

四边固支弹性矩形薄板的自由振动

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摘要 首先弹性矩形薄板的动力学方程表示成为 Hamilton 正则方程,然后采用辛几何方法对全状态相变量进行分离变量,并利用得到的共扼辛正交归一关系,求出四边固支弹性矩形薄板的固有频率和振型的解析解表达式.由于在求解过程中不需要事先人为的选取挠度函数,而是从弹性矩形薄板的动力学基本方程出发,直接利用数学的方法求出可以满足四边固支边界条件下薄板的固有频率和振型的解析解表达式,使得问题的求解更加理论化和合理化.此外,还给出了计算实例来验证本文所采用的方法以及所推导出公式的正确性.

关键词 四边固支矩形薄板,Hamilton 正则方程,固有频率和振型,辛几何方法

前言

弹性矩形薄板是土木工程中较为常用的一种结构形式,如在桥梁工程中的桥面板以及各种建筑工程中的楼板等.但其精确的理论解多年一直没有得到很好的解答.目前,通常用以求解此类问题的方法主要有两大类,即解析法和数值法.解析法主要是叠加法^[5]、富里哀级数法^[6].数值法主要有有限元法和边界法^[6].各种解析法中都要事先选定挠度函数,而这些函数的选取具有一定的任意性,无确定的规律可循.虽然,数值法可以完全解决这类问题,但要求输入数据多,比较繁杂.

对于四边固支弹性矩形薄板振动问题的研究文献远远少于静力问题.文献[7]利用叠加法分析了固支弹性矩形薄板的固有频率.文献[8]利用富里哀级数法分析了固支弹性矩形薄板的振动问题.

辛几何方法在弹性力学中的应用是由钟万勰教授首先提出的^[1].这一方法的应用为弹性力学的求解开辟了新的思路.本文从弹性薄板的平衡方程出发,利用纯数学的手段直接推导出完全满足四边固支边界条件下薄板的固有频率和振型的解析解表达式.本方法摒弃了挠度函数选取,概念清晰,易于应用.为了验证该方法的正确性,文中还给出了数值算例.

1 矩形薄板的 Hamilton 正则方程和辛几何解法

矩形薄板的振动方程为

$$D \nabla^4 W(x, y, t) = m \frac{\partial^2 W(x, y, t)}{\partial t^2} \quad (1)$$

其中 $\nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2}$; $D = \frac{Eh^3}{12(1-\nu^2)}$ 为板的抗弯刚度, E, ν, h, m 分别为材料的弹性模量、泊松比、厚度和板的单位面积质量. $W(x, y, t)$ 为板的动挠度.假设板在瞬时动挠度可表达为

$$W(x, y, t) = \sum_{m=1}^{\infty} (A_m \cos \omega_m t + B_m \sin \omega_m t) W_m(x, y) \quad (2)$$

式中 A_m, B_m 为待定系数, ω_m 为板自振频率, $W_m(x, y)$ 为振型函数.将(2)式代入(1)可得到弹性地基上矩形薄板振型函数的微分方程

$$D \nabla^4 W_m(x, y) + m \omega_m^2 W_m(x, y) = 0 \quad (3)$$

通过求解(3)式就可得到矩形薄板的自振频率和振型.同样,板的内力也可作与(2)一样的分解.例如,板的弯矩就可以利用振型函数表示为

$$M_x = -D \left(\frac{\partial^2 W_m}{\partial x^2} + \nu \frac{\partial^2 W_m}{\partial y^2} \right); \quad (4)$$

$$M_y = -D \left(\frac{\partial^2 W_m}{\partial y^2} + \nu \frac{\partial^2 W_m}{\partial x^2} \right) \quad (5)$$

由式(4)和式(5)相加可得

$$M_x + M_y = -D(1+\nu) \left[\frac{\partial^2 W_m}{\partial x^2} + \frac{\partial^2 W_m}{\partial y^2} \right] - D(1+\nu) \nabla^2 W_m \quad (6)$$

令 $M = -\frac{M_x + M_y}{D(1+\nu)}$ 则式(3)和式(6)可表示为

$$D \nabla^2 M + m\omega_m^2 W_m(x, y) = 0 \quad (7)$$

$$\nabla^2 W_m = M \quad (8)$$

令

$$\frac{\partial W_m}{\partial y} = 0 \quad (9)$$

$$\frac{\partial M}{\partial y} = -\varphi \quad (10)$$

式(7)和式(8)可写成

$$\frac{\partial \theta}{\partial y} = M - \frac{\partial^2 W_m}{\partial x^2} \quad (11)$$

$$\frac{\partial \varphi}{\partial y} = \frac{m\omega_m^2}{D} W_m + \frac{\partial^2 M}{\partial x^2} \quad (12)$$

