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膜受迫振动方程的多辛格式及其守恒律*

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摘要　基于 Hamilton 空间体系的多辛理论研究了膜强迫振动问题. 利用 Runge - Kutta 多辛格式构造了一种 9×3 点半隐式的多辛离散格式,该格式满足多辛守恒律. 数值算例结果表明该多辛离散格式不仅能够有效提高数值计算精度,而且能够保持膜振动系统的局部性质. 同时利用多辛格式模拟得到的波形图表明多辛方法具有较好的长时间数值稳定性.

关键词　多辛,　龙格 - 库塔方法,　非守恒型方程

引 言

1984 年冯康先生提出计算 Hamilton 系统的辛算法[1-3], 国内外学者把此方法推广到了无穷维的正则 Hamilton 方程组[4-5],但是这一理论在偏微分方程辛离散方面存在局限性. 为此, Bridges 和 Reich 等提出基于守恒型偏微分方程多辛结构的多辛积分概念,并针对 Zakharov - Kuznetsov 方程、shallow - water 方程给出了相应的多辛格式及其守恒律[6-9],但是在这些实例中,很少涉及到二维偏微分方程,因为二维偏微分方程无论是多辛格式的构造、多辛守恒律的证明还是算法的实现都存在很大困难,而对于非守恒型偏微分方程,其困难更大.

膜受迫振动方程正是一个非守恒型二维的偏微方程. 针对这一方程,本文导出了其多辛形式及多辛守恒律,并给出了多辛守恒律的数学证明;然后利用 Runge - Kutta 离散方法构造了其离散格式;最后以一个典型的数值算例验证了多辛算法的有效性.

1　膜受迫振动方程的多辛形式

考虑膜受迫振动问题:

$$\partial_{tt}u - c^2(\partial_{xx}u + \partial_{yy}u) = F'(u) \quad (x,y,t) \in u \subset R^3 \tag{1}$$

它具有典型的哈密尔顿体系下的多辛偏微分方程形式[6-7]:

$$M\partial_t z + K_1\partial_x z + K_2\partial_y z = \nabla_z S(z), z \in R^d \tag{2}$$

式中: $M, K_1, K_2 \in R^{d\times d}$ 为反对称矩阵(可逆); $S: R^d \to R$ 是光滑函数,为哈密尔顿函数,膜受迫振动方程(1)的多辛形式(2)推导过程如下:

引入正交动量:

$v = \partial_t u \quad w = \partial_x u \quad p = \partial_y u$

振动方程(1)就可以写成如下的偏微分方程形式:

$$\begin{cases} -\partial_t v + c^2\partial_x w + c^2\partial_y p = -F'(u) \\ \partial_t u = v \\ -c^2\partial_x u = -c^2 w \\ -c^2\partial_y u = -c^2 p \end{cases} \tag{3}$$

定义状态变量:

$z = [u, v, w, p]^T \in R^4$

就可以得到反对称矩阵:

$$M = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, K_1 = \begin{bmatrix} 0 & 0 & c^2 & 0 \\ 0 & 0 & 0 & 0 \\ -c^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 0 & 0 & 0 & c^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -c^2 & 0 & 0 & 0 \end{bmatrix}$$

以及哈密尔顿函数:

$$S(z) = \frac{1}{2}v^2 - \frac{1}{2}c^2(w^2 + p^2) - F(u)$$

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2 多辛守恒律

多辛形式(2)备受学术界关注的原因之一是其存在多辛守恒律. 将多辛形式(2)中的某些基态$z(x,y,z)$线性化后可得到其对应的变分方程:

$$M\partial_t Z + K_1\partial_x Z + K_2\partial_y Z = S_{zz}(z)Z \quad (4)$$

假定U,V是变分方程(4)的两个解,那么U,V必然满足:

$$\partial_t(U^T MV) + \partial_x(U^T K_1 V) + \partial_y(U^T K_2 V) = 0 \quad (5)$$

