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二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 定理*

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摘要　研究二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 定理. 首先,基于 Herglotz 广义变分原理,建立了二阶非完整系统 Vacco 动力学的 Herglotz 型运动微分方程. 其次,根据 Hamilton-Herglotz 作用量的非等时变分公式,给出了二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 对称性和准对称性的概念及其判据方程,并推导了系统的 Herglotz 型 Noether 定理及其逆定理. 最后,举例说明了该结果的应用.

关键词　二阶非完整系统,　Vacco 动力学,　Herglotz 变分原理,　Noether 定理

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Herglotz-Type Noether Theorem for Vacco Dynamics of Second-Order Nonholonomic Systems*

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Abstract　The Herglotz-type Noether theorems for the Vacco dynamics of second-order nonholonomic systems are studied. Firstly, based on the Herglotz generalized variational principle, the Herglotz-type differential equations of motion for Vacco dynamics of second-order nonholonomic systems are established. Secondly, according to the non-isochronous variation formulas of Hamilton-Herglotz action, the concepts of Herglotz-type Noether symmetry and quasi-symmetry and their criterion equations for Vacco dynamics of second-order nonholonomic systems are given, and the Herglotz-type Noether theorems and their inverse theorems are further derived. Finally, an example is given to illustrate the application of the results.

Key words　second-order nonholonomic systems,　Vacco dynamics,　Herglotz variational principle,　Noether's theorem

引言

非完整力学系统作为分析力学的重要分支,其理论广泛应用于轮式系统、电机系统等领域[1]. 梅凤翔先生曾指出:非完整系统动力学允许有多种模型,约束的物理实现不是唯一的[2]. 非完整力学模型主要有 Chetaev 型和 Vacco 型,分别以 d'Alembert-Lagrange 原理和嵌入约束的 Hamilton 原理为基础,前者是力学的,后者是数学的[3]. Vacco 动力学是由 Kozlov[4] 提出的研究不可积分约束系统动力学的一种新的数学模型,主要用于研究控制问题. 动力学系统对称性与守恒量的研究在现代数

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理科学中占有重要地位[5]. 近几十年来，关于 Vacco 动力学及其对称性的研究已经取得了较多成果，如 Vacco 动力学的几何理论[6,7]、Noether 对称性[8,9]、Mei 对称性[10]、Lie 对称性[11]和高阶 Vacco 动力学[12-14]等. 陈立群教授[12]在研究高阶非完整系统 Vacco 动力学时，还证明了微分算子 d 和变分算子 δ 顺序可交换. 然而，迄今关于 Vacco 动力学的研究很少涉及受高阶非完整约束的非保守系统. Herglotz[15]在研究接触变换及其与 Poisson 括号和 Hamilton 系统的关系时曾提出一类新的变分原理，该原理作为经典变分原理的推广可用于解决非保守问题. 2002 年，Georgieva 等[16]利用 Lagrange 函数在无穷小变换下的不变性建立了 Herglotz 型 Noether 定理. 此后，许多学者对 Herglotz 变分原理及其 Noether 定理展开了深入研究[17-28]. 2015 年，Santos 等从最优控制的角度研究了 Herglotz 型高阶变分问题[29]及其 Noether 定理[30]. 迄今为止，关于非保守系统的 Herglotz 型 Noether 守恒量的研究已经取得了重要进展，但研究仍局限于完整系统或一阶非完整系统层面. 然而，由于机器人系统的动力学控制和自动控制技术发展的需要，二阶或更高阶非完整约束系统动力学得到人们的关注[31-35]. 基于此，本文将从 Herglotz 变分原理出发，在文献[13]的基础上，研究二阶非完整非保守系统 Vacco 动力学的 Herglotz 型运动微分方程及其 Noether 理论.

为方便计算，文中采用爱因斯坦求和约定.

1　二阶非完整 Vacco 系统动力学的 Herglotz 型运动微分方程

研究 N 个质点构成的力学系统，其位形由 n 个广义坐标 $q_s\,(s=1,2,\ldots,n)$ 确定，假设系统的运动受到 g 个独立的二阶非完整约束

$$f_\beta(t,q_s,\dot q_s,\ddot q_s)=0,\ (\beta=1,2,\cdots,g) \tag{1}$$

下面研究二阶非完整系统 Vacco 动力学的 Herglotz 变分问题.

确定正轨 $q_s(t)\in C^2([t_0,t_1],\mathbb{R})$，由微分方程

$$\dot z(t)=L[t,q_s(t),\dot q_s(t),z(t)]+\lambda_\beta(t)f_\beta[t,q_s(t),\dot q_s(t),\ddot q_s(t)] \tag{2}$$

定义的作用量 $z(t)$ 在给定的边界条件

$$q_s(t)\big|_{t=t_0}=q_{s0},\ q_s(t)\big|_{t=t_1}=q_{s1} \tag{3}$$

和初始条件

$$z(t)\big|_{t=t_0}=z_{t_0} \tag{4}$$

$z(t_1)$ 取得极值. 其中，$\lambda_\beta=\lambda_\beta(t)$ 为引入的 Lagrange 乘子，$L(t,q_s,\dot q_s,z)$ 是 Herglotz 型 Lagrange 函数. q_{s0}，q_{s1}，z_{t_0} 均为常数.