为了能把式(9)~式(12)表示成 Hamilton 正则方程, 可把式(9)~式(12)改写成

$$\frac{\partial}{\partial y} \left(\frac{D}{m\omega_m^2} M \right) = \frac{D}{m\omega_m^2} (-\varphi) \quad (13)$$

$$\frac{\partial \theta}{\partial y} = \frac{m\omega_m^2}{D} \left(\frac{D}{m\omega_m^2} M \right) - \frac{\partial^2 W_m}{\partial x^2} \quad (14)$$

$$\frac{\partial \varphi}{\partial y} = \frac{m\omega_m^2}{D} W_m + \frac{m\omega_m^2}{D} \frac{\partial^2}{\partial x^2} \left(\frac{D}{m\omega_m^2} M \right) \quad (15)$$

由式(9), 式(13)~式(15)可写成

$$\frac{\partial Z}{\partial y} = HZ \quad (16)$$

其中

$$H = \begin{bmatrix} Q & A \\ B & -Q^T \end{bmatrix}; A = \begin{bmatrix} 1 & 0 \\ 0 & \frac{D}{m\omega_m^2} \end{bmatrix};$$

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}; N = \frac{D}{m\omega_m^2} M$$

$$B = \begin{bmatrix} -\frac{\partial^2}{\partial x^2} & \frac{m\omega_m^2}{D} \\ \frac{m\omega_m^2}{D} & \frac{m\omega_m^2}{D} \frac{\partial^2}{\partial x^2} \end{bmatrix};$$

$$Z = [W_m, N, \theta, -\varphi]^T$$

可见, 矩阵 A 和 B 是对称的, 由文献[1]可知是典型的 Hamilton 微分算子矩阵, 其特点是本征值正负成对出现, 本征向量相互辛正交。方程(16)为弹性矩形薄板动力学问题的 Hamilton 正则方程。其解可采用文献[3]中给出的辛几何法, 先利用分离变量法先求出微分方程式(16)一般解, 然后利用

边界条件就可得到弹性地基上矩形薄板的自振频率和振型。令

$$Z = -\bar{X}(x) Y(y) \quad (17)$$

其中

$$\bar{X}(x) = [\bar{W}_m(x) \quad \bar{N}(x) \quad \bar{\theta}(x) \quad -\bar{\varphi}(x)]^T$$

将式(17)代入式(16)可得到

$$Y(y) = e^{\mu y} \quad (18)$$

$$H\bar{X}(x) = \mu\bar{X}(x) \quad (19)$$

其中 μ 是本征值, 待求。而 $\bar{X}(x)$ 是与之对应的本征函数向量。Hamilton 型算子矩阵 H 具有如下性质。其本征值可以分成两组, 分别记为 μ_{ai} 和 μ_{bi} , 并且有

$$\mu_{ai}, \operatorname{Re}(\mu_{ai}) \leq 0;$$

$$\mu_{bi} = -\mu_{bi}; i = 1, 2, 3 \dots$$

与其对应的本征函数向量之间存在共扼辛正交归一的关系。

$$\bar{X}_{ai}^T J \bar{X}_{bj} = \delta_{ij}; \bar{X}_{bi}^T J \bar{X}_{aj} = -\delta_{ij};$$

$$\bar{X}_{ai}^T J \bar{X}_{aj} = 0; \bar{X}_{bi}^T J \bar{X}_{bj} = 0$$

其中 $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ 为辛几何中的度量矩阵, I 为单位矩阵。根据上述性质, 任意全状态向量可以用本征向量展开为

$$Z(x, y) = \sum_{i=1}^{\infty} [a_i \exp(\mu_{ai} y) \bar{X}_{ai} + b_i \exp(-\mu_{bi} y) \bar{X}_{bi}] \quad (20)$$

其中待定常数 a_i 和 b_i ($i = 1, 2, 3 \dots$) 可由板在 y 方向的两边边界条件确定出。

实际上, 方程(19)是特征值问题, 将它展开为

$$\begin{cases} \bar{\theta}(x) = \mu \bar{W}_m(x) \\ -\frac{D}{m\omega_m^2} \bar{\varphi}(x) = \mu \bar{N}(x) \\ \frac{m\omega_m^2}{D} \bar{N}(x) - \frac{\partial^2 \bar{W}_m(x)}{\partial x^2} = \mu \bar{\theta}(x) \\ \frac{m\omega_m^2}{D} \frac{\partial^2 \bar{N}(x)}{\partial x^2} + \frac{m\omega_m^2}{D} \bar{W}_m(x) = \mu \bar{\varphi}(x) \end{cases} \quad (21)$$

