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移动荷载作用下非线性基础梁内共振分析

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摘要 研究了有限长弹性基础上梁在移动载荷作用下的内共振响应.建立了移动集中力激励的非线性粘弹性基础支承的有限长 Euler-Bernoulli 梁模型,并对非线性偏微分方程进行离散,在第三阶固有频率与第一阶固有频率成三倍关系时,采用多尺度方法导出了 3:1 内共振的可解性条件,研究了有无移动载荷时基础阻尼和非线性刚度对梁内共振条件下自由振动响应和受迫振动响应的影响规律.在此基础上,应用 Lyapunov 第一方法确定了系统的稳定性条件.

关键词 非线性基础梁, 移动载荷, 内共振, 受迫振动, 可解性条件

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Internal Resonance Analysis of Nonlinear Foundation Beams under Moving Loads

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Abstract The internal external resonance response of a beam on a finite length elastic foundation under moving loads is studied. A finite length Euler-Bernoulli beam model supported on a nonlinear viscoelastic foundation excited by a moving concentrated force is established, and the nonlinear partial differential equation is discretized. When the third-order natural frequency and the first-order natural frequency are three times related, the solvability condition for 3:1 internal resonance is derived by using a multi-scale method. The effects of foundation damping and nonlinear stiffness on the free vibration response and forced vibration response of a beam under internal resonance conditions with or without moving loads are studied. On this basis, the stability conditions of the system are determined by using the first Lyapunov method.

Key words nonlinear basic beam, moving load, internal resonance, forced vibration, solvability condition

引言

移动荷载下基础梁能够作为很多工程结构的力学模型,例如,受到来自高速、重载车辆的作用的非线性基础路面结构.研究移动载荷作用下基础梁的振动分析是结构动力学中的一个重要课题,也是机械、桥梁和铁路工程等不同学科的研究热点.

Ghayesh 等[1]以 3:1 内共振为例,采用数值的方法研究了梁轴向运动引起的纵向和横向耦合非线性振动的动力学特性,根据不同的参数分析其表现出的动态特性.Hu 等[2]利用多尺度法,得到了空间轨道系留系统中两弹簧悬挂的空间柔性梁的内

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共振条件,研究了内共振对姿态稳定性的影响以及构件间能量传递趋势的影响.Ding 等[3]在 3:1 内共振条件下,研究了粘弹性移动梁受迫振动的稳态周期响应,采用数值算例分析了粘弹性行为对稳态周期响应的影响.Wang 等[4]采用输送流体线性梁和非线性弹簧来模拟在弹性基础上放置的流体输送管的振动,用多尺度法求出了各模态在稳态下的频率响应,并用各模态的振幅来检验内共振,通过改变流体的流速来分析系统的稳定性.Wang 和 Ding[5]研究了悬臂梁的非线性自由振动、1/3 超谐共振以及重力引起的 3:1 内共振,采用时间多尺度方法,获得其对应振动的模态响应,运用谐波平衡法求解超谐波共振中的激励分量和模型的挠度,研究了重力引起的内部共振对垂直悬臂梁非线性振动的动态影响.Guillot 等[6]对压电片梁中 1:3 内共振现象,以实验和理论的方法进行了研究.Ding 等[7]采用直接多尺度方法研究了非线性振动对轴向运动梁的应力分布和疲劳寿命的影响,揭示了内共振对 V 型带稳态响应和疲劳寿命的影响.Younesian 等[8]采用 Galerkin 方法建立了微梁的无量纲运动控制方程,结合多尺度方法求解非线性方程组,并确定了主共振和次共振条件,研究了不同谐振情况下的谐波响应.顾伟等[9]研究了预变形叶片在变速条件下的非线性动力学行为,详细研究了温度梯度、阻尼、转速扰动幅度等系统参数对叶片动态响应的影响,在 2:1 内共振条件下,研究了立方项对方程的影响.毕勤胜和陈予恕[10]分析了复摆自治系统在 1:1 内共振时的混沌和分岔特性.魏明海等[11]研究了内共振和外共振联合作用下的索一梁组合结构非线性振动问题.在众多关于飞线振动的研究成果中,与道路相关的非线性系统内一外联合共振的研究鲜有报道.

目前对于连续体的共振问题多是将连续体以梁或板的形式进行研究[12-14].杨予等[15]将桥简化为简支梁,研究其在均布移动载荷作用下的振动响应,指出梁的动态响应与其自振频率、载荷的行驶频率和荷载与梁质量的振动相关.刘涛等[16]分析了轴向运动速度和材料的非均匀性对轴向运动功能梯度梁发生共振时的影响.Martínez-Castro[17]以时变模态方程得到了移动荷载作用下非均匀多跨 Euler-Bernoulli 梁的时域半解析解,并证实了该方法具有很高的精度和鲁棒性.张弛等[18]针对轴向移动梁结构,讨论了边界支撑参数对振动的影响.Younesian 等[19]采用 Galerkin 方法结合多尺度方法求解非线性运动控制方程,研究了由非线性粘弹性基础支承的裂纹梁的频率响应,分析了不同参数对频率响应解的影响.Ding 等[20]着眼于车辆与路面的耦合非线性振动问题,将路面建模为非线性基础上的 Timoshenko 梁,研究了汽车一路面耦合系统的动力学响应.

本文建立了在移动载荷作用下的非线性粘弹性基础上的有限长 Euler-Bernoulli 梁,且系统的第三阶固有频率与第一阶固有频率成三倍关系,用多尺度法得到内共振条件下非线性系统的自由振动和强迫振动响应.通过参数分析,研究了不同参数对系统动态响应的影响规律.

1 非线性粘弹性基础梁模型

考虑在移动载荷作用下基于非线性粘弹性 Kelvin 基础的有限长 Euler-Bernoulli 梁模型,如图 1 所示.基础梁密度为 ρ,梁的宽度和厚度分别为 b 和 h,惯性矩为 I,路面长度为 L,弹性模量是 E.非线性 Kelvin 基础模型可以如下表示[21,22]:

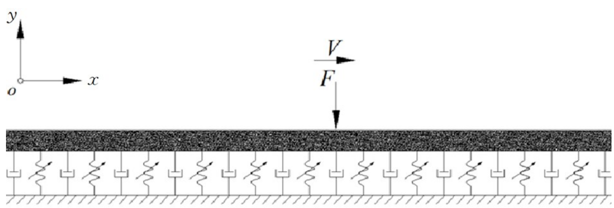


图 1 非线性粘弹性基础上的有限 Euler-Bernoulli 梁示意图
Fig.1 Finite Euler-Bernoulli beam on nonlinear viscoelastic foundation

$$P = k_1 Y + k_3 Y^3 + \mu \frac{\partial Y}{\partial T} \tag{1}$$

其中,P 是在梁上任意点发生位移的力,k_1 和 k_3 分别是基础的线性刚度和非线性刚度系数,μ 是基础的阻尼系数.根据 Euler-Bernoulli 梁理论,Kelvin 基础系统的控制微分方程可以表示为

$$EI \frac{\partial^4 w}{\partial X^4} + \rho A \frac{\partial^2 w}{\partial T^2} + k_1 w + k_3 w^3 + \mu \frac{\partial w}{\partial T} = F\delta(X - VT) \tag{2}$$

