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Poincaré-Chetaev 变量下广义 Routh 方程的对称性与守恒量

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摘要 根据 Rumyantsev 提出的 Poincaré-Chetaev 变量下的广义 Routh 方程,用无限小变换的方法研究它的对称性与守恒量. 得到守恒量存在的条件和形式. 该结果比以往的 Poincaré-Chetaev 方程的相关结论更一般. 最后,举例说明结果的应用.

关键词 对称性,守恒量,广义 Routh 方程

Hamilton 系统的近代理论所用的重要方法之一,不是采用正则变量,而是采用非正则变量. 这些非正则变量通常是系统的物理变量. Poincaré-Chetaev 变量下的方程已成为一种理论. 这种理论的高瞻远瞩在于恰好与 Hamilton 力学的近代理论相一致. Rumyantsev 的一系列重要工作[1~3]都在发展这一理论. 文[4]研究了 Poincaré-Chetaev 方程的李代数和 Poisson 积分法;文[5]研究了 Poincaré-Chetaev 方程的 Lie 对称性与守恒量;文[6]研究了 Poincaré-Chetaev 方程的 Noether 对称性与守恒量;文[3]建立了 Poincaré-Chetaev 变量下的广义 Routh 方程. 这是 Poincaré-Chetaev 方程理论的新发展. 本文从对称性与守恒量的角度研究广义 Routh 方程,得到守恒量存在的条件和守恒量的形式.

1 广义 Routh 方程

Rumyantsev 新近提出的 Poincaré-Chetaev 变量下的广义 Routh 方程如下[3]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial \eta_r}=\frac{\partial R}{\partial \eta_\beta}\left(C_{\alpha r}^{\beta}\eta_\alpha-C_{sr}^{\beta}\frac{\partial R}{\partial y_s}\right)+ y_\delta\left(C_{\alpha r}^{\delta}\eta_\alpha-C_{\gamma r}^{\delta}\frac{\partial R}{\partial y_\gamma}\right)+X_rR+Q_r,$$

$$\frac{\mathrm{d}y_s}{\mathrm{d}t}=\frac{\partial R}{\partial \eta_\beta}\left(C_{\alpha s}^{\beta}\eta_\alpha-C_{\gamma s}^{\beta}\frac{\partial R}{\partial y_\gamma}\right)+ y_\delta\left(C_{\alpha s}^{\delta}\eta_\alpha-C_{\gamma s}^{\delta}\frac{\partial R}{\partial y_\gamma}\right)+X_sR+Q_s, \quad (1)$$

$$\frac{\mathrm{d}x_i}{\mathrm{d}t}=B_{ir}\eta_r-B_{is}\frac{\partial R}{\partial y_s}$$

$$(i=1,\cdots,n;\alpha,\beta,r=1,\cdots,p;\ s,\delta,\gamma=p+1,\cdots,k)$$

其中 R 为广义 Routh 函数,有

$$R(t,x,\eta,y)=\frac{1}{2}a_{ij}^{*}\eta_i\eta_j+\gamma_{sj}\eta_jy_s-\frac{1}{2}b_{rs}y_ry_s+U(t,x) \quad (2)$$

$$(i,j=1,\cdots,p;r,s=p+1,\cdots,k)$$

而

$$y_s=a_{si}\eta_i,\eta_s=b_{sr}y_r-\gamma_{sj}\eta_j \quad (r,s=p+1,\cdots,k) \quad (3)$$

$$a_{ij}^{*}=a_{ij}\text{-}b_{rs}a_{ri}a_{sj}$$

这里

$$b_{rs}=\frac{A_{rs}}{D},\gamma_{sj}=b_{sr}a_{rj} \quad (4)$$

而 A_{rs}为行列式

$$D=\det\left(\frac{\partial^2 T}{\partial\eta_r\partial\eta_s}\right)_{r,s=p+1}^{k}\neq 0$$

的元素 a_{rs}的余子式;$a_{ij}(x)$为动能二次型系数,动能表示为

$$T(x,\eta)=\frac{1}{2}a_{ij}(x)\eta_i\eta_j \quad (i,j=1,\cdots,k) \quad (5)$$

$U(t,x)$为力函数,Q_i 为广义力;C_{rl}^{m}为结构系数,有

$$C_{rl}^{m}=-C_{lr}^{m} \quad (6)$$

而 X_r 为无限小线性算子,有

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$$X_r f = B_{ir}\frac{\partial f}{\partial x_i} \quad (r = 1,\cdots,k) \tag{7}$$