方程(2)两端同时进行等时变分运算，可得

$$\delta\dot z=\frac{\partial L}{\partial q_s}\delta q_s+\frac{\partial L}{\partial \dot q_s}\delta\dot q_s+\frac{\partial L}{\partial z}\delta z+\lambda_\beta\left(\frac{\partial f_\beta}{\partial q_s}\delta q_s+\frac{\partial f_\beta}{\partial \dot q_s}\delta\dot q_s+\frac{\partial f_\beta}{\partial \ddot q_s}\delta\ddot q_s\right) \tag{5}$$

根据文献[12]，对于二阶非线性非完整系统，满足微分运算 d 和变分运算 δ 的顺序可交换. 因此，我们解上述方程，可得

$$\delta z(t)\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)-\delta z(t_0)=\int_{t_0}^{t}\left[\left(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right)\delta q_s+\left(\frac{\partial L}{\partial \dot q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_s}\right)\delta\dot q_s+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}\delta\ddot q_s\right]\exp\left(-\int_{t_0}^{\tau}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\mathrm{d}\tau \tag{6}$$

因为 $z(t_1)$ 有极值，而在 $t=t_0$ 时满足条件(4)，故有

$$\delta z(t)\big|_{t=t_0}=\delta z(t)\big|_{t=t_1}=0 \tag{7}$$

方程(6)两端同时取 $t=t_1$，并利用式(7)可得

$$\int_{t_0}^{t_1}\left[\left(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right)\delta q_s+\left(\frac{\partial L}{\partial \dot q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_s}\right)\delta\dot q_s+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}\delta\ddot q_s\right]\exp\left(-\int_{t_0}^{\tau}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\mathrm{d}\tau=0 \tag{8}$$

由分部积分法可知

$$\int_{t_0}^{t_1}\delta\dot q_s\left(\frac{\partial L}{\partial \dot q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_s}\right)\exp\left(-\int_{t_0}^{\tau}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\mathrm{d}\tau=\left[\delta q_s\left(\frac{\partial L}{\partial \dot q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_s}\right)\exp\left(-\int_{t_0}^{\tau}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\Bigg|_{t_0}^{t_1}-\int_{t_0}^{t_1}\delta q_s\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(\frac{\partial L}{\partial \dot q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_s}\right)\exp\left(-\int_{t_0}^{\tau}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\mathrm{d}\tau \tag{9}$$

以及

$$\int_{t_0}^{t_1}\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}\delta\ddot q_s\exp\left(-\int_{t_0}^{\tau}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\mathrm{d}\tau=\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}\delta\dot q_s\exp\left(-\int_{t_0}^{\tau}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\Bigg|_{t_0}^{t_1}-\left\{\delta q_s\frac{\mathrm{d}}{\mathrm{d}\tau}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}\cdot\exp\left(-\int_{t_0}^{\tau}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\right\}\Bigg|_{t_0}^{t_1}+\int_{t_0}^{t_1}\delta q_s\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}\cdot\exp\left(-\int_{t_0}^{\tau}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\mathrm{d}\tau \tag{10}$$

将式(9)、式(10)代入式(8)，并利用式(3)，我们有

$$\int_{t_0}^{t_1}\delta q_s\left\{\left(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right)\cdot\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)-\right.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\Bigg\}\mathrm{d}t=0 \tag{11}$$

根据 Lagrange 乘子法[36]，且由于式(11)对任意积分区间$[t_0,t_1]$都满足，故有

$$\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]=0 \tag{12}$$

我们称式(12)为二阶非完整系统 Vacco 动力学的 Herglotz 型运动微分方程.

2 二阶非完整 Vacco 动力学的 Herglotz 型 Noether 对称性

引入无穷小变换为

$$t^*=t+\Delta t,\ q_s^*(t^*)=q_s(t)+\Delta q_s(t) \tag{13}$$

或其展开式

$$t^*=t+\varepsilon_\sigma\xi_0^\sigma(t,q_k,\dot{q}_k,z),$$
$$q_s^*(t^*)=q_s(t)+\varepsilon_\sigma\xi_s^\sigma(t,q_k,\dot{q}_k,z) \tag{14}$$

其中$\xi_0^\sigma,\xi_s^\sigma(s=1,2,\cdots,n)$为无穷小生成元，而$\varepsilon_\sigma(\sigma=1,2,\cdots,r)$无穷小参数.

由于泛函$z(t)$经过变换式(13)成为$z^*(t^*)$，它们之间的关系如下

$$z^*(t^*)=z(t)+\Delta z(t) \tag{15}$$

且对任意一个可微函数$F(t)$，有[37]

$$\Delta\dot{F}=\frac{\mathrm{d}}{\mathrm{d}t}\Delta F-\dot{F}\frac{\mathrm{d}}{\mathrm{d}t}\Delta t \tag{16}$$

从而有

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta z=\left(\frac{\partial L}{\partial t}+\lambda_\beta\frac{\partial f_\beta}{\partial t}\right)\Delta t+\left(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right)\Delta q_s+\frac{\partial L}{\partial z}\Delta z+\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)\Delta\dot{q}_s+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}\Delta\ddot{q}_s+\dot{\lambda}_\beta f_\beta\Delta t+(L+\lambda_\beta f_\beta)\frac{\mathrm{d}}{\mathrm{d}t}\Delta t \tag{17}$$

积分之，可得

$$\Delta z(t)\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)-\Delta z(t_0)=\int_{t_0}^{t}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\left(\frac{\partial L}{\partial t}+\lambda_\beta\frac{\partial f_\beta}{\partial t}\right)\Delta t+\left(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right)\Delta q_s+\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)\Delta\dot{q}_s+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}\Delta\ddot{q}_s+\dot{\lambda}_\beta f_\beta\Delta t+(L+\lambda_\beta f_\beta)\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right]\mathrm{d}t \tag{18}$$