假定梁两端均为简支,因此,边界条件如下:

$$w(X,T)\big|_{X=0} = w(X,T)\big|_{X=L} = 0,$$
$$EI \frac{\partial \psi}{\partial X}(X,T)\bigg|_{X=0} = EI \frac{\partial \psi}{\partial X}(X,T)\bigg|_{X=L} = 0 \tag{3}$$

根据边界条件,梁的位移函数的表达式如下:

$$w(X,T)=\sum_{i=1}^{\infty}q_i(T)\varphi_i(X) \tag{4}$$

其中,$\varphi_i(X)$ 表示第 i 阶模态函数,$q_i(T)$ 表示关于时间 T 的第 i 阶广义位移. 梁的模态函数表示为

$$\varphi_i(X)=\sin\frac{i\pi X}{L} \tag{5}$$

将式(4)代入式(2)得,

$$\sum_{i=1}^{n}[\ddot{q}_i(T)+2\varepsilon\hat{\mu}\dot{q}_i(T)+\omega_i^2q_i(T)]\sin\frac{i\pi X}{L}+\hat{k}_3[\sum_{i=1}^{n}q_i(T)\sin\frac{i\pi X}{L}]^3=\varepsilon Lf\delta(X-VT) \tag{6}$$

其中,$\frac{\mu}{\rho A}=2\varepsilon\hat{\mu}$,$\frac{k_1}{\rho A}+\frac{i^4\pi^4EI}{\rho AL^4}=\omega_i^2$,$\frac{k_3}{\rho A}=\hat{k}_3$,$\frac{F}{\rho A}=\frac{L}{2}\varepsilon f$,ε 为小参数($0<\varepsilon\ll1$).

根据三角函数的正交性,上式等号两边同时乘以 $\varphi_m(X)=\sin\frac{m\pi X}{L}$,然后在区间长度$[0,L]$上进行积分,取 Galerkin 截断前三阶来求解,可得:

$$\ddot{q}_1(T)+\omega_1^2q_1(T)=-2\varepsilon\hat{\mu}\dot{q}_1(T)+\varepsilon(-\alpha_1q_1^3-\alpha_2q_1^2q_3-\alpha_3q_1q_2^2-\alpha_4q_1q_3^2-\alpha_5q_2^2q_3)+\varepsilon f\sin(\Omega_1T) \tag{7}$$

$$\ddot{q}_2(T)+\omega_2^2q_2(T)=-2\varepsilon\hat{\mu}\dot{q}_2(T)+\varepsilon(-\alpha_6q_1^2q_2-\alpha_7q_1q_2q_3-\alpha_8q_2^3-\alpha_9q_2q_3^2)+\varepsilon f\sin(\Omega_2T) \tag{8}$$

$$\ddot{q}_3(T)+\omega_3^2q_3(T)=-2\varepsilon\hat{\mu}\dot{q}_3(T)+\varepsilon(-\alpha_{10}q_1^3-\alpha_{11}q_1^2q_3-\alpha_{12}q_1q_2^2-\alpha_{13}q_2^2q_3-\alpha_{14}\hat{k}_3q_3^3)+\varepsilon f\sin(\Omega_3T) \tag{9}$$

其中,$\Omega_i=(i\pi V)/L$,$\alpha_2=-3/4\hat{k}_3$,$\alpha_1=\alpha_5=\alpha_8=\alpha_{12}=\alpha_{14}=3/4\hat{k}_3$,$\alpha_3=\alpha_4=\alpha_6=\alpha_7=\alpha_9=\alpha_{11}=\alpha_{13}=3/2\hat{k}_3$,$\alpha_{10}=-1/4\hat{k}_3$.

运用多尺度法,将方程的解设为

$$q_i=q_{i0}(T_0,T_1)+\varepsilon q_{i1}(T_0,T_1)+\cdots \tag{10}$$

其中,$T_n=\varepsilon^nt$ ($n=1,2,3,\cdots$),将式(10)代入式(7)~(9)可得:

ε^0 阶:

$$D_0^2q_{10}+\omega_1^2q_{10}=0 \tag{11}$$

$$D_0^2q_{20}+\omega_2^2q_{20}=0 \tag{12}$$

$$D_0^2q_{30}+\omega_3^2q_{30}=0 \tag{13}$$

ε^1 阶:

$$D_0^2q_{11}+\omega_1^2q_{11}=-2D_0D_1q_{10}-2\hat{\mu}D_0q_{10}-\alpha_1q_{10}^3-\alpha_2q_{10}^2q_{30}-\alpha_3q_{10}q_{20}^2-\alpha_4q_{10}q_{30}^2-\alpha_5q_{20}^2q_{30}+f\sin(\Omega_1T) \tag{14}$$

$$D_0^2q_{21}+\omega_2^2q_{21}=-2D_0D_1q_{20}-2\hat{\mu}D_0q_{20}-\alpha_6q_{10}^2q_{20}-\alpha_7q_{10}q_{20}q_{30}-\alpha_8q_{20}^3-\alpha_9q_{20}q_{30}^2+2f\sin(\Omega_2T) \tag{15}$$

$$D_0^2q_{31}+\omega_3^2q_{31}=-2D_0D_1q_{30}-2\hat{\mu}D_0q_{30}-\alpha_{10}\hat{k}_3q_{10}^3-\alpha_{11}q_{10}^2q_{30}-\alpha_{12}q_{10}q_{20}^2-\alpha_{13}q_{20}^2q_{30}-\alpha_{14}q_{30}^3+2f\sin(\Omega_3T) \tag{16}$$

由式(11)~(13)可以得到下列解的形式.

$$q_{10}=A_1(T_1)e^{j\omega_1T_0}+cc \tag{17}$$

$$q_{20}=A_2(T_1)e^{j\omega_2T_0}+cc \tag{18}$$

$$q_{30}=A_3(T_1)e^{j\omega_2T_0}+cc \tag{19}$$

其中,为上式中所对应的共轭项.

