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非保守非完整系统的 Herglotz 型 Noether 定理*

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摘要　研究非保守非完整系统的 Herglotz 型 Noether 定理及其逆定理. 首先,将 Herglotz 变分原理推广到非保守非完整系统,并基于该原理推导出系统的带乘子型运动微分方程. 其次,引入无限小变换,研究 Herglotz 作用量的不变性,提出并证明非保守非完整系统的 Noether 定理. 再次,研究了对称性逆问题,给出了 Noether 逆定理. 最后,以受非保守力的 Appell-Hamel 问题为例,介绍 Herglotz 型 Noether 定理的应用.

关键词　非完整系统,　Herglotz 变分原理,　Noether 定理

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Herglotz-Type Noether Theorem for Nonconservative Nonholonomic System*

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Abstract　The Herglotz-type Noether theorem and its inverse theorem for non-conservative nonholonomic systems are studied. Firstly, the Herglotz variational principle is extended to non-conservative nonholonomic systems, and the differential equation of motion with multipliers is derived based on this principle. Secondly, the infinitesimal transformation is introduced to study the invariance of Lagrange-Herglotz action, and the Noether theorem for non-conservative nonholonomic systems is proposed and proved. Thirdly, the inverse problem of symmetry is studied and Noether's inverse theorem is given. Finally, taking Appell-Hamel problem subject to non-conservative forces as an example, we introduce the application of Herglotz type Noether theorem.

Key words　nonholonomic system,　Herglotz variational principle,　Noether's theorem

引言

对于常见的非保守系统,已不能将其纳入到 Hamilton 变分原理的经典框架中[1]. Herglotz[2] 在研究接触变换及其与 Hamilton 系统和 Poisson 括号时提出了一类新变分原理,不同于经典变分问题的作用量,Herglotz 作用量不仅与时间、曲线及其导数有关,还与泛函本身有关. Herglotz 广义变分原理为非保守系统提供了一个新的研究途径,如非保守 Lagrange 力学[3-6]、非保守 Hamilton 力学[7]、

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非保守 Birkhoff 力学[8]等. 最近,文献[9]和文献[10]将 Herglotz 变分问题推广到非完整力学系统.

众所周知,非完整系统是一类受到 Frobenius 意义下不可积微分约束的力学系统[11,12],它广泛存在于自然界和工程实际问题中.非完整力学是继 Lagrange 力学和 Hamilton 力学后,分析力学的第三个发展阶段[13]. 在经典力学中,经常涉及到非完整系统,如硬币沿斜面做纯滚动运动[14]、两个轴相联结的轮子沿水平面做纯滚动运动[15],以及冰橇[16]、滑冰[17]、自行车[18]以及蛇板运动[19]. 在工程领域,受非完整约束的系统也广泛存在,例如控制系统[20,21]、机器人系统[22-25]、航空航天[26,27]系统等新兴领域. 近年来,非完整约束系统的几何动力学引起了学者们的广泛兴趣,并取得了一系列重要进展[28-30]. 本文将进一步基于 Herglotz 变分问题,研究非保守非完整 Lagrange 系统的守恒定律,建立带乘子型的运动微分方程,给出并证明 Noether 定理及其逆定理.

1 Herglotz 型运动微分方程

位形空间中非保守系统的广义 Herglotz 原理为

$$z(b) \to \mathrm{extr} \tag{1}$$

$$\dot{z}(t) = L(t, q_s, \dot{q}_s, z), t \in [a, b] \tag{2}$$

$$z(t)\big|_{t=a} = z_a \tag{3}$$

其中,$a, b, z_a \in R$, $q_s (s=1,2,\cdots,n)$ 为系统的广义坐标,广义坐标 q_s 在端点处受到限制,即:$q_s(t)\big|_{t=a} = q_{sa}, q_s(t)\big|_{t=b} = q_{sb}$. $\dot{q}_s (s=1,2,\cdots,n)$ 为广义速度. $L(t, q_s, \dot{q}_s, z)$ 是 Herglotz 型 Lagrange 函数. 由式(1)~式(3)确定的泛函 z 被称为 Herglotz 作用量,$z = z[q_s; t]$ 是由函数 $q_s(t)$ 所确定的.

