2}}u_0|^2,$$

同理有

$$\Psi_4(t)\leqslant\frac{1}{\gamma}|A^{\frac{1}{2}}v_m(t)|^2+\frac{1}{\gamma}|A^{\frac{1}{2}}v_0|^2.$$

所以

$$|\dot{u}_m(t)|^2+|\dot{v}_m(t)|^2+(\kappa m_0-\frac{1}{\alpha})|A^{\frac{1}{2}}u_m(t)|^2+(\kappa m_0-\frac{1}{\gamma})|A^{\frac{1}{2}}v_m(t)|^2+c(\int_{M(s,\delta)}\frac{|A^{\frac{1}{2}}u_m(s)-A^{\frac{1}{2}}u_m(\tau)|^2}{(s-\tau)^{1+v}}d\tau ds)+(\int_{M(s,\delta)}\frac{|A^{\frac{1}{2}}v_m(s)-A^{\frac{1}{2}}v_m(\tau)|^2}{(s-\tau)^{1+w}}d\tau ds)\leqslant 2\beta+2\Psi(0)+\frac{1}{\alpha}|A^{\frac{1}{2}}u_0|^2+\frac{1}{\gamma}|A^{\frac{1}{2}}v_0|^2+\frac{1}{2\alpha^2}+\frac{1}{2\gamma^2}=C.$$

所以

$$|\dot{u}_m(t)|<C,\ |\dot{v}_m(t)|<C.$$

$$|A^{\frac{1}{2}}u_m(t)|<C,\ |A^{\frac{1}{2}}v_m(t)|<C,$$

$$\int_{M(s,\delta)}\frac{|A^{\frac{1}{2}}v_m(s)-A^{\frac{1}{2}}v_m(\tau)|^2}{|s-\tau|^{1+w}}d\tau ds<C,$$

$$\int_{M(s,\delta)}\frac{|A^{\frac{1}{2}}u_m(s)-A^{\frac{1}{2}}u_m(\tau)|^2}{|s-\tau|^{1+v}}d\tau ds<C.$$

故

$$\dot{u}_m(t)\in L^\infty(0,T;H),\ \dot{v}_m(t)\in L^\infty(0,T;H),$$

$$u_m(t)\in L^\infty(0,T;V),\ v_m(t)\in L^\infty(0,T;V).$$

因为

$$\|u_m(t)\|_{H^{\frac{w}{2}}(0,T;V)}=\int_0^T|u_m(t)|^2dt+\int_{T_0}^{T_0}\frac{|u_m(t)-u_m(\tau)|^2}{|t-\tau|^{1+v}}d\tau dt<C,$$

$$\|v_m(t)\|_{H^{\frac{w}{2}}(0,T;V)}=\int_0^T|v_m(t)|^2dt+\int_{T_0}^{T_0}\frac{|v_m(t)-v_m(\tau)|^2}{|t-\tau|^{1+w}}d\tau dt<C.$$

所以

$$u_m(t)\in H^{\frac{w}{2}}(0,T;V),\ v_m(t)\in H^{\frac{w}{2}}(0,T;V).$$

估计二 若$\phi\in V$，且有$V\to V_m:\phi\to\phi_m$，$V_m\subset V$，则(7)变为

$$\begin{cases}(\ddot{u}_m(t),\phi_m)=-M(|A^{\frac{1}{2}}u_m(t)|^2+|A^{\frac{1}{2}}v_m(t)|^2)(Au_m(t),\phi_m)-(\int_0^t h(t-\tau)Au_m(\tau)d\tau,\phi_m),\\(\ddot{v}_m(t),\phi_m)=-M(|A^{\frac{1}{2}}u_m(t)|^2+|A^{\frac{1}{2}}v_m(t)|^2)(Av_m(t),\phi_m)-(\int_0^t g(t-\tau)Av_m(\tau)d\tau,\phi_m).\end{cases}$$

因为

$$(\ddot{u}_m(t),\phi_m)=-M(|A^{\frac{1}{2}}u_m(t)|^2+|A^{\frac{1}{2}}v_m(t)|^2)(Au_m(t),\phi_m)-\int_0^t h(t-\tau)(Au_m(\tau),\phi_m)d\tau,$$