其解可归结为如下常微分方程的求解

$$\frac{d^4 \bar{W}_m(x)}{dx^4} + 2\mu^2 \frac{d^2 \bar{W}_m(x)}{dx^2} + (\mu^4 + k) \bar{W}_m(x) = 0 \quad (22)$$

其中 $k = m\omega_m^2/D$ 方程(22)的特征根为两组共扼复数 $r = \pm(\alpha + i\beta)$, 而

$$\alpha = \sqrt{\frac{1}{2}(\sqrt{\mu^4 + k} - \mu^2)};$$

$$\beta = \sqrt{\frac{1}{2}(\sqrt{\mu^4 + k} + \mu^2)}$$

所以其解为

$$\bar{W}_m(x) = A_1 \cos \alpha x \operatorname{sh} \beta x + B_1 \cos \alpha x \operatorname{ch} \beta x + C_1 \sin \alpha x \operatorname{sh} \beta x + D_1 \sin \alpha x \operatorname{ch} \beta x \quad (23)$$

式中的常数 A_1, B_1, C_1, D_1 可由板在 x 方向的边界条件决定出. 固支的边界条件为

$$\text{当 } x = 0; a; W_m = 0; \frac{\partial W_m}{\partial x} = 0 \quad (24)$$

将式(23)代入边界条件式(24)可得到 $B = 0$;

$$D_1 = -\frac{b}{a}A$$

$$\begin{cases} A_1[\alpha \cos(\alpha a) \operatorname{sh}(a\beta) - \beta \sin(\alpha a) \operatorname{ch}(a\beta)] + \\ C_1 \alpha \sin(\alpha a) \operatorname{sh}(a\beta) = 0 \\ C_1[\alpha \beta \sin(\alpha a) \operatorname{ch}(a\beta) + \alpha^2 \cos(\alpha a) \operatorname{sh}(a\beta) - \\ A_1[(\alpha_2 + \beta^2) \sin(\alpha a) \operatorname{sh}(a\beta)] = 0 \end{cases} \quad (25)$$

由上式存在非零解的条件,可以得到本征值的超越方程为

$$\beta^2 \sin^2(\alpha a) = \alpha^2 \operatorname{sh}^2(a\beta) \quad (26)$$

由式(21), 式(25)和式(26)可以得到本征函数向量

$$\bar{X}(x) = \begin{cases} A_1[\cos(\alpha x) \operatorname{sh}(\beta x) - \frac{\beta}{\alpha} \sin(\alpha x) \times \\ \operatorname{ch}(\beta x)] + C_1 \sin(\alpha x) \operatorname{sh}(\beta x) \\ \mu \{ A_1[\cos(\alpha x) \operatorname{sh}(\beta x) - \frac{\beta}{\alpha} \sin(\alpha x) \times \\ \operatorname{ch}(\beta x)] + C_1 \sin(\alpha x) \operatorname{sh}(\beta x) \} \\ \frac{D}{m\omega^2} (\beta A_2 \operatorname{ch} \beta x + B_2 \operatorname{sh} \beta x) \\ - \mu (\beta A_2 \operatorname{ch} \beta x + B_2 \operatorname{sh} \beta x) \end{cases} \quad (27)$$

式中

$$A_2 = [2\alpha^2 C_1 \cos(\alpha x) - A_1(\beta^2 + \alpha^2 + \mu^2) \sin(\alpha x)] / \alpha$$

$$B_2 = A_1(\mu^2 - \beta^2 - \alpha^2) \cos(\alpha x) + C_1(\beta^2 - \alpha^2 + \mu^2) \sin(\alpha x)$$

$$A_1 = \alpha \operatorname{ch}(\alpha a) [\beta \sin(\alpha a) + \alpha \cos(\alpha a)];$$

$$C_1 = (\alpha^2 + \beta^2) \sin^2(\alpha a)$$

对于式(20)可以求得矩形薄板的振型函数为

$$W_m(x, y) = \sum_{i=1}^{\infty} \{ a_i \exp(\mu_i y) \bar{X}_{ai} + b_i \exp(-\mu_i y) \bar{X}_{bi} \} \quad (28)$$

式中常数 a_i 和 b_i 可由 y 坐标方向的边界条件决定出. y 坐标方向的边界条件为

$$\text{当 } y = 0; b; W_m = 0; \frac{\partial W_m}{\partial y} = 0 \quad (29)$$

将式(28)代入边界条件式(29)可得到关于常数 a_i 和 b_i ($i = 1, 2, 3, \dots, \infty$) 的无穷维的代数方程. 在实际的计算中根据所需精度取有限项. 由于代数方程的系数与频率 ω 有关, 利用代数方程有非零解的条件, 即系数矩阵的行列式的值为零. 就可得到频率的值. 将求出的频率代入式(28)就可得到振型函数 $W_m(x, y)$ 的解析表达式.