由式(1)~式(3),易得 Herglotz 型微分变分原理[6]

$$\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_s}\right)\delta q_s = 0 \tag{4}$$

若该系统受 g 个一阶非线性非完整约束

$$f_\beta(t, q_s, \dot{q}_s) = 0, (\beta = 1,2,\cdots,g; s = 1,2,\cdots,n) \tag{5}$$

约束(5)对虚位移的限制满足 Appell-Chetaev 条件

$$\sum_{s=1}^n \frac{\partial f_\beta}{\partial \dot{q}_s}\delta q_s = 0 \tag{6}$$

利用交换关系的 Hölder 定义,所有的微分运算 d 和变分运算 δ 均可交换,即满足

$$\mathrm{d}\delta = \delta\mathrm{d} \tag{7}$$

考虑式(4)和式(6),利用通常的 Lagrange 乘子法,得

$$\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_s} + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}\right) = 0 \tag{8}$$

方程(8)是所论非完整系统的 Herglotz 型运动微分方程. 其中,$\lambda_\beta = \lambda_\beta(t, q_s, \dot{q}_s, z)$ 是约束乘子,与约束(5)相应的约束反力为 $\Lambda_s = \lambda_\beta \dfrac{\partial f_\beta}{\partial \dot{q}_s}$. 方程(8)可写为

$$\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_s} + \Lambda_s\right) = 0 \tag{9}$$

方程(9)是与非完整系统(5),(8)相应的完整系统的 Herglotz 型运动微分方程.

2 Herglotz 型 Noether 定理

引入无穷小变换

$$t^* = t + \Delta t, q_s^*(t^*) = q_s(t) + \Delta q_s \tag{10}$$

或其展开式

$$t^* = t + \varepsilon\xi_0(t, q_k, \dot{q}_k),$$
$$q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, q_k, \dot{q}_k)\ (s, k = 1,2,\cdots,n) \tag{11}$$

其中,ε 是无限小参数,ξ_0 和 $\xi_s (s=1,2,\cdots,n)$ 分别是无限小变换的时间生成元和空间生成元.

由等时变分与非等时变分之间的关系[31]

$$\Delta(\cdot) = \delta(\cdot) + \frac{\mathrm{d}}{\mathrm{d}t}(\cdot)\Delta t \tag{12}$$

我们有

$$\delta q_s = \varepsilon(\xi_s - \dot{q}_s\xi_0) \tag{13}$$

将式(13)代入 Appell-Chetaev 条件(6),并由 ε 的任意性,得

$$\sum_{s=1}^n \frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = 0 \tag{14}$$

设 Δz 为 Herglotz 作用量 z 在变换前后之差为 ε 的主线性部分,由关系(12),有

$$\Delta z = \delta z + \dot{z}\Delta t \tag{15}$$

$$\Delta\dot{z} = \frac{\mathrm{d}}{\mathrm{d}t}\Delta z - L\frac{\mathrm{d}}{\mathrm{d}t}\Delta t \tag{16}$$

由方程(2)，得

$$\Delta\dot{z}=\Delta L=\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial q_s}\Delta q_s+\frac{\partial L}{\partial \dot{q}_s}\Delta\dot{q}_s+\frac{\partial L}{\partial z}\Delta z \tag{17}$$

考虑到式(16)，方程(17)可表示为关于 Δz 的一阶常微分方程

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta z=\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial q_s}\Delta q_s+\frac{\partial L}{\partial \dot{q}_s}\Delta\dot{q}_s+\frac{\partial L}{\partial z}\Delta z+L\frac{\mathrm{d}}{\mathrm{d}t}\Delta t \tag{18}$$

方程(18)有解

$$\Delta z(t)\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=\int_a^t\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial t}\Delta t+\frac{\partial L}{\partial q_s}\Delta q_s+\frac{\partial L}{\partial \dot{q}_s}\Delta\dot{q}_s+L\frac{\mathrm{d}}{\mathrm{d}t}\Delta t\right)\mathrm{d}t \tag{19}$$

利用关系(12)，式(19)可表为

$$\Delta z(t)\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=\int_a^t\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(L\Delta t+\frac{\partial L}{\partial \dot{q}_s}\delta q_s\right)\right]+\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}_s}\right)\delta q_s\right\}\mathrm{d}t \tag{20}$$

由于

$$\Delta t=\varepsilon\xi_0,\quad \Delta q_s=\varepsilon\xi_s \tag{21}$$

将式(21)代入式(19)、式(20)，并考虑到式(13)，再将等式(14)两边同时乘 $\varepsilon\lambda_\beta$ 分别与式(22)、式(23)相加可以得到

$$\Delta z(t)\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=\varepsilon\int_a^t\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial L}{\partial t}\xi_0+\frac{\partial L}{\partial q_s}\xi_s+\frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s+\left(L-\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s\right)\dot{\xi}_0+\Lambda_s(\xi_s-\dot{q}_s\xi_0)\right]\mathrm{d}t \tag{22}$$