将式(17)~(19)代入到式(14)~(16)可得:

$$\begin{aligned}D_0^2q_{11}+\omega_1^2q_{11}=&-2j\omega_1A_1'e^{j\omega_1T_0}+2j\omega_1\bar{A}_1'e^{-j\omega_1T_0}-2j\hat{\mu}\omega_1A_1e^{j\omega_1T_0}+2j\hat{\mu}\omega_1\bar{A}_1e^{-j\omega_1T_0}+\\&(-\alpha_1A_1^3e^{3j\omega_1T_0}-3\alpha_1A_1^2\bar{A}_1e^{j\omega_1T_0}-3\alpha_1A_1\bar{A}_1^2e^{-j\omega_1T_0}-\alpha_1\bar{A}_1^3e^{-3j\omega_1T_0}-\\&\alpha_2A_1^2A_3e^{j(2\omega_1+\omega_3)T_0}-2\alpha_2A_1\bar{A}_1A_3e^{j\omega_3T_0}-\alpha_2\bar{A}_1^2A_3e^{-j(2\omega_1-\omega_3)T_0}-\alpha_2A_1^2\bar{A}_3e^{j(2\omega_1-\omega_3)T_0}-\\&2\alpha_2A_1\bar{A}_1\bar{A}_3e^{-j\omega_3T_0}-\alpha_2\bar{A}_1^2\bar{A}_3e^{-j(2\omega_1+\omega_3)T_0}-\alpha_3A_1A_2^2e^{j(\omega_1+2\omega_2)T_0}-2\alpha_3A_1A_2\bar{A}_2e^{j\omega_1T_0})-\\&\alpha_3A_1\bar{A}_1^2e^{j(\omega_1-2\omega_2)T_0}-\alpha_3\bar{A}_1A_2^2e^{-j(\omega_1-2\omega_2)T_0}-2\alpha_3\bar{A}_1A_2\bar{A}_2e^{-j\omega_1T_0}-\alpha_3\bar{A}_1\bar{A}_2^2e^{-j(\omega_1+2\omega_2)T_0}-\\&\alpha_4A_1A_3^2e^{j(\omega_1+2\omega_3)T_0}-2\alpha_4A_1A_3\bar{A}_3e^{j\omega_1T_0})-\alpha_4A_1\bar{A}_3^2e^{j(\omega_1-2\omega_3)T_0}-\alpha_4\bar{A}_1A_3^2e^{-j(\omega_1-2\omega_3)T_0}-\\&2\alpha_4\bar{A}_1A_3\bar{A}_3e^{-j\omega_1T_0}-\alpha_4\bar{A}_1\bar{A}_3^2e^{-j(\omega_1+2\omega_3)T_0}-\alpha_5A_2^2A_3e^{j(2\omega_2+\omega_3)T_0}-2\alpha_5A_2\bar{A}_2A_3e^{j\omega_3T_0})-\\&\alpha_5\bar{A}_2^2A_3e^{-j(2\omega_2-\omega_3)T_0}-\alpha_5A_2^2\bar{A}_3e^{j(2\omega_2-\omega_3)T_0}-2\alpha_5A_2\bar{A}_2\bar{A}_3e^{-j\omega_3T_0}-\alpha_5\bar{A}_2^2\bar{A}_3e^{-j(2\omega_2+\omega_3)T_0}+\\&1/2f(e^{j(\Omega_1T-\frac{\pi}{2})}+e^{-j(\Omega_1T-\frac{\pi}{2})})\end{aligned} \tag{20}$$

$$\begin{aligned}D_0^2q_{21}+\omega_2^2q_{21}=&-2j\omega_2A_2'e^{j\omega_2T_0}+2j\omega_2\bar{A}_2'e^{-j\omega_2T_0}-2j\hat{\mu}\omega_2A_2e^{j\omega_2T_0}+2j\hat{\mu}\omega_2\bar{A}_2e^{-j\omega_2T_0}+\\&(-\alpha_6\hat{k}_3A_1^2A_2e^{j(2\omega_1+\omega_2)T_0}-2\alpha_6\bar{A}_1A_1A_2e^{j\omega_2T_0}-\alpha_6\bar{A}_1^2A_2e^{-j(2\omega_1-\omega_2)T_0}-\alpha_6A_1^2\bar{A}_2e^{j(2\omega_1-\omega_2)T_0}-\\&2\alpha_6\bar{A}_1A_1\bar{A}_2e^{-j\omega_2T_0}-\alpha_6\bar{A}_1^2\bar{A}_2e^{-j(2\omega_1+\omega_3)T_0}-\\&\alpha_7A_1A_2A_3e^{j(\omega_1+\omega_2+\omega_3)T_0}-\alpha_7\bar{A}_1A_2A_3e^{-j(\omega_1-\omega_2-\omega_3)T_0})-\end{aligned}$$

$$\alpha_7 A_1 \bar{A}_1^2 e^{j(\omega_1-2\omega_2)T_0} - \alpha_3 \bar{A}_1 A_2^2 e^{-j(\omega_1-2\omega_2)T_0} -$$
$$\alpha_7 A_1 A_2 \bar{A}_3 e^{j(\omega_1+\omega_2-\omega_3)T_0} - \alpha_7 \bar{A}_1 A_2 \bar{A}_3 e^{-j(\omega_1-\omega_2+\omega_3)T_0} -$$
$$\alpha_7 A_1 \bar{A}_2 \bar{A}_3 e^{j(\omega_1-\omega_2-\omega_3)T_0} - \alpha_7 \bar{A}_1 \bar{A}_2 \bar{A}_3 e^{-j(\omega_1+\omega_2+\omega_3)T_0}) -$$
$$\alpha_8 A_2^3 e^{3j\omega_2 T_0} - 3\alpha_8 A_2^2 \bar{A}_2 e^{j\omega_2 T_0} -$$
$$3\alpha_8 A_2 \bar{A}_2^2 e^{-j\omega_2 T_0} - \alpha_8 \bar{A}_2^3 e^{-3j\omega_2 T_0} -$$
$$\alpha_9 A_2 A_3^2 e^{j(\omega_2+2\omega_3)T_0} - 2\alpha_9 A_2 \bar{A}_3 A_3 A_3 e^{j\omega_2 T_0}) -$$
$$\alpha_9 A_2 \bar{A}_3^2 e^{j(\omega_2-2\omega_3)T_0} - \alpha_9 \bar{A}_2 A_3^2 e^{-j(\omega_2-2\omega_3)T_0} -$$
$$2\alpha_9 \bar{A}_2 \bar{A}_3 A_3 e^{-j\omega_2 T_0} - \alpha_9 \bar{A}_2 \bar{A}_3^2 e^{-j(\omega_2+2\omega_3)T_0} +$$
$$f(e^{j(\Omega_2 T-\frac{\pi}{2})} + e^{-j(\Omega_2 T-\frac{\pi}{2})}) \tag{21}$$