所以

$$|\ddot{u}_m(t)|_{V'}\leqslant M(|A^{\frac{1}{2}}u_m|^2+|A^{\frac{1}{2}}v_m|^2)|Au_m|_{V'}+\int_0^t h(t-\tau)|Au_m(\tau)|_{V'}d\tau\leqslant M(|A^{\frac{1}{2}}u_m|^2+$$

$$|A^{\frac{1}{2}}v_m|^2)\|u_m(t)\|+\int_0^t h(t-\tau)\|u_m(\tau)\|d\tau.$$

因为 V 中的范数与 H 中的范数等价,所以

$$|\ddot{u}_m(t)|_{V'}\leqslant CM(|A^{\frac{1}{2}}u_m(t)|^2+|A^{\frac{1}{2}}v_m(t)|^2)\times |A^{\frac{1}{2}}u_m(t)|+C\int_0^t h(t-\tau)|A^{\frac{1}{2}}u_m(\tau)|d\tau,\quad(8)$$

同理有

$$|\ddot{v}_m(t)|_{V'}\leqslant CM(|A^{\frac{1}{2}}u_m(t)|^2+|A^{\frac{1}{2}}v_m(t)|^2)\times |A^{\frac{1}{2}}v_m(t)|+C\int_0^t g(t-\tau)|A^{\frac{1}{2}}v_m(\tau)|d\tau.\quad(9)$$

(8)和(9)相加得

$$|\ddot{u}_m(t)|+|\ddot{v}_m(t)|\leqslant CM(|A^{\frac{1}{2}}u_m(t)|^2+|A^{\frac{1}{2}}v_m(t)|^2)(|A^{\frac{1}{2}}u_m(t)|+|A^{\frac{1}{2}}v_m(t)|)+C\int_0^t h(t-\tau)|A^{\frac{1}{2}}u_m(\tau)|d\tau+C\int_0^t g(t-\tau)|A^{\frac{1}{2}}v_m(\tau)|d\tau\leqslant C+C\frac{1}{\alpha}+C\frac{1}{\gamma},$$

所以

$$|\ddot{u}_m(t)|_{V'}<C,\ |\ddot{v}_m(t)|_{V'}<C,$$

$$\ddot{u}_m(t)\in L^\infty(0,T;V'),\ \ddot{v}_m(t)\in L^\infty(0,T;V').$$

接下来的估计 $|\ddot{u}_m(t_1)-\ddot{u}_m(t_2)|_{V'}$,便可以估计到记忆项,对于 $0<t_2<t_1<T$

$$\left|\int_0^{t_1}h(t_1-\tau_1)A^{\frac{1}{2}}u_m(\tau_1)d\tau-\int_0^{t_2}h(t_2-\tau_2)A^{\frac{1}{2}}u_m(\tau_2)d\tau\right|_{V'}=\left|\int_0^{t_1}2h(t_1-\tau_1)A^{\frac{1}{2}}u_m(\tau_1)d\tau\right|\leqslant C(\int_2^{t_1}h(t_1-\tau_1)^2d\tau)^{1/2}(\int_2^{t_1}|A^{\frac{1}{2}}u_m(\tau_1)|^2d\tau)^{1/2}\leqslant C(t_1-t_2)^{1/2}(\sup_{0\leqslant t\leqslant T}|A^{\frac{1}{2}}u_m(\tau)|^2)^{1/2},$$

所以

$$\frac{|\ddot{u}_m(t_1)-\ddot{u}_m(t_2)|_{V'}^2}{|t_1-t_2|}\leqslant\sup_{0\leqslant t\leqslant T}|A^{\frac{1}{2}}u_m(\tau)|^2,$$

同理

$$\frac{|\ddot{v}_m(t_1)-\ddot{v}_m(t_2)|_{V'}^2}{|t_1-t_2|}\leqslant\sup_{0\leqslant t\leqslant T}|A^{\frac{1}{2}}v_m(\tau)|^2.$$