由于$\Delta z(t_0)=0$，则式(18)还可以写成

$$\Delta z(t)\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{\Bigg[\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)(\Delta q_s-\dot{q}_s\Delta t)+(L+\lambda_\beta f_\beta)\cdot\Delta t+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}(\Delta\dot{q}_s-\ddot{q}_s\Delta t)\Bigg]\cdot\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right](\Delta q_s-\dot{q}_s\Delta t)\Bigg\}+\Bigg\{\left(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right)\cdot\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]+\frac{\mathrm{d}^2}{\mathrm{d}^2t}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\Bigg\}(\Delta q_s-\dot{q}_s\Delta t)\mathrm{d}t \tag{19}$$

将生成元ξ_0^σ和ξ_s^σ代入式(18)和式(19)，可得

$$\Delta z(t)\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=\int_{t_0}^{t}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\cdot\Bigg[\left(\frac{\partial L}{\partial t}+\lambda_\beta\frac{\partial f_\beta}{\partial t}\right)\xi_0^\sigma+\left(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right)\xi_s^\sigma+(L+\lambda_\beta f_\beta)\dot{\xi}_0^\sigma+\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma)+\dot{\lambda}_\beta f_\beta\xi_0^\sigma+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}(\ddot{\xi}_s^\sigma-2\ddot{q}_s\dot{\xi}_0^\sigma-\dot{q}_s\ddot{\xi}_0^\sigma)\Bigg]\varepsilon_\sigma\mathrm{d}t \tag{20}$$

以及

$$\Delta z(t)\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=\int_{t_0}^{t}\varepsilon_\sigma\frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\cdot\Bigg[\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)\bar{\xi}_s^\sigma+(L+\lambda_\beta f_\beta)\xi_0^\sigma+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma-\ddot{q}_s\xi_0^\sigma)\Bigg]-\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\bar{\xi}_s^\sigma\Bigg\}+\varepsilon_\sigma\Bigg\{\left(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right)\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]+$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\Bigg\}\bar{\xi}_s^\sigma\mathrm{d}t \tag{21}$$

其中 $\bar{\xi}_s^\sigma=\xi_s^\sigma-\dot{q}_s\xi_0^\sigma$. 方程(20)和(21)是二阶非完整系统 Vacco 动力学的 Hamilton-Herglotz 作用量变分的两个基本公式.

如果对于系统的 Hamilton-Herglotz 作用量泛函 z,有

$$\Delta z(t_1)=0 \tag{22}$$

成立,则称变换式(14)为 Herglotz 型 Noether 意义下的对称变换.

此时根据式(20)可得

$$\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\cdot\Bigg[\left(\frac{\partial L}{\partial t}+\lambda_\beta\frac{\partial f_\beta}{\partial t}\right)\xi_0^\sigma+\left(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right)\xi_s^\sigma+(L+\lambda_\beta f_\beta)\dot{\xi}_0^\sigma+\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma)+\dot{\lambda}_\beta f_\beta\xi_0^\sigma+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}(\ddot{\xi}_s^\sigma-2\ddot{q}_s\dot{\xi}_0^\sigma-\dot{q}_s\ddot{\xi}_0^\sigma)\Bigg]=0 \tag{23}$$

由二阶非完整约束方程(1)和 Vacco 动力学的 Herglotz 型方程(12)确定的系统

$$\Delta z(t_1)\exp\left(-\int_{t_0}^{t_1}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=\int_{t_0}^{t_1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\varepsilon_\sigma G^\sigma\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\mathrm{d}t \tag{24}$$

其中 $G^\sigma=G^\sigma(t,q_s,\dot{q}_s,z)$ 为规范函数,则称变换式(14)为 Herglotz 型 Noether 意义下的准对称变换.

此时由方程(20)得到

$$\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\Bigg[\left(\frac{\partial L}{\partial t}+\lambda_\beta\frac{\partial f_\beta}{\partial t}\right)\xi_0^\sigma+\dot{\lambda}_\beta f_\beta\xi_0^\sigma+\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma)+(L+\lambda_\beta f_\beta)\dot{\xi}_0^\sigma+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}(\ddot{\xi}_s^\sigma-2\ddot{q}_s\dot{\xi}_0^\sigma-\dot{q}_s\ddot{\xi}_0^\sigma)+\left(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right)\xi_s^\sigma\Bigg]=\frac{\mathrm{d}}{\mathrm{d}t}\left[G^\sigma\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right] \tag{25}$$

式(23)和式(25)称为二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 判据方程.

3　二阶非完整 Vacco 动力学的 Herglotz 型 Noether 定理

在对称变换下,由式(21)和二阶非完整 Vacco 动力学的 Herglotz 型方程(12)可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\Bigg\{\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\cdot\Bigg[\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)\bar{\xi}_s^\sigma+(L+\lambda_\beta f_\beta)\xi_0^\sigma+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma-\ddot{q}_s\xi_0^\sigma)\Bigg]-\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\bar{\xi}_s^\sigma\Bigg\}=0 \tag{26}$$

对上式进行积分可得 Herglotz 型 Noether 守恒量

$$\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\cdot\Bigg[\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)\bar{\xi}_s^\sigma+(L+\lambda_\beta f_\beta)\xi_0^\sigma+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma-\ddot{q}_s\xi_0^\sigma)\Bigg]-\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\bar{\xi}_s^\sigma=\mathrm{const.} \tag{27}$$

在准对称变换下,还可得 Herglotz 型 Noether 守恒量

$$\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\cdot\Bigg[\left(\frac{\partial L}{\partial \dot{q}_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\right)\bar{\xi}_s^\sigma-G^\sigma+(L+\lambda_\beta f_\beta)\xi_0^\sigma+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}(\dot{\xi}_s^\sigma-\dot{q}_s\dot{\xi}_0^\sigma-\ddot{q}_s\xi_0^\sigma)\Bigg]-\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}\exp\left(-\int_{t_0}^{t}\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\bar{\xi}_s^\sigma=\mathrm{const.} \tag{28}$$

定理 1　如果对于由二阶非完整约束方程(1)和 Vacco 动力学的 Herglotz 型方程(12)确定的系统,生成元函数 ξ_0^σ,ξ_s^σ 和规范函数 G^σ 满足 Noether 判据方程(25),则此系统存在 Herglotz 型 Noether 守恒量式(28).