$$D_0^2 q_{31} + \omega_3^2 q_{31} = -2j\omega_3 A_3' e^{j\omega_3 T_0} + 2j\omega_3 \bar{A}_3' e^{-j\omega_3 T_0} -$$
$$2j\hat{\mu}\omega_3 A_3 e^{j\omega_3 T_0} + 2j\hat{\mu}\omega_3 \bar{A}_3 e^{-j\omega_3 T_0} +$$
$$(-\alpha_{10} A_1^3 e^{3j\omega_1 T_0} - 3\alpha_{10} A_1^2 \bar{A}_1 e^{j\omega_1 T_0} -$$
$$3\alpha_{10} A_1 \bar{A}_1^2 e^{-j\omega_1 T_0} - \alpha_{10} \bar{A}_1^3 e^{-3j\omega_1 T_0} -$$
$$\alpha_{11} A_1^2 A_3 e^{j(2\omega_1+\omega_3)T_0} - 2\alpha_{11} A_1 \bar{A}_1 A_3 e^{j\omega_3 T_0} -$$
$$\alpha_{11} \bar{A}_1^2 A_3 e^{-j(2\omega_1-\omega_3)T_0} - \alpha_{11} A_1^2 \bar{A}_3 e^{j(2\omega_1-\omega_3)T_0} -$$
$$2\alpha_{11} A_1 \bar{A}_1 \bar{A}_3 e^{-j\omega_3 T_0} - \alpha_{11} \bar{A}_1^2 \bar{A}_3 e^{-j(2\omega_1+\omega_3)T_0} -$$
$$\alpha_{12} A_1 A_2^2 e^{j(\omega_1+2\omega_2)T_0} - 2\alpha_{12} A_1 A_2 \bar{A}_2 e^{j\omega_1 T_0}) -$$
$$\alpha_{12} A_1 \bar{A}_2^2 e^{j(\omega_1-2\omega_2)T_0} - \alpha_{12} \bar{A}_1 A_2^2 e^{-j(\omega_1-2\omega_2)T_0} -$$
$$2\alpha_{12} \bar{A}_1 A_2 \bar{A}_2 e^{-j\omega_1 T_0} - \alpha_{12} \bar{A}_1 \bar{A}_2^2 e^{-j(\omega_1+2\omega_2)T_0} -$$
$$\alpha_{13} A_2^2 A_3 e^{j(2\omega_2+\omega_3)T_0} - 2\alpha_{13} A_2 \bar{A}_2 A_3 e^{j\omega_3 T_0}) -$$
$$\alpha_{13} \bar{A}_2^2 A_3 e^{-j(2\omega_2-\omega_3)T_0} - \alpha_{13} A_2^2 \bar{A}_3 e^{j(2\omega_2-\omega_3)T_0} -$$
$$2\alpha_{13} A_2 \bar{A}_2 \bar{A}_3 e^{-j\omega_3 T_0} - \alpha_{13} \bar{A}_2^2 \bar{A}_3 e^{-j(2\omega_2+\omega_3)T_0} -$$
$$\alpha_{14} A_3^3 e^{3j\omega_3 T_0} - 3\alpha_{14} A_3^2 \bar{A}_3 e^{j\omega_3 T_0}) -$$
$$3\alpha_{14} A_3 \bar{A}_3^2 e^{-j\omega_3 T_0} - \alpha_{14} \bar{A}_3^3 e^{-3j\omega_3 T_0} -$$
$$f(e^{j(\Omega_3 T-\frac{\pi}{2})} + e^{-j(\Omega_3 T-\frac{\pi}{2})}) \tag{22}$$

2　3:1 内共振条件下的稳定性分析

为研究 3:1 内共振下基础梁的自由振动和强迫振动响应，将系统第 3 阶模态的共振频率与第 1 阶模态共振频率的 3 倍设为相近，即，进而可以确定基础的线性刚度.其中，以及下文提到的均为引入的调谐因子.

2.1　3:1 内共振下的自由振动

分析自由振动，即考虑在无移动载荷作用下的振动响应.消除久期项的条件为：

$$-2j\omega_1 A_1' - 2j\hat{\mu}\omega_1 A_1 - 3\alpha_1 A_1^2 \bar{A}_1 -$$
$$\alpha_2 \bar{A}_1^2 A_3 e^{j\sigma_1 T_1} - 2\alpha_3 A_1 A_2 \bar{A}_2 -$$
$$2\alpha_4 A_1 A_3 \bar{A}_3 = 0 \tag{23}$$

$$-2j\omega_2 A_2' - 2j\hat{\mu}\omega_2 A_2 - 2\alpha_6 \bar{A}_1 A_1 A_2 -$$
$$3\alpha_8 A_2^2 \bar{A}_2 - 2\alpha_9 A_2 \bar{A}_3 A_3 = 0 \tag{24}$$

$$-2j\omega_3 A_3' - 2j\hat{\mu}\omega_3 A_3 - \alpha_{10} A_1^3 e^{-j\sigma_1 T_1} -$$
$$2\alpha_{11} A_1 \bar{A}_1 A_3 - 2\alpha_{13} A_2 \bar{A}_2 A_3 - 3\alpha_{14} A_3^2 \bar{A}_3 = 0 \tag{25}$$

令 $A_1 = 1/2 a_1 e^{j\beta_1}$ 、$A_2 = 1/2 a_2 e^{j\beta_2}$ 和 $A_3 = 1/2 a_3 e^{j\beta_3}$ ，并代入到上式，再分离实部和虚部，可得：

$$\omega_1 \beta_1' a_1 - 3/8\alpha_1 a_1^3 - 1/8\alpha_2 a_1^2 a_3 \cos\gamma -$$
$$1/4\alpha_4 a_1 a_2^2 - 1/4\alpha_4 a_1 a_3^2 = 0 \tag{26}$$

$$-\omega_1 a_1' - a_1 \hat{\mu}\omega_1 - 1/8\alpha_2 a_1^2 a_3 \sin\gamma = 0 \tag{27}$$

$$\omega_2 \beta_2' a_2 - 1/4\alpha_6 a_1^2 a_2 - 1/4\alpha_9 a_2 a_3^2 -$$
$$3/8\alpha_8 a_2^3 = 0 \tag{28}$$

$$-\omega_2 a_2' - \hat{\mu}\omega_2 a_2 = 0 \tag{29}$$

$$\omega_3 \beta_3' a_3 - 1/8\alpha_{10} a_1^3 \cos\gamma - 1/4\alpha_{11} a_1^2 a_3 -$$
$$1/4\alpha_{13} a_2^2 a_3 - 3/8\alpha_{14} a_3^3 = 0 \tag{30}$$

$$-\omega_3 a_3' - \hat{\mu}\omega_3 a_3 + 1/8\alpha_{10} a_1^3 \sin\gamma = 0 \tag{31}$$

其中，$\gamma = \sigma_1 T_1 + \beta_3 - 3\beta_1$.