对于方程(5),引入预辛形式:

$$\omega(U,V) = U^T MV, k_1(U,V) = U^T K_1 V, k_2(U,V) = U^T K_2 V$$

那么方程(5)就等价于多辛守恒律(CLS):

$$\partial_t\omega + \partial_x k_1 + \partial_y k_2 = 0 \quad (6)$$

将ω、k_1、k_2采用更为抽象的定义方式,即采用外积定义方式[10]:

$$\omega = \frac{1}{2}\mathrm{d}z\Lambda M\mathrm{d}z, k_1 = \frac{1}{2}\mathrm{d}z\Lambda K_1\mathrm{d}z, k_2 = \frac{1}{2}\mathrm{d}z\Lambda K_2\mathrm{d}z$$

则对于膜受迫振动方程(1),多辛守恒律的具体形式为:

$$\partial_t(\mathrm{d}v\Lambda\mathrm{d}u) + c^2[\partial_x(\mathrm{d}v\Lambda\mathrm{d}w) + \partial_y(\mathrm{d}v\Lambda\mathrm{d}p)] = 0 \quad (7)$$

3 利用 Runge – Kutta 多辛格式离散多辛偏微分方程组

考虑二维相空间上标准的哈密尔顿系统:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = J\nabla_z H(z) \quad (8)$$

其中z为状态变量$z=(p,q)^T$,J为 Jacobian 矩阵$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,$H(z)\in C^\infty(R^2)$为哈密尔顿函数.

假定哈密尔顿系统是可分的,所谓可分的哈密尔顿系统,是指该系统的哈密尔顿函数可以分成

$$H(z) = H(p,q) = U(p) + V(q)$$

这样,可分的哈密尔顿系统的运动方程为:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} p \\ q \end{bmatrix} = J\begin{bmatrix} U'_p \\ V'_q \end{bmatrix} = \begin{bmatrix} f(q) \\ g(p) \end{bmatrix} \quad (9)$$

系统(9)一个相容的m级 Runge – Kutta 格式为:

$$\begin{cases} z_{k+1} = z_k + h\sum_{i=1}^{m} b_i J\nabla H(k_i) \\ k_i = z_k + h\sum_{j=1}^{m} a_{i,j} J\nabla H(k_i) \end{cases} \quad for\ i = 1,2,\cdots,m \quad (10)$$

其中h是时间步长,系数b_i,$a_{i,j}$满足阶条件[11-12].

格式(10)成为辛格式当且仅当:

$$b_i a_{i,j} + b_j a_{j,i} - b_i b_j = 0,\ \forall i,j$$

为了构造格式的形式简单,再次引入3个中间变量:$T=\partial_t v$,$Q=\partial_x w$和$R=\partial_y p$. 利用这些中间变量,则多辛偏微分方程组(3)可以写成如下的3个不同方向上的微分方程组的联合形式:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}v = T \\ \frac{\mathrm{d}}{\mathrm{d}t}u = v \end{cases} (a) \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}x}w = Q \\ \frac{\mathrm{d}}{\mathrm{d}x}u = w \end{cases} (b) \quad \begin{cases} \frac{\mathrm{d}}{\mathrm{d}y}p = R \\ \frac{\mathrm{d}}{\mathrm{d}y}u = p \end{cases} (c) \quad (11)$$

其中式(11 – a)对应时间t方向,式(11 – b)对应空间x方向,式(11 – c)对应空间y方向,同时中间变量T、Q、R满足:

$$T - c^2 Q - c^2 R = F'(u) \quad (12)$$

下面我们就利用格式(10)离散联合的微分方程组(11),假定三个方向的离散都是从第n层到第$n+1$层,Δt为时间步长,Δx为空间x方向的步长,Δy为空间y方向的步长. 先采用r级 Runge – Kutta 辛格式离散时间t方向的方程(11 – a),分别采用s级和l级 Runge – Kutta 辛格式离散x方向和y方向的方程.