特别地,当 $G^\sigma\equiv 0$ 时,上述定理成为:

定理 2　若对于由二阶非完整约束方程(1)和 Vacco 动力学的 Herglotz 型方程确定的系统,生成元函数 ξ_0^σ,ξ_s^σ 满足 Noether 判据方程(23),则此系统存在 Herglotz 型 Noether 守恒量式(27).

定理 1 和定理 2 为二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 定理. 若 $L(t,q_s,\dot{q}_s,z)$ 不再依赖于 $z(t)$,即满足 $\partial L(t,q_s,\dot{q}_s,z)/\partial z=0$,则退化为经典 Lagrange 函数 $L(t,q_s,\dot{q}_s)$,此时二阶非完整系统 Vacco 动力学的 Herglotz 型方程(12)将退化为[12,13]:

$$\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\lambda_\beta\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f_\beta}{\partial \dot{q}_s}-\dot{\lambda}_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}+\ddot{\lambda}_\beta\frac{\partial f_\beta}{\partial \ddot{q}_s}+2\dot{\lambda}_\beta\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f_\beta}{\partial \ddot{q}_s}\right)+\lambda_\beta\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial f_\beta}{\partial \ddot{q}_s}\right)=0 \tag{29}$$

特别地,若约束是一阶非完整的,则定理 2 退化为一阶非完整系统 Vacco 动力学的 Herglorz 型

Noether 定理;若 Herglotz 型 Lagrange 函数同时也不含作用量 z,那么定理 2 将退化为经典的 Vacco 动力学系统的 Noether 定理[8],即为如下推论.

推论 1 如果对于一阶非完整 Vacco 动力学系统,变换式(14)是 Noether 准对称变换,则该系统存在 r 个线性独立的第一积分,形如

$$(L+\lambda_\beta f_\beta)\xi_0^\sigma+\left(\frac{\partial L}{\partial \dot q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_s}\right)\bar\xi_s^\sigma-G^\sigma=\text{const.},\ (\sigma=1,2,\cdots,r) \tag{30}$$

反过来,已知二阶非完整 Vacco 动力学系统的个第一积分为

$$I^\sigma(t,q_i,\dot q_i,z)=\text{const.},(\sigma=1,2,\cdots,r) \tag{31}$$

对式(31)两端关于 t 求导,并与式(12)两端同时乘以 $\bar\xi_s^\sigma$ 后相加,可得

$$\frac{\mathrm{d}I^\sigma}{\mathrm{d}t}+\left\{\left(\frac{\partial L}{\partial q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right)\exp\left(-\int_{t_0}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)-\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\partial L}{\partial \dot q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_s}\right)\exp\left(-\int_{t_0}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}\exp\left(-\int_{t_0}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\right\}\bar\xi_s^\sigma=0 \tag{32}$$

令 $\dddot q_s$ 项的系数为 0,即

$$\frac{\partial I^\sigma}{\partial \dot q_i}-\left(\frac{\partial^2 L}{\partial \dot q_s\partial \dot q_i}+\lambda_\beta\frac{\partial^2 f_\beta}{\partial \dot q_s\partial \dot q_i}+2\lambda_\beta\frac{\partial L}{\partial z}\frac{\partial^2 f_\beta}{\partial \ddot q_s\partial \dot q_i}+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}\frac{\partial^2 L}{\partial z\partial \dot q_i}-\lambda_\beta\frac{\partial^3 f_\beta}{\partial \ddot q_s\partial \dot q_i\partial t}-\dot\lambda_\beta\frac{\partial^3 f_\beta}{\partial \ddot q_s\partial t\partial \dot q_i}-\lambda_\beta\frac{\partial^2 f_\beta}{\partial \ddot q_s\partial q_i}-2\dot\lambda_\beta\frac{\partial^2 f_\beta}{\partial \ddot q_s\partial \dot q_i}\right)\exp\left(-\int_{t_0}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\bar\xi_s^\sigma=0 \tag{33}$$

从上述方程反解出 $\bar\xi_s^\sigma$ 可得

$$\bar\xi_s^\sigma=\tilde b_{si}\cdot\exp\left(\int_{t_0}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\cdot\frac{\partial I^\sigma}{\partial \dot q_i} \tag{34}$$

其中

$$b_{si}=\frac{\partial^2 L}{\partial \dot q_s\partial \dot q_i}+\lambda_\beta\frac{\partial^2 f_\beta}{\partial \dot q_s\partial \dot q_i}+2\lambda_\beta\frac{\partial L}{\partial z}\cdot\frac{\partial^2 f_\beta}{\partial \ddot q_s\partial \dot q_i}+\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}\cdot\frac{\partial^2 L}{\partial z\partial \dot q_i}-2\dot\lambda_\beta\frac{\partial^2 f_\beta}{\partial \ddot q_s\partial \dot q_i}-\dot\lambda_\beta\frac{\partial^3 f_\beta}{\partial \ddot q_s\partial t\partial \dot q_i}-\lambda_\beta\frac{\partial^2 f_\beta}{\partial \ddot q_s\partial q_i}-\lambda_\beta\frac{\partial^3 f_\beta}{\partial \ddot q_s\partial \dot q_i\partial t},\ (s,i=1,2,\cdots,n) \tag{35}$$