式(29)在稳态条件下的响应为

$$a_2 = 0 \tag{32}$$

由式(27)、式(31)和式(32)可得：

$$a_1 a_1' + \frac{\omega_3 \alpha_2}{\omega_1 \alpha_{10}} a_3 a_3' = -\hat{\mu} a_1^2 - \frac{\omega_3 \alpha_2}{\omega_1 \alpha_{10}} \hat{\mu} a_3 \tag{33}$$

令 $f_{13}(T_1) = a_1^2 + \frac{\omega_3 \alpha_2}{\omega_1 \alpha_{10}} a_3^2$，将其代入式(33)，可得：

$$\frac{df_{13}(T_1)}{dT_1} = -2\hat{\mu} f_{13}(T_1) \tag{34}$$

由式(34)可得：

$$f_{13}(T_1) = C e^{-2\hat{\mu} T_1} \tag{35}$$

其中 C 为大于 0 的常数.

从式(35)可以看出，当 $T_1 \to +\infty$ 时，$f_{13}(+\infty) = 0$，从而理论分析上可以说明系统是随着时间逐渐衰减为 0.同样，在时域下变量 a_1 和 a_3 的数值解可由式(26)～式(31)得到，如图 2～图 7 所示.图 2 通过数值仿真给出了系统在自由振动下的衰减过程.

阻尼和非线性刚度对系统自由振动的影响如图 3 所示.从图 3(a)和 3(c)可以看出随着阻尼增加，系统的衰减速度就会越快；图 3(b)和 3(d)可以看出，非线性刚度会增加系统的振动频率，但对其

衰减速度基本无影响.

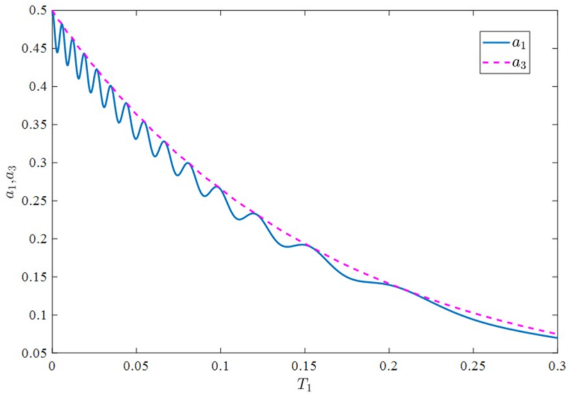
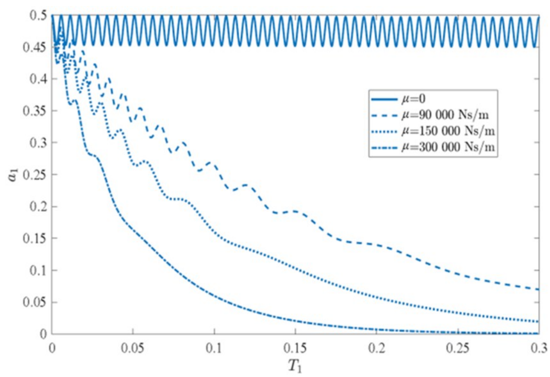
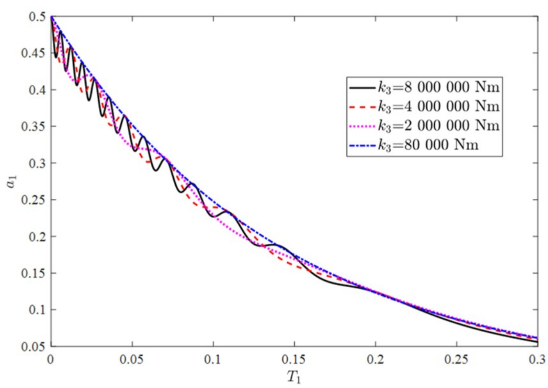


图 2　自由振动下幅值的衰减过程

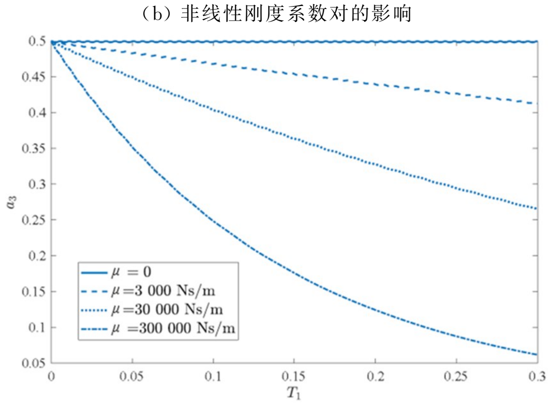
Fig.2　Attenuation process of amplitude under free vibration



(a) 阻尼系数对的影响



(b) 非线性刚度系数对的影响



(c) 阻尼系数对的影响

(d) 非线性刚度系数对的影响

图 3　自由振动下参数对系统响应的影响

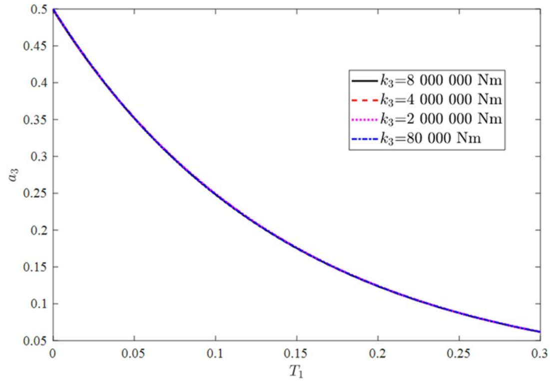


Fig.3　Influence of parameters on system response under free vibration

2.2　3:1 内共振下的受迫振动分析

在 3:1 内共振(即 $\omega_3 \approx 3\omega_1$ 时)下分别分析 $\Omega_1 \approx \omega_1$、$\Omega_2 \approx \omega_2$ 和 $\Omega_3 \approx \omega_3$ 时系统发生强迫共振条件.

(1)情况 I:$\Omega_1 \approx \omega_1$

假定　$\Omega_1 = \omega_1 + \varepsilon\sigma_2$　(36)

消除久期项,并分离实部和虚部,可以得到:

$$\omega_1\beta_1' a_1 - 3/8\alpha_1 a_1^3 - 1/4/\alpha_4 a_1 a_2^2 - 1/4\alpha_4 a_1 a_3^2 - 1/8\alpha_2 a_1^2 a_3 \cos\gamma_1 + f/2\cos\gamma_2 = 0 \quad (37)$$

$$-\omega_1 a_1' - \hat{\mu}\omega_1 a_1 - 1/8\alpha_2 a_1^2 a_3 \sin\gamma_1 + f/2\sin\gamma_2 = 0 \quad (38)$$

$$\omega_2\beta_2' a_2 - 1/4\alpha_6 a_1^2 a_2 - 1/4\alpha_9 a_2 a_3^2 - 3/8\alpha_8 a_2^3 = 0 \quad (39)$$

$$-\omega_2 a_2' - \hat{\mu}\omega_2 a_2 = 0 \quad (40)$$

$$\omega_3\beta_3' a_3 - 1/8\alpha_{10} a_1^3 \cos\gamma_1 - 1/4\alpha_{11} a_1^2 a_3 - 1/4\alpha_{13} a_2^2 a_3 - 3/8\alpha_{14} a_3^3 = 0 \quad (41)$$

$$-\omega_3 a_3' - \hat{\mu}\omega_3 a_3 + 1/8\alpha_{10} a_1^3 \sin\gamma_1 = 0 \quad (42)$$

其中,$\gamma_1 = \sigma_1 T_1 + \beta_3 - 3\beta_1$,$\gamma_2 = \sigma_2 T_1 - \beta_1 - \pi/2$.