以及

$$\Delta z(t)\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=\varepsilon\int_a^t\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(L\xi_0+\frac{\partial L}{\partial \dot{q}_s}\bar{\xi}_s\right)\right]+\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial q_s}+\Lambda_s\right)(\xi_s-\dot{q}_s\xi_0)\right\}\mathrm{d}t \tag{23}$$

其中，$\bar{\xi}_s=\xi_s-\dot{q}_s\xi_0$. 式(22)和式(23)是系统的 Herglotz 作用量的变分的两个基本公式.

根据 Noether 对称性的概念[31,32]，下面给出非保守非完整系统 Herglotz 型 Noether 对称变换及准对称变换的判据.

如果

$$\Delta z(b)\exp\left(-\int_a^b\frac{\partial L}{\partial z}\mathrm{d}\theta\right)=0 \tag{24}$$

成立，则称无限小变换(11)为 Noether 对称变换. 由式(24)及式(22)得

$$\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial L}{\partial t}\xi_0+\frac{\partial L}{\partial q_s}\xi_s+\frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s+\left(L-\frac{\partial L}{\partial \dot{q}_s}\dot{q}_s\right)\dot{\xi}_0+\Lambda_s(\xi_s-\dot{q}_s\xi_0)\right]=0 \tag{25}$$

式(25)称为 Noether 等式，可得以下定理.

定理 1　设无限小变换(11)满足 Noether 等式(25)及限制条件(14)，则非保守非完整 Lagrange 系统(5)，(8)存在 Noether 守恒量

$$I=\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\cdot\left[L\xi_0+\frac{\partial L}{\partial \dot{q}_s}(\xi_s-\dot{q}_s\xi_0)\right]=\mathrm{const.} \tag{26}$$

证明：　在式(23)中，令 $t=b$，并利用式(24)，得到

$$\varepsilon\int_a^b\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(L\xi_0+\frac{\partial L}{\partial \dot{q}_s}\bar{\xi}_s\right)\right]+\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial q_s}+\Lambda_s\right)(\xi_s-\dot{q}_s\xi_0)\right\}\mathrm{d}t=0 \tag{27}$$

由积分区间$[a,b]$和 ε 的任意性，可得

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[L\xi_0+\frac{\partial L}{\partial \dot{q}_s}(\xi_s-\dot{q}_s\xi_0)\right]\right\}+\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial q_s}+\Lambda_s\right)(\xi_s-\dot{q}_s\xi_0)=0 \tag{28}$$

将方程(9)代入上式，有

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[L\xi_0+\frac{\partial L}{\partial \dot{q}_s}(\xi_s-\dot{q}_s\xi_0)\right]\right\}=0 \tag{29}$$

对其进行积分，得到守恒量式(26). 证毕.

如果

$$\Delta z(b)\exp\left(-\int_a^b\frac{\partial L}{\partial z}\mathrm{d}\theta\right)$$

$$=-\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left[G\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\mathrm{d}t \tag{30}$$

成立，则称无限小变换(11)为 Noether 准对称变换. 由式(30)及式(22)得

$$\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial L}{\partial t}\xi_0+\frac{\partial L}{\partial q_s}\xi_s+\frac{\partial L}{\partial \dot q_s}\dot\xi_s+\left(L-\frac{\partial L}{\partial \dot q_s}\dot q_s\right)\dot\xi_0+\Lambda_s(\xi_s-\dot q_s\xi_0)\right]$$
$$=-\frac{\mathrm{d}}{\mathrm{d}t}\left[G\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right] \tag{31}$$

式(31)亦称为 Noether 等式. 由 Noether 准对称性，亦可得到 Noether 守恒量. 有以下定理.