若令式(31)等于守恒量式(28),可得

$$\xi_0^\sigma=\left\{I^\sigma\cdot\exp\left(\int_{t_0}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)-\left(\frac{\partial L}{\partial \dot q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_s}\right)\bar\xi_s^\sigma-\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}(\dot\xi_s^\sigma-\dot q_s\dot\xi_0^\sigma-\ddot q_s\xi_0^\sigma)+G^\sigma-\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}\exp\left(\int_{t_0}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\bar\xi_s^\sigma\right\}(L+\lambda_\beta f_\beta)^{-1} \tag{36}$$

因此,在已知 $I^\sigma(t,q,\dot q,z)=\text{const.}$,并选定合适的规范函数后,可由式(34)和式(36)找到相应生成元函数 $\bar\xi_0^\sigma$ 和 $\bar\xi_s^\sigma$. 于是有如下定理:

定理 3 对于二阶非完整 Vacco 动力学的 Herglotz 型系统式(12),若存在 r 个线性独立的第一积分式(31),那么系统存在由方程式(34)和式(36)确定的生成元函数使其满足 Herglotz 型 Noether 准对称性.

若 $G^\sigma\equiv 0$,则式(36)成为

$$\xi_0^\sigma=\left\{I^\sigma\cdot\exp\left(\int_{t_0}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)-\left(\frac{\partial L}{\partial \dot q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_s}\right)\bar\xi_s^\sigma-\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}(\dot\xi_s^\sigma-\dot q_s\dot\xi_0^\sigma-\ddot q_s\xi_0^\sigma)-\frac{\mathrm{d}}{\mathrm{d}t}\left[\lambda_\beta\frac{\partial f_\beta}{\partial \ddot q_s}\exp\left(\int_{t_0}^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\bar\xi_s^\sigma\right\}(L+\lambda_\beta f_\beta)^{-1} \tag{37}$$

此时,定理 3 退化为:

定理 4 对于二阶非完整 Vacco 动力学的 Herglotz 型系统式(12),若存在 r 个线性独立的第一积分式(31),那么该系统存在由方程式(34)和式(37)确定的生成元函数使其满足 Herglotz 型 Noether 对称性.

定理 3 和定理 4 可称为二阶非完整 Vacco 动力学系统的 Herglotz 型 Noether 逆定理.

4 算例

一单位质量的质点在空间中运动. 若该质点受到非保守力为

$$F_x=-\dot x,F_y=-\dot y,F_\upsilon=-\dot\upsilon \tag{38}$$

令 $q_1=x,q_2=y,q_3=\upsilon$,它的运动受有一个双面理想二阶线性非完整约束为

$$f=-q_1\ddot q_1-q_2\ddot q_2+q_3\ddot q_3=0 \tag{39}$$

试研究该系统的守恒量.

如果已知系统的 Herglotz 型 Lagrange 函数为

$$L=\frac{1}{2}(\dot q_1^2+\dot q_2^2+\dot q_3^2)-z \tag{40}$$

将式(39)、式(40)均代入二阶非完整 Vacco 动力学的 Herglotz 型系统式(12)可得

$$\mathrm{e}^{t-t_0}(-2\lambda\ddot q_1-\ddot q_1+\dot q_1-\ddot\lambda q_1-2\dot\lambda\dot q_1+2\dot\lambda q_1+$$

$$2\lambda\dot{q}_1-\lambda q_1)=0 \quad (41)$$

$$e^{t-t_0}(-2\lambda\ddot{q}_2-\ddot{q}_2+\dot{q}_2-\ddot{\lambda}q_2-2\dot{\lambda}\dot{q}_2+2\dot{\lambda}q_2+$$
$$2\lambda\dot{q}_2-\lambda q_2)=0 \quad (42)$$

$$e^{t-t_0}(2\lambda\ddot{q}_3-\ddot{q}_3+\dot{q}_3+\ddot{\lambda}q_3+2\dot{\lambda}\dot{q}_3-2\dot{\lambda}q_3-$$
$$2\lambda\dot{q}_3+\lambda q_3)=0 \quad (43)$$

根据判据方程式(25)可得

$$\lambda(-\ddot{q}_1\xi_1-\ddot{q}_2\xi_2+\ddot{q}_3\xi_3)+\dot{\lambda}(-q_1\ddot{q}_1-q_2\ddot{q}_2+q_3\ddot{q}_3)\xi_0+$$
$$\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2-2z)\dot{\xi}_0-\lambda(q_1\ddot{q}_1+q_2\ddot{q}_2-q_3\ddot{q}_3)\dot{\xi}_0+$$
$$\dot{q}_1(\dot{\xi}_1-\dot{q}_1\dot{\xi}_0)+\dot{q}_2(\dot{\xi}_2-\dot{q}_2\dot{\xi}_0)+\dot{q}_3(\dot{\xi}_3-\dot{q}_3\dot{\xi}_0)-$$
$$\lambda q_1(\ddot{\xi}_1-2\ddot{q}_1\dot{\xi}_0-\dot{q}_1\ddot{\xi}_0)-\lambda q_2(\ddot{\xi}_2-2\ddot{q}_2\dot{\xi}_0-\dot{q}_2\ddot{\xi}_0)+$$
$$\lambda q_3(\ddot{\xi}_3-2\ddot{q}_3\dot{\xi}_0-\dot{q}_3\ddot{\xi}_0)-G-\dot{G}=0 \quad (44)$$