(2)情况 II:$\Omega_2 \approx \omega_2$

假定　$\Omega_2 = \omega_2 + \varepsilon\sigma_2$　(43)

消除久期项,并分离实部和虚部,可以得到:

$$\omega_1\beta_1' a_1 - 3/8\alpha_1 a_1^3 - 1/4\alpha_4 a_1 a_2^2 - 1/4\alpha_4 a_1 a_3^2 - 1/8\alpha_2 a_1^2 a_3 \cos\gamma_1 = 0 \quad (44)$$

$$-\hat{\mu}\omega_1 a_1 - 1/8\alpha_2 a_1^2 a_3 \sin\gamma_1 = 0 \quad (45)$$

$$\omega_2\sigma_2 a_2 - 1/4/\alpha_6 a_1^2 a_2 - 1/4\alpha_9 a_2 a_3^2 - 3/8\alpha_8 a_2^3 + f/2\cos\gamma_2 = 0 \quad (46)$$

$$-\hat{\mu}\omega_2 a_2 + f/2\sin\gamma_2 = 0 \quad (47)$$

$$\omega_3(3\beta_1' - \sigma_1)a_3 - 1/8\alpha_{10} a_1^3 \cos\gamma_1 - 1/4\alpha_{11} a_1^2 a_3 - 1/4\alpha_{13} a_2^2 a_3 - 3/8\alpha_{14} a_3^3 = 0 \quad (48)$$

$$-\hat{\mu}\omega_3 a_3 + 1/8\alpha_{10} a_1^3 \sin\gamma_1 = 0 \quad (49)$$

其中,$\gamma_1 = \sigma_1 T_1 + \beta_3 - 3\beta_1$,$\gamma_2 = \sigma_2 T_1 - \beta_2 - \pi/2$.

(3)情况 III:$\Omega_3 \approx \omega_3$

假定 $\Omega_3=\omega_3+\varepsilon\sigma_3$ (50)

消除久期项,并分离方程的实部和虚部可以得到式(51)~式(56).

$$\omega_1\beta_1'a_1-3/8\alpha_1a_1^3-1/8\alpha_2a_1^2a_3\cos\gamma-1/4\alpha_4a_1a_2^2-1/4\alpha_4a_1a_3^2=0 \tag{51}$$

$$-\omega_1a_1'-a_1\hat{\mu}\omega_1-1/8\alpha_2a_1^2a_3\sin\gamma=0 \tag{52}$$

$$\omega_2\beta_2'a_2-1/4\alpha_6a_1^2a_2-1/4\alpha_9a_2a_3^2-3/8\alpha_8a_2^3=0 \tag{53}$$

$$-\omega_2a_2'-\hat{\mu}\omega_2a_2=0 \tag{54}$$

$$\omega_3(3\beta_1'-\sigma_1)a_3-1/8\alpha_{10}a_1^3\cos\gamma_1-1/4\alpha_{11}a_1^2a_3-1/4\alpha_{13}a_2^2a_3-3/8\alpha_{14}a_3^3+1/2f\cos\gamma_2=0 \tag{55}$$

$$-\hat{\mu}\omega_3a_3+1/8\alpha_{10}a_1^3\sin\gamma_1+1/2f\sin\gamma_2=0 \tag{56}$$

其中,$\gamma_1=\sigma_1T_1+\beta_3-3\beta_1$,$\gamma_2=\sigma_2T_1-\beta_3-\pi/2$.

(4)稳定性分析

有限长梁和非线性基础的物理几何参数如表 1 所示.以情况 I 为例分析系统的稳定性,观察方程组式(37)~式(42)可以看出稳态下 $a_2'=a_2=0$,则方程组可以写为

$$\omega_1\beta_1'a_1-3/8\alpha_1a_1^3-1/4\alpha_4a_1a_3^2-1/8\alpha_2a_1^2a_3\cos\gamma_1+f/2\cos\gamma_2=0 \tag{57}$$

$$-\omega_1a_1'-\hat{\mu}\omega_1a_1+f/2\sin\gamma_2=0 \tag{58}$$

$$\omega_3\beta_3'a_3-1/8\alpha_{10}a_1^3\cos\gamma_1-1/4\alpha_{11}a_1^2a_3-3/8\alpha_{14}a_3^3=0 \tag{59}$$

$$-\omega_3a_3'-\hat{\mu}\omega_3a_3+1/8\alpha_{10}a_1^3\sin\gamma_1=0 \tag{60}$$

又因为

$$\beta_3'=\gamma_1'-3\gamma_2'+3\sigma_2-\sigma_1 \tag{61}$$

$$\beta_1'=\sigma_2-\gamma_2' \tag{62}$$

表 1 基础梁和移动载荷的物理几何参数[1]

Table 1 Physical geometric parameters of foundation beams and moving load

参数	符号	数值
弹性模量(GPa)	E	6.998
密度(kg/m^3)	ρ	2373
厚度(m)	h	0.15
宽度(m)	b	1.0
长度(m)	L	160
移动荷载(kN)	F	50

将式(61)和式(62)代入式(57)~式(60)可得,

$$a_1'=-\hat{\mu}a_1-\frac{\alpha_2}{8\omega_1}a_1^2a_3\sin\gamma_1+\frac{f}{2\omega_1}\sin\gamma_2 \tag{63}$$

$$a_3'=-\hat{\mu}a_3+\frac{\alpha_{10}}{8\omega_3}\hat{k}_3a_1^3\sin\gamma_1 \tag{64}$$

$$\gamma_1'=\sigma_1+(\frac{\alpha_{11}}{4\omega_3}-\frac{9\alpha_1}{8\omega_1}\hat{k}_3)a_1^2+(\frac{3\alpha_{14}}{8\omega_3}-\frac{3\alpha_4}{4\omega_1})a_3^2+(\frac{\alpha_{10}}{8\omega_3a_3}a_1^3-\frac{3\alpha_2}{8\omega_1}a_1a_3)\cos\gamma_1+\frac{3f}{2a_1\omega_1}\cos\gamma_2 \tag{65}$$

$$\gamma_2'=\sigma_2-\frac{3\alpha_1}{8\omega_1}a_1^2-\frac{\alpha_4}{4\omega_1}a_3^2-\frac{\alpha_2}{8\omega_1}a_1a_3\cos\gamma_1+\frac{f}{2a_1\omega_1}\cos\gamma_2 \tag{66}$$