采用r级 Runge – Kutta 辛格式离散时间t方向的方程得到t方向r级 Runge – Kutta 格式,该格式满足守恒律:

$$\frac{\mathrm{d}u_{i,j}^{n+1}\Lambda\mathrm{d}v_{i,j}^{n+1} - \mathrm{d}u_{i,j}^{n}\Lambda\mathrm{d}v_{i,j}^{n}}{\Delta t} = \sum_{m=1}^{r} b_m(\mathrm{d}U_{i,j}^{m}\Lambda\mathrm{d}T_{i,j}^{m}) \quad (13)$$

采用s级 Runge – Kutta 辛格式离散时间x方向的方程得到x方向s级 Runge – Kutta 格式,该格式满足守恒律:

$$\frac{\mathrm{d}u_{n+1,j}^{k}\Lambda\mathrm{d}w_{n+1}^{k} - \mathrm{d}u_{n,j}^{k}\Lambda\mathrm{d}w_{n,j}^{k}}{\Delta x} = \sum_{m=1}^{s} d_m(\mathrm{d}U_{m,j}^{k}\Lambda\mathrm{d}Q_{m,j}^{k}) \quad (14)$$

采用l级 Runge – Kutta 辛格式离散时间y方向的方程得到y方向l级 Runge – Kutta 格式,该格式满足守恒律:

$$\frac{\mathrm{d}u_{i,n+1}^{k}\Lambda\mathrm{d}p_{i,n+1}^{k} - \mathrm{d}u_{i,n}^{k}\Lambda\mathrm{d}p_{i,n}^{k}}{\Delta y} = \sum_{m=1}^{l} \hat{d}_m(\mathrm{d}U_{i,m}^{k}\Lambda\mathrm{d}R_{i,m}^{k}) \quad (15)$$

在任意网格点上,中间变量满足:

$$T_{i,j}^{k} - c^2 Q_{i,j}^{k} - c^2 R_{i,j}^{k} = F'(u_{i,j}^{k}) \quad (16)$$

联立 t 方向的 r 级 Runge – Kutta 格式、x 方向的 s 级 Runge – Kutta 格式、y 方向的 l 级 Runge – Kutta 格式和式(16)构成多辛偏微分方程组(3)的一个差分格式,其截断误差为 $O(\Delta t^{2r})+O(\Delta x^{2s})+O(\Delta y^{2l})$. 下面首先说明该格式是多辛的.

方程(12)对应的变分方程的离散形式是:

$$dT_{i,j}^{k}-c^{2}dQ_{i,j}^{k}-c^{2}dR_{i,j}^{k}=F(u_{i,j}^{k})du_{i,j}^{k} \tag{17}$$

联立(13),(14),(15)和(17)式得到离散的多辛守恒律:

$$\sum_{i=0}^{s-1}\sum_{j=0}^{l-1}e_{i+1}\hat{e}_{j+1}\frac{du_{i,j}^{n+1}\wedge dv_{i,j}^{n+1}-du_{i,j}^{n}\wedge dv_{i,j}^{n}}{\Delta t}=c^{2}\sum_{k=1}^{r}\sum_{j=0}^{l-1}b_{k}\hat{e}_{j}\frac{du_{n+1,j}^{k}\wedge dw_{n+1,j}^{k}-du_{n,j}^{k}\wedge dw_{n,j}^{k}}{\Delta x}+c^{2}\sum_{i=0}^{s-1}\sum_{k=1}^{r}e_{i}b_{k}\frac{du_{i,n+1}^{k}\wedge dp_{i,n+1}^{k}-du_{i,n}^{k}\wedge dp_{i,n}^{k}}{\Delta y} \tag{18}$$

上式中,由 $\sum_{i=0}^{s-1}e_{i+1}$ 和 $\sum_{j=0}^{l-1}\hat{e}_{j+1}=1$ 可以直接推得 $\sum_{i=0}^{s-1}\sum_{j=0}^{l-1}e_{i+1}\hat{e}_{j+1}=1$,同理可得,$\sum_{k=1}^{r}\sum_{j=0}^{l-1}b_{k}\hat{e}_{j+1}=1$ 和 $\sum_{i=0}^{s-1}\sum_{k=1}^{r}e_{i+1}b_{k}=1$,因此可以肯定(18)式至少是(7)式的一阶离散逼近. 因此,由 t 方向的 r 级 Runge – Kutta 格式、x 方向的 s 级 Runge – Kutta 格式、y 方向的 l 级 Runge – Kutta 格式和式(16)构成的差分格式是多辛的.