因为

$$|\ddot{u}_m(t)|_{H^{\frac{w}{2}}(0,T;V')}=\int_0^T|u_m(t)|_{V'}^2dt+\int_0^T\int_0^T\frac{|\ddot{u}_m(t_1)-\ddot{u}_m(t_2)|_{V'}^2}{|t_1-t_2|^{1+2\times\frac{v}{2}}}dt_1dt_2\leqslant C(T),$$

$$|\ddot{v}_m(t)|_{H^{\frac{w}{2}}(0,T;V')}=\int_0^T|v_m(t)|_{V'}^2dt+\int_0^T\int_0^T\frac{|\ddot{v}_m(t_1)-\ddot{v}_m(t_2)|_{V'}^2}{|t_1-t_2|^{1+2\times\frac{w}{2}}}dt_1dt_2\leqslant C(T).$$

所以

$$\ddot{u}_m(t)\in H^{\frac{v}{2}}(0,T;V'),\ \ddot{v}_m(t)\in H^{\frac{w}{2}}(0,T;V').$$

因为

$$|\dot{u}_m(t)|_{H^{1+\frac{v}{2}}(0,T;V')}=\int_0^T|\dot{u}_m(t)|_{V'}^2dt+\int_0^T\int_0^T\frac{|\dot{u}_m(t_1)-\dot{u}_m(t_2)|_{V'}^2}{|t-\tau|^{1+2\times(1+\frac{v}{v})}}dtd\tau\leqslant C,$$

$$|\dot{v}_m(t)|_{H^{1+\frac{w}{2}}(0,T;V')}=\int_0^T|\dot{v}_m(t)|_{V'}^2dt+\int_0^T\int_0^T\frac{|\dot{v}_m(t_1)-\dot{v}_m(t_2)|_{V'}^2}{|t-\tau|^{1+2\times(1+\frac{w}{v})}}dtd\tau\leqslant C,$$

所以

$$\dot{u}_m(t)\in H^{1+\frac{v}{2}}(0,T;V'),\ \dot{v}_m(t)\in H^{1+\frac{w}{2}}(0,T;V').$$

对于 $\frac{1}{1+\frac{v}{2}}<\gamma<1$, $\frac{1}{1+\frac{w}{2}}<\gamma<1$,根据引理知 $(\dot{u}_m)_{m\in N}$ 在 $H^{(1+\frac{v}{v})\gamma}(0,T;H^{-\gamma})$ 和 $\dot{v}_m)_{m\in N}$ 在 $H^{(1+\frac{w}{v})\gamma}(0,T;H^{-\gamma})$ 中收敛,所以,

$$H^{(1+\frac{v}{2})\gamma}(0,T;V')\subset H^1(0,T;V'),$$

$$H^{(1+\frac{w}{2})\gamma}(0,T;V')\subset H^1(0,T;V').$$

4　收敛性

由 Aubin-Lions 紧性理论知,分别存在 $\{u_m\}$ 的子序列 $\{u_\mu\}$,$\{v_m\}$ 的子序列 $\{v_\mu\}$($\{u_m\}$,$\{v_m\}$ 在不同空间的子序列都统一记为 $\{u_\mu\}$,$\{v_\mu\}$),因为可分的赋范空间的一致有界的线性泛函序列中必可取出一个弱 * 收敛的子序列,所以,$u_\mu\to u$,$v_\mu\to v$ 在 $L^2(0,T;V)$ 中弱 * 收敛;$\dot{u}_\mu\to\dot{u}$,$\dot{v}_\mu\to\dot{v}$ 在 $L^2(0,T;H)$ 中弱 * 收敛;$\ddot{u}_\mu\to\ddot{u}$,$\ddot{v}_\mu\to\ddot{v}$ 在 $L^2(0,T;V')$ 中弱 * 收敛;$u_\mu\to u$ 在 $H^{\frac{v}{2}}(0,T;V)$ 中弱 * 收敛;$v_\mu\to v$ 在 $H^{\frac{w}{2}}(0,T;V)$ 中弱 * 收敛;$\ddot{u}_\mu\to\ddot{u}$ 在 $H^{\frac{v}{2}}(0,T;V')$ 中弱 * 收敛;$\ddot{v}_\mu\to\ddot{v}$ 在 $H^{\frac{w}{2}}(0,T;V')$ 中弱 * 收敛.