以 $[a_1,a_3,\gamma_1,\gamma_2]^{\mathrm{T}}$ 为状态向量,通过线性化式(63)~式(66)可以得到 Jacobian 矩阵

$$\boldsymbol{J}=\begin{bmatrix}J_{11}&J_{12}&J_{13}&J_{14}\\J_{21}&J_{22}&J_{23}&J_{24}\\J_{31}&J_{32}&J_{33}&J_{34}\\J_{41}&J_{42}&J_{43}&J_{44}\end{bmatrix} \tag{67}$$

其中

$$J_{11}=-\hat{\mu}-\frac{\alpha_2}{4\omega_1}a_1a_3\sin\gamma_1,$$

$$J_{12}=-\frac{\alpha_2}{8\omega_1}a_1^2\sin\gamma_1,$$

$$J_{13}=-\frac{\alpha_2}{8\omega_1}a_1^2a_3\cos\gamma_1,$$

$$J_{14}=\frac{f}{2\omega_1}\cos\gamma_2,\ J_{21}=\frac{3\alpha_{10}}{8\omega_3}\hat{k}_3a_1\sin\gamma_1,$$

$$J_{22}=-\hat{\mu},\ J_{23}=\frac{\alpha_{10}}{8\omega_3}\hat{k}_3a_1^3\cos\gamma_1,\ J_{24}=0,$$

$$J_{31}=(\frac{\alpha_{11}}{2\omega_3}-\frac{9\alpha_1}{4\omega_1}\hat{k}_3)a_1+(\frac{2\alpha_{10}}{8\omega_3a_3}a_1^2-\frac{3\alpha_2}{8\omega_1}a_3)\cos\gamma_1-\frac{3f}{2a_1^2\omega_1}\cos\gamma_2,$$

$$J_{32}=(\frac{3\alpha_{14}}{4\omega_3}-\frac{3\alpha_4}{2\omega_1})a_3+(-\frac{\alpha_{10}}{8\omega_3a_3^2}a_1^3-\frac{3\alpha_2}{8\omega_1}a_1)\cos\gamma_1,$$

$$J_{33}=-(\frac{\alpha_{10}}{8\omega_3a_3}a_1^3-\frac{3\alpha_2}{8\omega_1}a_1a_3)\sin\gamma_1,$$

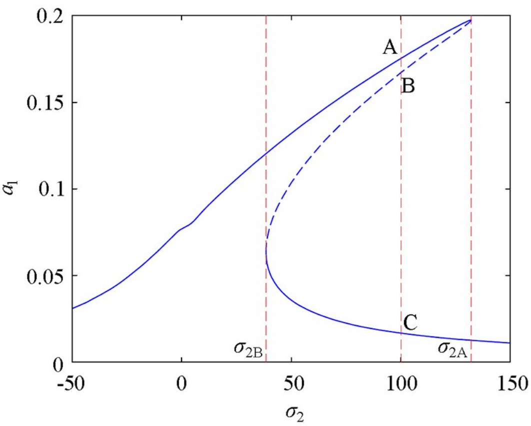
$$J_{34}=-\frac{3f}{2a_1\omega_1}\sin\gamma_2,$$

$$J_{41}=-\frac{3\alpha_1}{4\omega_1}a_1-\frac{\alpha_2}{8\omega_1}a_3\cos\gamma_1-\frac{f}{2a_1^2\omega_1}\cos\gamma_2,$$

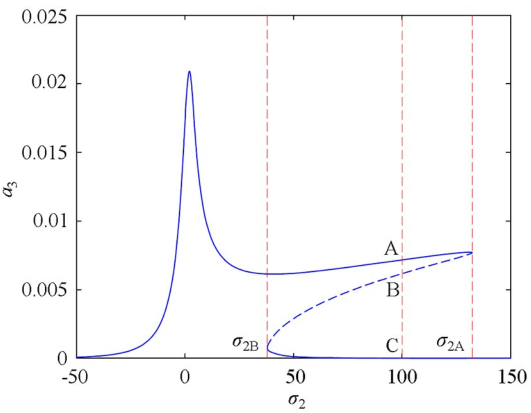
$$J_{42}=-\frac{\alpha_4}{2\omega_1}a_3-\frac{\alpha_2}{8\omega_1}a_1\cos\gamma_1,$$

$$J_{43}=\frac{\alpha_2}{8\omega_1}a_1a_3\sin\gamma_1,\ J_{44}=-\frac{f}{2a_1\omega_1}\sin\gamma_2$$

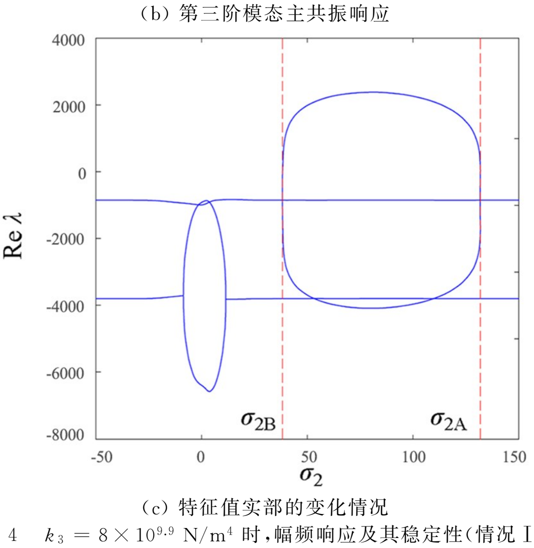
根据 Lyapunov 第一方法,可以判定:若矩阵 $\boldsymbol{J}$ 的特征值的实部全为负则系统稳定,反之系统不稳定.本文根据以上所述的方法,分别给出了在情况 I、情况 II 和情况 III 下当 $k_3=8\times10^{9.9}$ N/m^4 时的稳定性分析图,如图 4～图 6 所示.