在该多辛格式中令 $r=s=l=1$,并以此离散膜受迫振动方程及其对应的多辛偏微分方程组,(11 – a)及其守恒律对应的离散形式为:

$$\begin{aligned}
&v_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}=v_{i+\frac{1}{2},j+\frac{1}{2}}^{k}+\frac{1}{2}\Delta tT_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\\
&u_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}=u_{i+\frac{1}{2},j+\frac{1}{2}}^{k}+\frac{1}{2}\Delta tv_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\\
&v_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1}=v_{i+\frac{1}{2},j+\frac{1}{2}}^{k}+\Delta tT_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\\
&u_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1}=u_{i+\frac{1}{2},j+\frac{1}{2}}^{k}+\Delta tv_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}
\end{aligned} \tag{19}$$

$$\frac{du_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1}\wedge dv_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1}-du_{i+\frac{1}{2},j+\frac{1}{2}}^{k}\wedge dv_{i+\frac{1}{2},j+\frac{1}{2}}^{k}}{\Delta t}=dU_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\wedge dT_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}} \tag{20}$$

(11 – b)及其守恒律对应的离散形式为:

$$\begin{aligned}
&w_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}=w_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}+\frac{1}{2}\Delta xQ_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\\
&u_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}=u_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}+\frac{1}{2}\Delta xw_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\\
&w_{i+1,j+\frac{1}{2}}^{k+\frac{1}{2}}=w_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}+\Delta xQ_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\\
&u_{i+1,j+\frac{1}{2}}^{k+\frac{1}{2}}=u_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}+\Delta xw_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}
\end{aligned} \tag{21}$$

$$\frac{du_{i+1,j+\frac{1}{2}}^{k+\frac{1}{2}}\wedge dw_{i+1,j+\frac{1}{2}}^{k+\frac{1}{2}}-du_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}\wedge dw_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}}{\Delta x}=dU_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\wedge dQ_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}} \tag{22}$$

(11 – c)及其守恒律对应的离散形式为:

$$\begin{aligned}
&p_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}=p_{i+\frac{1}{2},j}^{k+\frac{1}{2}}+\frac{1}{2}\Delta yR_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\\
&u_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}=u_{i+\frac{1}{2},j}^{k+\frac{1}{2}}+\frac{1}{2}\Delta yp_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\\
&p_{i+\frac{1}{2},j+1}^{k+\frac{1}{2}}=p_{i+\frac{1}{2},j}^{k+\frac{1}{2}}+\Delta yR_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\\
&u_{i+\frac{1}{2},j+1}^{k+\frac{1}{2}}=u_{i+\frac{1}{2},j}^{k+\frac{1}{2}}+\Delta yp_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}
\end{aligned} \tag{23}$$

$$\frac{du_{i+1,j+\frac{1}{2}}^{k+\frac{1}{2}}\wedge dw_{i+1,j+\frac{1}{2}}^{k+\frac{1}{2}}-du_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}\wedge dw_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}}{\Delta x}=dU_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\wedge dQ_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}} \tag{24}$$

离散的多辛守恒律为:

$$\frac{du_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1}\wedge dv_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1}-du_{i+\frac{1}{2},j+\frac{1}{2}}^{k}\wedge dv_{i+\frac{1}{2},j+\frac{1}{2}}^{k}}{\Delta t}=c^{2}\frac{du_{i+1,j+\frac{1}{2}}^{k+\frac{1}{2}}\wedge dw_{i+1,j+\frac{1}{2}}^{k+\frac{1}{2}}-du_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}\wedge dw_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}}{\Delta x}+c^{2}\frac{du_{i+\frac{1}{2},j+1}^{k+\frac{1}{2}}\wedge dp_{i+\frac{1}{2},j+1}^{k+\frac{1}{2}}-du_{i+\frac{1}{2},j}^{k+\frac{1}{2}}\wedge dp_{i+\frac{1}{2},j}^{k+\frac{1}{2}}}{\Delta y}$$