2 算例

为了证明本文所推导出的公式的正确性, 取文献[8]中的算例, 同时分别列出了本文计算出的固有频率结果和文献[8]的计算结果, 以便加以对比. 表1列出固有频率, 图1至图4分别给出了与固有频率所对应的振型.

表1 四边固支弹性矩形薄板固有频率

Table 1 The frequencies of a rectangular thin plate with completed clamped supported

item	λ_{11}	λ_{12}	λ_{22}	λ_{13}	λ_{23}
reference[8]	5.999	8.567	10.40	11.50	12.84
this paper	5.997	8.574	10.53	11.64	12.82

其中固有频率为 $\omega_{mn} = \frac{\lambda_{mn}^2}{a^2} \sqrt{D/m}$. 由上表可知, 本文的计算结果同文献[8]的计算结果是很接近的. 从而表明本文所推导出的四边固支矩形薄板的固有频率和振型是正确的.

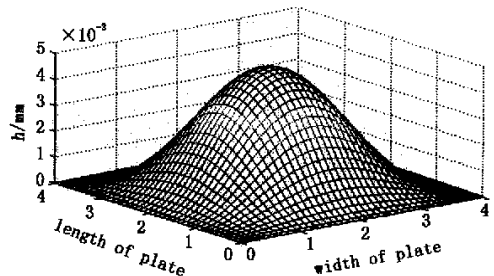


图1 11阶振型

Fig. 1 The vibration model of 11 orders

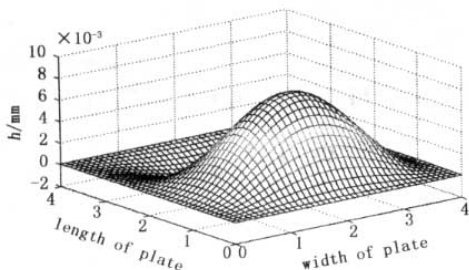


图2 12阶振型

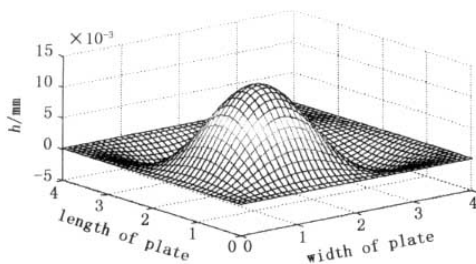


图3 22阶振型

Fig.3 The vibration model of 22 orders

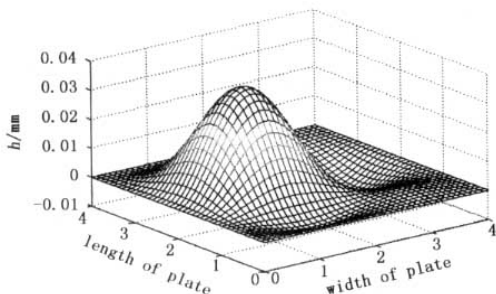


图4 23阶振型

Fig.4 The vibration model of 23 orders

3 结束语

矩形薄板的动力学方程导向 Hamilton 体系的正则方程,在辛几何空间中用分离变量法,就可以推导出了四边固支边界条件下,矩形薄板的固有频率和振型的解析表达式.由于不需要人为选取位移函数,而是直接从弹性矩形薄板的动力学方程出发

来推导出问题的解析解表达式,使得本文的方法更加合理化和理论化.通过数值算例也证明了本文方法的正确性.

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FREE VIBRATION OF RECTANGULAR THIN PLATE WITH COMPLETED CLAMPED SUPPORTED

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Abstract The analytic solution of the eigenfrequencies and models of a rectangular thin plate with completed clamped supported was derived by the symplectic geometry method. Firstly, the basic dynamic equations for the elastic thin plate were transferred into Hamilton canonical equations. And then the whole variables were separated. Finally, according to the eigenfunction expansion method in the symplectic geometry, the explicit solutions of the eigenfrequencies and models of the rectangular thin plate with completed clamped supported were obtained. Because only the basic dynamic elasticity equations of the thin plate were used, it does not need to select the deformation function arbitrarily. Therefore, the solution is reasonable and theoretical. Moreover, some numerical results were presented to demonstrate the correction of formulations.

Key words rectangular thin plate with completed clamped supported, hamilton canonical equations, eigenfrequencies and models, symplectic geometry method