5　(u,v) 满足方程

下证 (u,v) 满足方程(4). $\{\omega_j(x)\}$ 为 V 和 H 的规范正交基,现在固定 j,取 $\mu>j$

$$\frac{d}{dt}(\dot{u}_\mu,\omega_j)\to\frac{d}{dt}(\dot{u},\omega_j)\text{在 }D'(0,T)\text{中收敛},$$

$\frac{d}{dt}(\dot{v}_\mu,\omega_j)\to\frac{d}{dt}(\dot{v},\omega_j)$ 在 $D'(0,T)$ 中收敛,

$M(|A^{\frac{1}{2}}u_\mu|^2+|A^{\frac{1}{2}}v_\mu|^2)(Au_\mu,\omega_j)\to M(|A^{\frac{1}{2}}u|^2+|A^{\frac{1}{2}}v|^2)(Au,\omega_j)$ 在 $L^2(0,T)$ 中弱 * 收敛,

$M(|A^{\frac{1}{2}}u_\mu|^2+|A^{\frac{1}{2}}v_\mu|^2)(Au_\mu,\omega_j)\to M(|A^{\frac{1}{2}}u|^2+|A^{\frac{1}{2}}v|^2)(Av,\omega_j)$ 在 $L^2(0,T)$ 中弱 * 收敛,

$(\int_0^t h(t-\tau)Au_\mu(\tau)d\tau,\omega_j)\to(\int_0^t h(t-\tau)Au(\tau)d\tau,\omega_j)$ 在 $L^2(0,T)$ 中弱 * 收敛,

$(\int_0^t g(t-\tau)Av_\mu(\tau)d\tau,\omega_j)\to(\int_0^t g(t-\tau)Av(\tau)d\tau,\omega_j)$ 在 $L^2(0,T)$ 中弱 * 收敛.

令 $\mu\to\infty$,则 $m\to\infty$,再由基 $\{\omega_j(x)\}$ 在 V 中稠密,当 $\sum_{j=1}^{m}c_j\omega_j\to\phi$ 在 V 中时,有

$$\begin{cases}\frac{d}{dt}(\dot{u},\phi)+M(|A^{\frac{1}{2}}u|^2+|A^{\frac{1}{2}}v|^2)(Au,\phi)+\\ \quad(\int_0^t h(t-\tau)Au(\tau)d\tau,\phi)=0,\\ \frac{d}{dt}(\dot{v},\phi)+M(|A^{\frac{1}{2}}u|^2+|A^{\frac{1}{2}}v|^2)(Av,\phi)+\\ \quad(\int_0^t g(t-\tau)Av(\tau)d\tau,\phi)=0.\end{cases}$$

6 (u,v) 满足初始条件

引入连续函数空间 $C_T^1[0,T]=\{\phi\in C^1[0,T];\ \phi(T)=\phi(0)=0\}$,作 $\varphi=\sum_{j=1}^{k}\phi_j\omega_j$,$\forall\phi_j\in C_T^1[0,T]$,有 $\varphi(T)=0,\varphi'(T)=0$,对 $\int_0^T(\ddot{u}_m,\varphi)dt$ 和 $\int_0^T(\ddot{u},\varphi)dt$ 利用分部积分,得 $\dot{u}(0)=u_1,\dot{v}(0)=v_1$,再对 $\int_0^T(\dot{u}_m,\dot{\varphi})dt$ 和 $\int_0^T(\dot{u},\dot{\varphi})dt$ 利用分部积分,得 $u(0)=u_0,v(0)=v_0$,所以 (u,v) 满足初始条件(5).

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THE GLOBAL SOLUTION FOR A CLASS COUPLED OF NONLINEAR ABSTRACT EQUATIONS WITH MEMORY TERM *

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Abstract This paper studied an initial boundary value problem for a coupled system of nonlinear abstract equations with memory term in Galerkin method. Two equations were added in view of the character of the system and the results were obtained by combing with the nature of the calculus. The convergence was studied, and the existence of a global weak solution for the abstract equations was proven.

Key words memory term, coupling, nonlinear, abstract equations, global solution

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