定理 2 设无限小变换(11)满足 Noether 等式(31)及限制条件(14)，则非保守非完整 Lagrange 系统(5)，系统(8)存在 Noether 守恒量

$$I=\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\cdot\left[L\xi_0+\frac{\partial L}{\partial \dot q_s}(\xi_s-\dot q_s\xi_0)+G\right]=\mathrm{const.} \tag{32}$$

证明： 在式(23) 中，令 $t=b$，并利用(30)，得到

$$\varepsilon\int_a^b\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(L\xi_0+\frac{\partial L}{\partial \dot q_s}\bar\xi_s\right)\right]+\exp\left(-\int_a^b \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial q_s}+\Lambda_s\right)(\xi_s-\dot q_s\xi_0)\right\}\mathrm{d}t$$
$$=-\int_a^b \frac{\mathrm{d}}{\mathrm{d}t}\left[G\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right]\mathrm{d}t \tag{33}$$

由积分区间 $[a,b]$ 和 ε 的任意性，得到

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(L\xi_0+\frac{\partial L}{\partial \dot q_s}\bar\xi_s\right)\right]+\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial q_s}+\Lambda_s\right)(\xi_s-\dot q_s\xi_0)$$
$$=-\frac{\mathrm{d}}{\mathrm{d}t}\left[G\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right] \tag{34}$$

将方程(9)代入上式，有

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[L\xi_0+\frac{\partial L}{\partial \dot q_s}(\xi_s-\dot q_s\xi_0)\right]\right\}$$
$$=-\frac{\mathrm{d}}{\mathrm{d}t}\left[G\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\right] \tag{35}$$

对其进行积分，得到守恒量(32). 证毕.

3 Herglotz 型 Noether 逆定理

已知系统(5)，系统(8)有守恒量

$$I=I(t,q_s,\dot q_s,z)=\mathrm{const.} \tag{36}$$

求解生成元 ξ_0 和 $\xi_s(s=1,2,\cdots,n)$，使它们成为 Herglotz 型 Noether 准对称变换.

将方程(8)乘以 $\bar\xi_s$，并对 s 求和，可得

$$\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_s}\right)\bar\xi_s=0 \tag{37}$$

将式(36)对 t 求导，再将结果与式(37)相加，得

$$\frac{\partial I}{\partial t}+\frac{\partial I}{\partial q_s}\dot q_s+\frac{\partial I}{\partial \dot q_s}\ddot q_s+\frac{\partial I}{\partial z}L+\exp\left(-\int_a^b \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(\frac{\partial L}{\partial q_s}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_s}+\frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot q_s}+\lambda_\beta\frac{\partial f_\beta}{\partial \dot q_s}\right)\bar\xi_s=0 \tag{38}$$

其中，

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot q_s}=\frac{\partial^2 L}{\partial t\partial \dot q_s}+\frac{\partial^2 L}{\partial q_k\partial \dot q_s}\dot q_k+\frac{\partial^2 L}{\partial \dot q_k\partial \dot q_s}\ddot q_k+\frac{\partial^2 L}{\partial z\partial \dot q_s}L \tag{39}$$

令式(38)中项 $\ddot q_s$ 的系数为 0，得

$$\frac{\partial I}{\partial \dot q_s}-\frac{\partial^2 L}{\partial \dot q_k\partial \dot q_s}\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\bar\xi_k=0 \tag{40}$$

令

$$I(t,q_s,\dot q_s,z)=\exp\left(-\int_a^t \frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left(L\xi_0+\frac{\partial L}{\partial \dot q_s}\bar\xi_s+G\right) \tag{41}$$

通过方程(40)和方程(41)，可以解出生成元 ξ_0 和 ξ_s. 可得以下定理.

定理 3 对于非保守非完整系统(5)和(8)，若有守恒量(36)，则由式(40)和式(41)可寻得无限小变换的空间和时间的生成元 ξ_0，ξ_s.

4 算例

研究 Appell-Hamel 问题[31,32]. 考虑系统还受到非保守力的作用，记非保守力为

$$F_x=-m\dot q_1,\ F_y=-m\dot q_2,\ F_z=-m\dot q_3 \tag{42}$$

则该系统在 Herglotz 意义下的 Lagrange 函数为

$$L=\frac{1}{2}m(\dot q_1^2+\dot q_2^2+\dot q_3^2)-mgq_3-z \tag{43}$$

受到的非完整约束为

$$\dot q_3-\sqrt{\dot q_1^2+\dot q_2^2}=0 \tag{44}$$

Herglotz 型方程(8)为

$$m\ddot{q}_1=-m\dot{q}_1-\lambda\frac{\dot{q}_1}{\dot{q}_3},\ m\ddot{q}_2=-m\dot{q}_2-\lambda\frac{\dot{q}_2}{\dot{q}_3},$$
$$m\ddot{q}_3=-m\dot{q}_3-mg+\lambda \tag{45}$$