上述方程有解

$$\xi_0=0,\ \xi_1=2\dot{q}_1,\ \xi_2=2\dot{q}_2,\ \xi_3=2\dot{q}_3,$$
$$G=\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2-2z \quad (45)$$

由定理 1，将方程式(45)代入式(28)，可得相应的守恒量为

$$I_1=e^{t-t_0}[(1+2\lambda)(\dot{q}_1^2+\dot{q}_2^2)+(1-2\lambda)\dot{q}_3^2+2z+$$
$$(2\dot{\lambda}-2\lambda)(q_1\dot{q}_1+q_2\dot{q}_2-q_3\dot{q}_3)]=\text{const.} \quad (46)$$

特殊的，由于 λ 的初值可以任意选取，因而能将式(46)中的 $\dot{\lambda}$ 换成 λ，此时守恒量变成

$$I_1=e^{t-t_0}[(1+2\lambda)(\dot{q}_1^2+\dot{q}_2^2)+$$
$$(1-2\lambda)\dot{q}_3^2+2z]=\text{const.} \quad (47)$$

其次，当 $G\equiv 0$ 时，由判据方程式(23)可得

$$-\lambda(\ddot{q}_1\xi_1+\ddot{q}_2\xi_2-\ddot{q}_3\xi_3)+\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2-$$
$$2z)\dot{\xi}_0-\lambda(q_1\ddot{q}_1+q_2\ddot{q}_2-q_3\ddot{q}_3)\dot{\xi}_0-\dot{\lambda}(q_1\ddot{q}_1+$$
$$q_2\ddot{q}_2-q_3\ddot{q}_3)\xi_0+\dot{q}_1(\dot{\xi}_1-\dot{q}_1\dot{\xi}_0)+\dot{q}_2(\dot{\xi}_2-$$
$$\dot{q}_2\dot{\xi}_0)+\dot{q}_3(\dot{\xi}_3-\dot{q}_3\dot{\xi}_0)-\lambda q_1(\ddot{\xi}_1-2\ddot{q}_1\dot{\xi}_0-$$
$$\dot{q}_1\ddot{\xi}_0)-\lambda q_2(\ddot{\xi}_2-2\ddot{q}_2\dot{\xi}_0-\dot{q}_2\xi_0)+\lambda q_3(\ddot{\xi}_3-$$
$$2\ddot{q}_3\dot{\xi}_0-\dot{q}_3\ddot{\xi}_0)=0 \quad (48)$$

方程(48)有解

$$\xi_0=1,\ \xi_1=0,\ \xi_2=0,\ \xi_3=0 \quad (49)$$

由定理 2，将方程式(49)的特解代入式(27)，可得相应守恒量为

$$I_2=e^{t-t_0}\left[\left(\frac{1}{2}+\lambda\right)\dot{q}_1^2+\left(\frac{1}{2}+\lambda\right)\dot{q}_2^2+\right.$$
$$\left.\left(\frac{1}{2}-\lambda\right)\dot{q}_3^2+z-(\lambda-\dot{\lambda})(q_1\dot{q}_1+q_2\dot{q}_2+\right.$$
$$\left.q_3\dot{q}_3)\right]=\text{const.} \quad (50)$$

最后，根据已知守恒量由定理 3 寻找相应的无穷小生成元函数. 假设系统有守恒量如下

$$I=e^{t-t_0}[(1+2\lambda)(\dot{q}_1^2+\dot{q}_2^2)+(1-2\lambda)\dot{q}_3^2+2z]$$
$$=\text{const.} \quad (51)$$

再由式(35)可知

$$b_{11}=1+\lambda,b_{22}=1+\lambda,b_{33}=1-\lambda,$$
$$\tilde{b}_{11}=\frac{1}{1+\lambda},\tilde{b}_{22}=\frac{1}{1+\lambda},\tilde{b}_{33}=\frac{1}{1-\lambda} \quad (52)$$

将式(51)、式(52)代入式(34)分别可得

$$\bar{\xi}_1=\tilde{b}_{11}\cdot\exp\left(\int_{t_0}^{t}\frac{\partial L}{\partial z}d\theta\right)\cdot\frac{\partial I}{\partial\dot{q}_1}=\frac{2+4\lambda}{1+\lambda}\dot{q}_1 \quad (53)$$

$$\bar{\xi}_2=\tilde{b}_{22}\cdot\exp\left(\int_{t_0}^{t}\frac{\partial L}{\partial z}d\theta\right)\cdot\frac{\partial I}{\partial\dot{q}_2}=\frac{2+4\lambda}{1+\lambda}\dot{q}_2 \quad (54)$$

$$\bar{\xi}_3=\tilde{b}_{33}\cdot\exp\left(\int_{t_0}^{t}\frac{\partial L}{\partial z}d\theta\right)\cdot\frac{\partial I}{\partial\dot{q}_3}=\frac{2-4\lambda}{1-\lambda}\dot{q}_3 \quad (55)$$

将式(51)～式(55)均代入式(36)，并选取

$$G=(1+2\lambda)(\dot{q}_1^2+\dot{q}_2^2)+(1-2\lambda)\dot{q}_3^2-2z \quad (56)$$