联立式(19),(21)和(23),消去中间变量 T,Q,R,得到多辛偏微分方程组(2)的多辛格式:

$$\begin{cases}
\dfrac{v_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1}-v_{i+\frac{1}{2},j+\frac{1}{2}}^{k}}{\Delta t}-c^{2}\dfrac{w_{i+1,j+\frac{1}{2}}^{k+\frac{1}{2}}-w_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}}{\Delta x}-c^{2}\dfrac{p_{i+\frac{1}{2},j+1}^{k+\frac{1}{2}}-p_{i+\frac{1}{2},j}^{k+\frac{1}{2}}}{\Delta y}=F'(u_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}})\\
\dfrac{u_{i+\frac{1}{2},j+\frac{1}{2}}^{k+1}-u_{i+\frac{1}{2},j+\frac{1}{2}}^{k}}{\Delta t}=v_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\\
\dfrac{u_{i+1,j+\frac{1}{2}}^{k+\frac{1}{2}}-u_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}}{\Delta x}=w_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}\\
\dfrac{u_{i+\frac{1}{2},j+1}^{k+\frac{1}{2}}-u_{i+\frac{1}{2},j}^{k+\frac{1}{2}}}{\Delta y}=p_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}}
\end{cases} \tag{25}$$

式中:$u_{i+\frac{1}{2},j}^{k+\frac{1}{2}}=\frac{1}{2}(u_{i+\frac{1}{2},j}^{k+1}+u_{i+\frac{1}{2},j}^{k})=\frac{1}{4}(u_{i+1,j}^{k+1}+u_{i,j}^{k+1}+u_{i+1,j}^{k}+u_{i,j}^{k})$ 等等.

进一步消去变量 v,w 和 p 就得到一个 9×3 点得的多辛格式:

$$\delta_{t}^{2}(u_{i+\frac{1}{2},j+\frac{1}{2}}^{k}+u_{i+\frac{1}{2},j-\frac{1}{2}}^{k}+u_{i-\frac{1}{2},j+\frac{1}{2}}^{k}+u_{i-\frac{1}{2},j-\frac{1}{2}}^{k})=$$

$$c^2\delta_x^2(u_{i,j+\frac{1}{2}}^{k+\frac{1}{2}}+u_{i,j-\frac{1}{2}}^{k+\frac{1}{2}}+u_{i,j+\frac{1}{2}}^{k-\frac{1}{2}}+u_{i,j-\frac{1}{2}}^{k-\frac{1}{2}})+$$
$$c^2\delta_y^2(u_{i+\frac{1}{2},j}^{k+\frac{1}{2}}+u_{i+\frac{1}{2},j}^{k-\frac{1}{2}}+u_{i-\frac{1}{2},j}^{k+\frac{1}{2}}+u_{i-\frac{1}{2},j}^{k-\frac{1}{2}})+$$
$$\frac{1}{2}[F'(u_{i-\frac{1}{2},j-\frac{1}{2}}^{k-\frac{1}{2}})+F'(u_{i-\frac{1}{2},j+\frac{1}{2}}^{k-\frac{1}{2}})+$$
$$F'(u_{i+\frac{1}{2},j-\frac{1}{2}}^{k+\frac{1}{2}})+F'(u_{i+\frac{1}{2},j+\frac{1}{2}}^{k+\frac{1}{2}})] \quad (26)$$

式中:δ^2 为二阶中心差商算子,例如:$\delta_x^2 u_{i,j}^k=(u_{i+1,j}^k-2u_{i,j}^k+u_{i-1,j}^k)/\Delta x^2$

4 数值实验

考虑如下的膜受迫振动问题:

$$\partial_{tt}u-4(\partial_{xx}u+\partial_{yy}u)=\cos(u)$$
$$u(x,y,0)=4\arctan(e^{\frac{\sqrt{6}}{4}x+\frac{\sqrt{2}}{4}y})$$
$$u(x,y,0)|_t=0$$
$$u(0,y,t)=u(a,y,t)=u(x,0,t)=u(x,b,t)=0 \quad (27)$$