利用约束方程(44)和方程(45),解得

$$\lambda=\frac{mg}{2} \tag{46}$$

故该系统的 Herglotz 型运动微分方程为

$$\ddot{q}_1=-\dot{q}_1-\frac{g\dot{q}_1}{2\dot{q}_3},\ \ddot{q}_2=-\dot{q}_2-\frac{g\dot{q}_2}{2\dot{q}_3},$$
$$\ddot{q}_3=-\dot{q}_3-\frac{g}{2} \tag{47}$$

由限制条件(14)以及 Noether 等式(31),可得

$$\dot{q}_1\xi_1+\dot{q}_2\xi_2-\dot{q}_3\xi_3=0 \tag{48}$$
$$-mg\xi_3+m\dot{q}_1\dot{\xi}_1+m\dot{q}_2\dot{\xi}_2+m\dot{q}_3\dot{\xi}_3-(m\dot{q}_3^2+mgq_3+z)\dot{\xi}_0+G+\dot{G}=0 \tag{49}$$

方程(48)和(49)有解

$$\xi_1=0,\xi_2=0,\xi_3=0,\xi_0=-1,G=0 \tag{50}$$
$$\xi_1=-\frac{1}{\dot{q}_2},\xi_2=\frac{\dot{q}_1}{\dot{q}_2^2},\xi_3=0,\xi_0=0,G=\frac{\dot{q}_1}{\dot{q}_2}e^{-t} \tag{51}$$
$$\xi_1=\frac{\dot{q}_3}{\dot{q}_1}e^{-t},\quad \xi_2=0,\quad \xi_3=e^{-t},\quad \xi_0=0,$$
$$G=(2mq_3+mgt)e^{-t} \tag{52}$$

生成元(50)～(52)是该系统的 Herglotz 型 Noether 准对称性. 由定理 2,求得守恒量

$$I_1=e^{t-a}\left[\frac{1}{2}(m\dot{q}_1^2+m\dot{q}_2^2+m\dot{q}_3^2)+mgq_3+z\right] \tag{53}$$
$$I_2=\frac{\dot{q}_1}{\dot{q}_2}e^{-a} \tag{54}$$
$$I_3=(2m\dot{q}_3+2mq_3+mgt)e^{-a} \tag{55}$$

守恒量(53)～(55)分别是由生成元(50)～(52)生成的 Noether 守恒量.

其次,研究逆问题. 假设系统有守恒量

$$I=(2m\dot{q}_3+2mq_3+mgt)e^{-a} \tag{56}$$

由式(40)和式(41),可知

$$2e^{-a}=e^{t-a}(\xi_3-\dot{q}_3\xi_0) \tag{57}$$
$$e^{t-a}[m\dot{q}_1\xi_1+m\dot{q}_2\xi_2+m\dot{q}_3\xi_3-(m\dot{q}_3^2+mgq_3+z)\xi_0+G]=(2m\dot{q}_3+2mq_3+mgt)e^{-a} \tag{58}$$

求解方程(57)和方程(58),但有 5 个未知量 $\xi_s(s=1,2,3)$,ξ_0 和 G ,故解并不唯一.

若令 $\xi_1=0$,有解

$$\xi_0=0,\xi_2=\frac{2\dot{q}_3}{\dot{q}_2}e^{-t},\ \xi_3=2e^{-t},$$
$$G=(2mq_3+mgt-2m\dot{q}_3)e^{-t} \tag{59}$$

若令 $\xi_1=\frac{2\dot{q}_3}{\dot{q}_1}e^{-t}$,则有解

$$\xi_0=0,\xi_2=0,\xi_3=2e^{-t},$$
$$G=(2mq_3+mgt-2m\dot{q}_3)e^{-t} \tag{60}$$

生成元(59)和生成元(60)亦满足限制条件(48),因此它们相应于该系统的 Herglotz 型 Noether 准对称性.

5　结论

Herglotz 广义变分原理为非保守系统动力学研究提供了一个新方法,不同于文献[9]基于 Herglotz 变分原理对线性非完整系统守恒定律的研究,本文利用对称变换来建立一般非完整系统的 Noether 定理,进而找到守恒量. 本文的主要贡献在于:区别于文献[9]基于 Herglotz 变分问题建立 Chaplygin 型运动微分方程,本文建立了一般非保守非完整系统的带乘子型运动微分方程,并进一步给出了 Herglotz 作用量的非等时变分式;建立系统基于 Herglotz 变分问题的 Noether 定理(定理 1 和定理 2),并相应地给出 Herglotz 型 Noether 定理的逆定理(定理 3). 文章结果和方法可进一步推广到 Birkhoff 系统、高阶微商系统或高阶非完整约束系统等.

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