则方程(53)～(56)有解

$$\xi_0=0,\ \xi_1=\frac{(2+4\lambda)\dot{q}_1}{1+\lambda}$$
$$\xi_2=\frac{(2+4\lambda)\dot{q}_2}{1+\lambda},\xi_3=\frac{(2-4\lambda)\dot{q}_3}{1-\lambda} \quad (57)$$

5　结论

本文基于 Herglotz 变分原理研究了二阶非完整系统的 Vacco 动力学及其 Noether 定理，主要结果如下：首先，基于 Herglotz 变分问题建立了二阶非完整系统 Vacco 动力学的 Herglotz 型运动微分方程式(12)；其次，根据 Hamilton-Herglotz 作用量变分的两个基本公式(20)和式(21)，推导了二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 定理(定理 1 和定理 2)；最后，还给出了二阶非完整系统 Vacco 动力学的 Herglotz 型 Noether 定理的逆定理(定理 3 和定理 4). 本文的研究方法和结果，可以进一步推广至任意阶非完整约束系统.

参考文献

[1]　梅凤翔，吴惠彬，李彦敏. 分析力学史略[M]. 北京：科学出版社，2019.
MEI F X, WU H B, LI Y M. A brief history of analytical mechanics [M]. Beijing: Science Press, 2019. (in Chinese)

[2]　梅凤翔. 关于非完整系统的 Paradoxical 行为——分

析力学札记之二 [J]. 力学与实践，1999，21(5)：65－66＋74.
MEI F X. The Paradoxical behavior of nonholonomic systems-notes on analytical mechanics (Ⅱ) [J]. Mechanics and Engineering, 1999, 21(5): 65－66＋74. (in Chinese)

[3] 梅凤翔. 分析力学(下卷)[M]. 北京：北京理工大学出版社，2013.
MEI F X. Analytical mechanics II [M]. Beijing: Beijing Institute of Technology Press, 2013. (in Chinese)

[4] KOZLOV V V. Dynamics of systems with nonintegrable constraints III [J]. Moscow University Mechanics Bulletin, 1983, 38(3): 40－51.

[5] 李燕，方建会，张克军. 高阶非完整系统的共形不变性与 Noether 守恒量[J]. 动力学与控制学报，2010，8(4)：300－304.
LI Y, FANG J H, ZHANG K J. Conformal invariance and Noether conserved quantity of higher-order nonholonomic system [J]. Journal of Dynamics and Control, 2010, 8(4): 300－304. (in Chinese)

[6] MARTÍNEZ S, CORTÉS J, DE LEÓN M. The geometrical theory of constraints applied to the dynamics of Vakonomic mechanical systems: the Vakonomic bracket [J]. Journal of Mathematical Physics, 2000, 41(4): 2090－2120.

[7] GUO Y X, WANG Y, CHEE G Y, et al. Nonholonomic versus Vakonomic dynamics on a Riemann-Cartan manifold [J]. Journal of Mathematical Physics, 2005, 46(6): 062902.

[8] 张解放. Vacco 动力学的 Noether 理论[J]. 应用数学和力学，1993，14(7)：635－641.
ZHANG J F. Noether's theory of Vacco dynamics [J]. Applied Mathematics and Mechanics, 1993, 14(7): 635－641. (in Chinese)

[9] MARTÍNEZ S, CORTÉS J, DE LEÓN M. Symmetries in Vakonomic dynamics: applications to optimal control [J]. Journal of Geometry and Physics, 2001, 38(3/4): 343－365.

[10] 顾书龙，张宏彬. Vacco 动力学方程的 Mei 对称性、Lie 对称性和 Noether 对称性[J]. 物理学报，2005，54(9)：3983－3986.
GU S L, ZHANG H B. Mei symmetry, Noether symmetry and Lie symmetry of a Vacco system [J]. Acta Physica Sinica, 2005, 54(9): 3983－3986. (in Chinese)

[11] DING N, FANG J H. Lie symmetry and conserved quantities for nonholonomic vacco dynamical systems [J]. Communications in Theoretical Physics, 2006, 46(2): 265－268.

[12] 陈立群. 高阶非完整系统的 Vacco 动力学[J]. 鞍山钢铁学院学报，1992，15(1)：34－39.
CHEN L Q. The Vacco dynamics for high-order nonholonomic systems [J]. Journal of Anshan Institute of I & S Technology, 1992, 15(1): 34－39. (in Chinese)

[13] 张毅. 二阶非完整 Vacco 动力学及其 Noether 定理[J]. 苏州科技大学学报(自然科学版)，2023，40(3)：29－35.
ZHANG Y. Second-order nonholonomic Vacco dynamics and its Noether's theorem [J]. Journal of Suzhou University of Science and Technology (Natural Science Edition), 2023, 40(3): 29－35. (in Chinese)

[14] 宋端，刘畅，郭永新. 高阶非完整约束系统嵌入变分恒等式的积分变分原理[J]. 物理学报，2013，62(9)：323－328.
SONG D, LIU C, GUO Y X. The integral variational principles for embedded variation identity of high-order nonholonomic constrained systems [J]. Acta Physica Sinica, 2013, 62(9): 323－328. (in Chinese)

[15] HERGLOTZ G. Gesammelte schriften [M]. Göttingen: Vandenhoeck & Ruprecht, 1979.

[16] GEORGIEVA B, GUENTHER R B. First Noether-type theorem for the generalized variational principle of Herglotz [J]. Topological Methods in Nonlinear Analysis, 2002, 20(2): 261.

[17] ALMEIDA R, MALINOWSKA A B. Fractional variational principle of Herglotz [J]. Discrete and Continuous Dynamical Systems-B, 2014, 19(8): 2367－2381.