有关该问题的精确解见参考文献[13]. 利用格式(26)在空间区域 D:[−40,10]×[−10,10]内取时间步长 $\Delta t=0.025$,空间步长 $\Delta x=0.1$、$\Delta y=0.02$ 对其进行数值模拟,得到不同时刻的数值解,用以验证理论分析的正确性以及所构造格式具有良好的长时间数值行为,图 1 和图 2 分别给出了方程(1)在 $t=1$ 和 $t=30$ 时刻的数值解. 表 1 给出了膜受迫振动问题(27)在 $t=30s$ 时刻部分网格点处的精确解、利用本文的多辛算法得到的数值解以及利用一般差分方法得到的数值解.

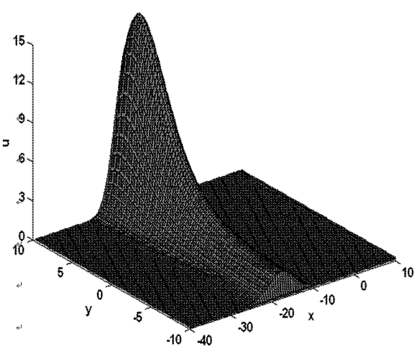

图 1 时刻的数值波形图

Fig. 1 Numerical wave form at $t=1$

表 1 的结果显示利用本文构造的多辛格式得到的数值结果在计算精度方面与一般差分方法相比有很大的提高. 从图 1 和图 2 可以看出:本文利用 Runge − Kutta 辛格式构造的多辛格式能够有效地模拟膜强迫振动的时间演化,并表现出了良好的长时间数值稳定性.

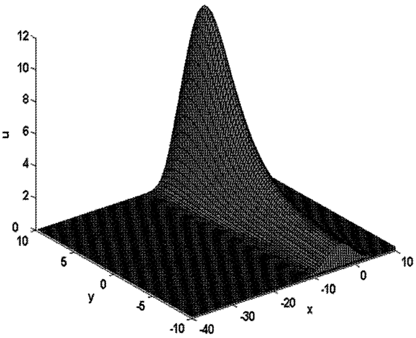

图 2 $t=30$ 时刻的数值波形图

Fig. 2 Numerical wave form at $t=30$

表 1 精确解与数值解结果比较

Table 1 Comparison between the numerical solution and exact solution

Exact solution	Numerical solution by multi-symplectic method	Numerical solution by ordinary difference method
0.06535626273721	0.0653562472515	0.0620149720527
0.42443929034227	0.4244386417539	0.4283182754461
1.08945804775808	1.0894579845470	1.0873684273192
2.79643959650320	2.79643961048027	2.7672235041813
4.48025681115531	4.48025680580735	4.4870016493370
7.17794910321090	7.17794904076376	7.1068143648713
11.57218547282356	11.5721857560819	11.2073548219107

5 结论

本文基于哈密尔顿空间体系的多辛理论研究了膜受迫振动方程,讨论了利用 Runge − Kutta 辛格式离散多辛偏微分方程组得到相应多辛格式的方法,并构造了一种 9×3 点半隐式的多辛离散格式. 数值实验的结果表明:利用本文得到的辛格式构造方法构造的多辛格式是有效的,该格式具有良好的长时间数值行为及稳定性.

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MULTI-SYMPLECTIC METHODS FOR MEMBRACE FORCED VIBRATION EQUATION *

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Abstract The multi-symplectic formulations of the membrane forced vibration equation with periodic boundary conditions in Hamilton space were considered. Using the Runge-Kutta multi-symplectic method, a semi-implicit nine-multiply three-point scheme with a symplectic conservation law was constructed to discrete the partial differential equation (PDE), which was derived from the membrane forced vibration equation. The results of the numerical experiments show that the multi-symplectic scheme can not only improve the numerical accuracy effectively but also maintain the local properties of the vibration system. From the simulation results, we can conclude that the multi-symplectic method has excellent long-time numerical behavior.

Key words multi-symplectic, runge-kutta methods, non-conservative equation

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