广义 Routh 方程(1)比 Poincaré 方程更一般. 因为，若考虑到如下关系

$$\frac{\partial R}{\partial y_s} = -\eta_s, \frac{\partial R}{\partial \eta_r} = \frac{\partial L}{\partial \eta_r},$$

$$X_l R = X_l L, \frac{\partial R}{\partial t} = \frac{\partial L}{\partial t} \tag{8}$$

则方程(1)给出 Poincaré 方程

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \eta_l} = C_{rl}^{m}\eta_r\frac{\partial L}{\partial \eta_m} + X_l L + Q_l \tag{9}$$

$$(l,m,r = 1,\cdots,k)$$

2 对称性与守恒量

关于 Poincaré-Chetaev 变量下的广义 Routh 方程(1)的对称性与守恒量的关系有如下结果.

定理 若存在规范函数 $G=G(t,x,\eta,y)$，使时间 t，参数 η_r 和 y_s 的无限小生成函数 $\xi_0(t,x,\eta,y)$，$\xi_r(t,x,\eta,y)$和 $\xi_s(t,x,\eta,y)$满足如下等式

$$\left(R - \frac{\partial R}{\partial \eta_r}\eta_r\right)\dot{\xi}_0 + \frac{\partial R}{\partial \eta_r}\dot{\xi}_r + y_s\dot{\xi}_s + \left(\frac{\partial R}{\partial t} - Q_r\eta_r + Q_s\frac{\partial R}{\partial y_s}\right)\xi_0 +$$
$$\left\{\begin{array}{l}\frac{\partial R}{\partial \eta_\beta}\left(C_{\alpha r}^{\beta}\eta_\alpha - C_{sr}^{\beta}\frac{\partial R}{\partial y_s}\right) + \\ y_\delta\left(C_{\alpha r}^{\delta}\eta_\alpha - C_{\gamma r}^{\delta}\frac{\partial R}{\partial y_\gamma}\right) + X_r R + Q_r\end{array}\right\}\xi_r +$$
$$\left\{\begin{array}{l}\frac{\partial R}{\partial \eta_\beta}\left(C_{\alpha s}^{\beta}\eta_\alpha - C_{\gamma s}^{\beta}\frac{\partial R}{\partial y_\gamma}\right) + \\ y_\delta\left(C_{\alpha s}^{\delta}\eta_\alpha - C_{\gamma s}^{\delta}\frac{\partial R}{\partial y_\gamma}\right) + X_s R + Q_s\end{array}\right\}\xi_s +$$
$$\dot{G} = 0 \tag{10}$$

则广义 Routh 方程(1)有守恒量

$$I = \left(R - \frac{\partial R}{\partial \eta_r}\eta_r\right)\xi_0 + \frac{\partial R}{\partial \eta_r}\xi_r + y_s\xi_s + G = \text{const.} \tag{11}$$

证 将式(11)按方程(1)求关于时间 t 的导数，得

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \left(R - \frac{\partial R}{\partial \eta_r}\eta_r\right)\dot{\xi}_0 + \left(\frac{\partial R}{\partial t} + \frac{\partial R}{\partial x_i}\dot{x}_i + \frac{\partial R}{\partial \eta_r}\dot{\eta}_r + \frac{\partial R}{\partial y_s}\dot{y}_s\right)\xi_0 -$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial \eta_r}\eta_r\xi_0 - \frac{\partial R}{\partial \eta_r}\dot{\eta}_r\xi_0 + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial \eta_r}\xi_r + \frac{\partial R}{\partial \eta_r}\dot{\xi}_r + \dot{y}_s\xi_s + y_s\dot{\xi}_s + \dot{G} \tag{12}$$