(a) 第一阶模态主共振响应



(b) 第三阶模态主共振响应



(c) 特征值实部的变化情况

图 4　$k_3=8\times10^{9.9}$ N/m^4 时,幅频响应及其稳定性(情况 Ⅰ)

Fig.4　Amplitude-frequency response and its stability at $k_3=8\times10^{9.9}$ N/m^4 (Case Ⅰ)

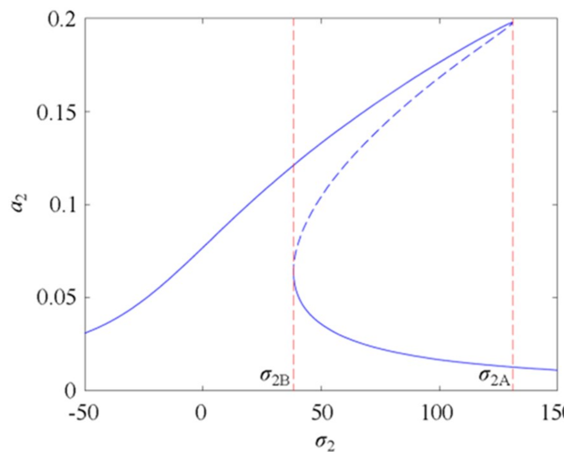
图 4(c)画出了情况 Ⅰ 下特征值实部的变化情况,可以看出在区间 $(\sigma_{2B},\sigma_{2A})$,即 $38.45<\sigma_2<132.15$ 时,系统存在正实特征值,则该部分存在不稳定解,其它部分为 1 个稳定解.与图 4(c)相对应的,图 4(a)和图 4(b)分别给出了第一阶模态和第三阶模态的幅频相应图.同时,结合表 2 可知由 A 和 C 点计算得到的 Jacobian 矩阵特征值都具有负实部,可以判定 A 和 C 点所在的曲线为稳定解用实线表示;由 B 点计算得到的 Jacobian 矩阵特征值实部中有两个具有正实部,可以判定 B 点所在的曲线为不稳定解用虚线表示.

表 2　图 4 标注各点所对应的 Jacobian 矩阵特征值

Table 2　Figure 4 shows the eigenvalues of Jacobian matrix corresponding to each point

	A	B	C
λ_1	−845.0+3030.2j	2269.20+0.00j	−842.8+9811.0j
λ_2	−845.0−3030.2j	−3960.50+0.00j	−842.8−9811.0j
λ_3	−3790.5+28045.1j	−3790.39+28201.7j	−3792.7+29835.7j
λ_4	−3790.5−28045.1j	−3790.39−28201.7j	−3792.7−29835.7j

运用同样的方法分别判断了系统在情况 Ⅱ 和情况 Ⅲ 下的稳定性.从图 5(a)和图 5(b)可以看出在区间 $(\sigma_{2B},\sigma_{2A})$,即 $38.16<\sigma_2<131.67$ 时,系统存在 2 个稳定解和 1 个不稳定解,其它部分为稳定解.从图 6(a)和图 6(b)可以看出在区间 $(\sigma_{2A},\sigma_{2C})$ 和 $(\sigma_{2F},\sigma_{2B})$,系统存在 2 个稳定解和 1 个不稳定解;在区间 $(\sigma_{2C},\sigma_{2E})$,系统存在 3 个稳定解和 2 个不稳定解;在区间 $(\sigma_{2E},\sigma_{2D})$,系统存在 3 个稳定解和 4 个不稳定解;在区间 $(\sigma_{2D},\sigma_{2F})$,系统存在 2 个稳定解和 3 个不稳定解,其它区间只存在 1 个稳定解.

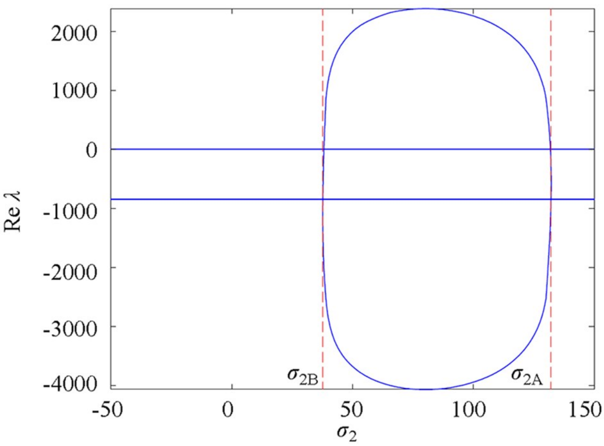


(a) 第二阶模态主共振响应

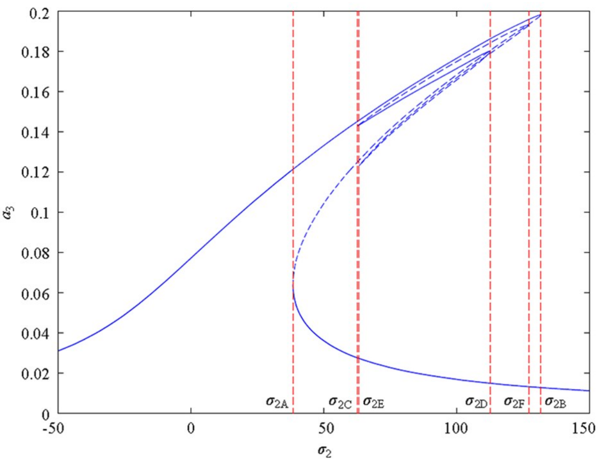
(b) 第二阶模态特征值实部的变化情况

图 5　$k_3=8\times10^{9.9}$ N/m^4 时,幅频响应及其稳定性(情况 Ⅱ)

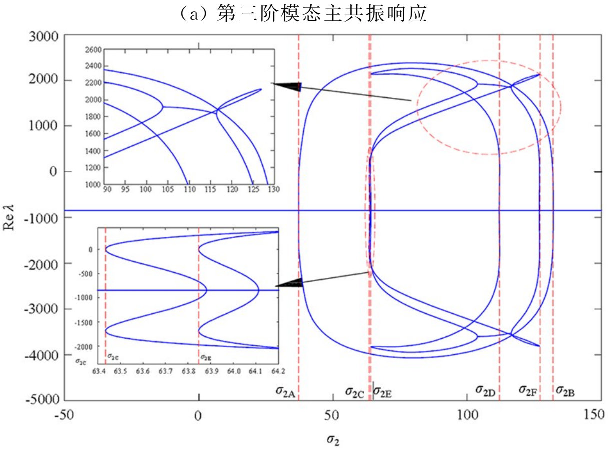
Fig.5　Amplitude-frequency response and its stability at



$k_3 = 8 \times 10^{9.9}$ N/m^4 (Case Ⅱ)



(a) 第三阶模态主共振响应



(b) 第三阶模态特征值实部的变化情况

图 6 $k_3 = 8 \times 10^{9.9}$ N/m^4 时,幅频响应及其稳定性(情况 Ⅲ)

Fig.6 Amplitude-frequency response and its stability at $k_3 = 8 \times 10^{9.9}$ N/m^4 (Case Ⅲ)

3 结论

研究了在移动荷载作用下非线性粘弹性基础梁的振动问题.采用 Galerkin 方法对非线性偏微分运动方程进行离散,并利用多尺度法求解了非线性运动控制方程.在此基础上,研究了非线性基础梁在 3:1 内共振下的自由振动,以及在内共振作用下梁受外激励作用的第一、第二和第三阶模态主共振的非线性动力学行为及其动力学响应的稳定性问题.研究揭示了基础参数对内共振下弹性梁振动响应的影响规律,并发现在发生第三阶主共振时,幅频响应中会出现 7 个解支,包括 3 个稳定解支和 4 个不稳定解支.

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