[18] GARRA R, TAVERNA G S, TORRES D F M. Fractional Herglotz variational principles with generalized Caputo derivatives [J]. Chaos, Solitons & Fractals, 2017, 102: 94－98.

[19] TIAN X, ZHANG Y. Noether's theorem for fractional Herglotz variational principle in phase space [J]. Chaos, Solitons & Fractals, 2019, 119: 50－54.

[20] 田雪，张毅. 非保守 Lagrange 系统的 Herglotz 型广义变分原理及其 Noether 理论[J]. 南京理工大学学报，2019，43(6)：765－770＋799.
TIAN X, ZHANG Y. Generalized variational principle of Herglotz type for non-conservative Lagrangian systems and its Noether's theory [J]. Journal of

Nanjing University of Science and Technology, 2019, 43(6): 765－770＋799. (in Chinese)

[21] ZHANG Y, TIAN X. Conservation laws of nonholonomic nonconservative system based on Herglotz variational problems [J]. Physics Letters A, 2019, 383: 691－696.

[22] ZHANG Y. Variational problem of Herglotz type for Birkhoffian system and its Noether's theorems [J]. Acta Mechanica, 2017, 228(4): 1481－1492.

[23] 蔡铭俣，张毅. 变质量力学系统的 Herglotz 型 Lagrange 方程与 Noether 对称性和守恒量[J]. 动力学与控制学报，2022，20(6)：106－113.
CAI M Y, ZHANG Y. Herglotz type Lagrange equations and Noether symmetry and conserved quantity for mechanical systems with variable mass [J]. Journal of Dynamics and Control, 2022, 20(6): 106－113. (in Chinese)

[24] ZHANG Y. Recent advances on Herglotz's generalized variational principle of nonconservative dynamics [J]. Transactions of Nanjing University of Aeronautics and Astronautics, 2020, 37(1): 13－26.

[25] DING J J, ZHANG Y. Conserved quantities and adiabatic invariants of fractional Birkhoffian system of Herglotz type [J]. Chinese Physics B, 2020, 29(4): 044501.

[26] 张毅，蔡锦祥. 事件空间中非完整力学系统的 Herglotz-d'Alembert 原理与守恒律[J]. 动力学与控制学报，2022，20(2)：15－21.
ZHANG Y, CAI J X. Herglotz-d'Alembert principle and conservation law for nonholonomic mechanical systems in event space [J]. Journal of Dynamics and Control, 2022, 20(02): 15－21. (in Chinese)

[27] DE LEÓN M, LAINZ M, MUÑOZ-LECANDA M C. Optimal control, contact dynamics and Herglotz variational problem [J]. Journal of Nonlinear Science, 2022, 33(1): 9.

[28] CRUZ F, ALMEIDA R, MARTINS N. Herglotz variational problems involving distributed-order fractional derivatives with arbitrary smooth kernels [J]. Fractal and Fractional, 2022, 6(12): 731.

[29] SANTOS S P S, MARTINS N, TORRES D F M. Higher-order variational problems of Herglotz type [J]. Vietnam Journal of Mathematics, 2014, 42(4): 409－419.

[30] SANTOS S P S, MARTINS N, TORRES D F M. Noether's theorem for higher-order variational problems of Herglotz type [C]//Dynamical Systems, Dierential Equations and Applications AIMS Proceedings. Berkeley, California: AIMS Press, 2015.

[31] ZHENG X Y, WU Y Q. Controller design of high order nonholonomic system with nonlinear drifts [J]. International Journal of Automation and Computing, 2009, 6(3): 240－244.

[32] 高芳征，袁付顺，高存臣. 一类高阶非完整系统的鲁棒适应模糊控制 [J]. 模糊系统与数学，2009，23(3)：158－163.
GAO F Z, YUAN F S, GAO C C. Robust adaptive fuzzy control for a class of uncertain high-order nonholonomic systems [J]. Fuzzy Systems and Mathematics, 2009, 23(3): 158－163. (in Chinese)

[33] 宋端，崔金超，刘世兴，等. 高阶非完整系统的广义 Birkhoff 表示[J]. 动力学与控制学报，2013，11(2)：97－101.
SONG D, CUI J C, LIU S X, et al. Generalized Birkhoffian representation of high-order nonholonomic systems [J]. Journal of Dynamics and Control, 2013, 11(2): 97－101. (in Chinese)

[34] WU J D, YE W J, WANG Y W, et al. A general position control method for planar underactuated manipulators with second-order nonholonomic constraints [J]. IEEE Transactions on Cybernetics, 2021, 51(9): 4733－4742.

[35] 李娜，赵铁石，姜海勇，等. 无根欠驱动冗余机器人动力耦合特性研究[J]. 机械工程学报，2016，52(9)：49－55.
LI N, ZHAO T S, JIANG H Y, et al. Research on dynamic coupling characteristic of rootless underactuated redundant robots [J]. Journal of Mechanical Engineering, 2016, 52(9): 49－55. (in Chinese)

[36] 梅凤翔. 非完整系统力学基础 [M]. 北京：北京工业学院出版社，1985.
MEI F X. Foundations of mechanics of nonholonomic systems [M]. Beijing: Beijing Institute of Technology Press, 1985. (in Chinese)

[37] 梅凤翔. 李群和李代数对约束力学系统的应用[M]. 北京：科学出版社，1999.
MEI F X. Applications of Lie groups and Lie algebras to constrained mechanical systems [M]. Beijing: Science Press, 1999. (in Chinese)