利用方程(1)，经计算得

$$\frac{\partial R}{\partial x_i}\dot{x}_i - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial R}{\partial \eta_r}\eta_r + \frac{\partial R}{\partial y_s}\dot{y}_s =$$
$$\frac{\partial R}{\partial x_i}\left(B_{ir}\eta_r - B_{is}\frac{\partial R}{\partial y_s}\right) -$$
$$\left\{\frac{\partial R}{\partial \eta_\beta}\left(C_{\alpha r}^{\beta}\eta_\alpha - C_{sr}^{\beta}\frac{\partial R}{\partial y_s}\right) + y_\delta\left(C_{\alpha r}^{\delta}\eta_\alpha - C_{\gamma r}^{\delta}\frac{\partial R}{\partial y_\gamma}\right) + X_r R + Q_r\right\}\eta_r +$$
$$\frac{\partial R}{\partial y_s}\left\{\frac{\partial R}{\partial \eta_\beta}\left(C_{\alpha s}^{\beta}\eta_\alpha - C_{rs}^{\beta}\frac{\partial R}{\partial y_\gamma}\right) + y_\delta\left(C_{\alpha s}^{\delta}\eta_\alpha - C_{\gamma s}^{\delta}\frac{\partial R}{\partial y_\gamma}\right) + X_s R + Q_s\right\} =$$
$$\frac{\partial R}{\partial y_s}Q_s - Q_r\eta_r \tag{13}$$

将式(13)代入式(12)并利用式(10)，便得

$$\left.\frac{\mathrm{d}I}{\mathrm{d}t}\right|_{(1)} = 0 \tag{14}$$

证毕.

上述定理比文[5]的定理2和文[6]的定理更一般. 事实上，将式(8)代入式(10)和式(11)，分别得到

$$\left(L - \frac{\partial L}{\partial \eta_l}\eta_l\right)\dot{\xi}_0 + \frac{\partial L}{\partial \eta_l}\dot{\xi}_l + \left(\frac{\partial L}{\partial t} - Q_l\eta_l\right)\xi_0 + \frac{\partial L}{\partial \eta_m}C_{rl}^{m}\eta_r\xi_m +$$
$$(X_l L + Q_l)\xi_l + \dot{G} = 0 \tag{15}$$

和

$$I = \left(L - \frac{\partial L}{\partial \eta_l}\eta_l\right)\xi_0 + \frac{\partial L}{\partial \eta_l}\xi_l + G = \text{const.} \tag{16}$$

当取

$$C_{0l}^{m} = 0 \tag{17}$$

时，式(15)和(16)就是文[5]中的式(14)和(15)，也就是文[6]中的式(15)和(16).

3 算例

例1 假设 Routh 函数 R 不显含时间 t，且所有广义力为零，即

$$\frac{\partial R}{\partial t} = 0, Q_r = Q_s = 0$$

$$(r = 1,\cdots,p; s = p+1,\cdots,k) \tag{18}$$

试研究此情形下系统的对称性与守恒量.

取生成元为

$$\xi_0 = 1, \xi_r = \xi_s = 0 \tag{19}$$

将式(19)代入式(10)，得到

$$G = 0 \tag{20}$$

而守恒量式(11)给出

$$I = R - \frac{\partial R}{\partial \eta_r}\eta_r = \text{const.} \tag{21}$$

例 2　假设系统有循环位移,使得[3]

$$C^{\beta}_{\alpha s} = 0, X_s R = 0, Q_l = 0$$
$$(\alpha, l = 1, \cdots, k; s = p+1, \cdots, k) \tag{22}$$

试研究系统的对称性与守恒量.

取生成元为

$$\xi_0 = 0, \xi_r = 0 \quad (r = 1, \cdots, p),$$
$$\xi_{p+1} = 1, \xi_{p+2} = \cdots = \xi_k = 0 \tag{23}$$

将式(23)代入式(10),得到

$$G = 0 \tag{24}$$

而守恒量式(11)给出

$$y_{p+1} = \text{const.} \tag{25}$$

类似地,可求得

$$y_{p+2} = \text{const}, \cdots, y_k = \text{const.} \tag{26}$$

文献[3]已给出积分(21),(25)和(26). 本文作为例子用对称性理论重新得到这些积分. 本文的定理可进一步研究具体的广义 Routh 方程的对称性与守恒量.

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SYMMETRIES AND CONSERVED QUANTITIES FOR GENERALIZED ROUTH'S EQUATIONS IN POINCARÉ-CHETAEV VARIABLES*

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Abstract　According to the generalized Routh's equations in Poincaré-Chetaev variables proposed by Rumyantsev, the symmetries and the conserved quantities of the equations were studied by using the method of infinitesimal transformation. The existence condition and the form of the conserved quantities were obtained. This result is more general than the past corresponding conclusions for Poincaré-Chetaev equations. Two examples were given to illustrate the application of the results.

Key words　symmetry, conserved quantity, generalized Routh's equation

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