

单向耦合 FitzHugh-Nagumo 神经元的滞后同步研究^{*}

袁韦欣¹ 镇斌^{1†} 徐鉴²

(1.上海理工大学环境与建筑学院,上海 200093)

(2.同济大学航空航天与力学学院,上海 200092)

摘要 本文提出一种新的研究思路,将两个单向耦合的 FitzHugh-Nagumo 神经元之间的滞后同步视为一种特殊的广义同步,通过辅助系统方法来获得滞后同步发生的条件.首先建立原响应系统的辅助系统,将原系统中驱动系统和响应系统的滞后同步问题转换为响应系统与辅助系统之间的误差系统原点稳定问题.通过拉普拉斯变换方法,将原本用微分方程表示的误差系统等效为 Volterra 积分方程表示,而后利用积分方程理论中的逐次逼近方法解析地得到原误差系统原点稳定性条件.本文给出的滞后同步判据与信号传送中的时间滞后量大小无关,数值模拟验证了本文提出判据的有效性.

关键词 FitzHugh-Nagumo 神经元, 时滞, 滞后同步, 广义同步, 单向耦合

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The Study for Lag Synchronization between Two Fitzhugh-Nagumo Neurons with Unidirectional Coupling^{*}

Yuan Weixin¹ Zhen Bin^{1†} Xu Jian²

(1. School of Environment and Architecture, University of Shanghai for Science and Technology, Shanghai 200093, China)

(2. School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai 200092, China)

Abstract In this paper, a new method is proposed based on the auxiliary system approach to investigate lag synchronization between two identical FitzHugh-Nagumo (FHN) neurons with unidirectional coupling. The lag synchronization can be achieved if the synchronization error system between the response and auxiliary systems has an asymptotically stable origin. Different from other studies, the synchronization error system is converted into a set of Volterra integral equations according to the Laplace transform. By using the successive approximation method in the theory of integral equations, an analytical criterion for the detection of lag synchronization between two FHN neurons is obtained. The value of the time delay in transmission has no relation to the lag synchronization condition, which has been verified by numerical simulations for the original system.

Key words FitzHugh-Nagumo neuron, time delay, lag synchronization, generalized synchronization, unidirectional coupling

引言

耦合神经元之间的同步现象一直是脑科学、医学、人工智能等领域学者持续关注的热点.单个神经元细胞并不具有智能表现,神经元之间的同步则与大脑的认知和行为具有相关性^[1-4],而进一步通过研究神经元之间的同步来理解大脑中的信息处理也被证实是可行的^[5-9].Hodgkin-Huxley(HH)方程^[10]以及它的简化版:FitzHugh-Nagumo(FHN)模型^[11],Hindmarsh-Rose(HR)模型^[12]常用于从神经动力学研究的层面上构建神经网络.尤其是FHN模型,其方程形式简单,更适于从非线性动力学的角度研究神经元之间的同步行为.需要注意的是,神经元之间的信息传递并非瞬时完成,通过突触传递的神经信号都会存在不同程度的滞后现象,在某些情况下这种滞后量甚至可以达到系统发放周期的量级^[13].因此,在讨论耦合神经元同步行为时,信号传递的时滞因素影响不能轻易忽视.

一般而言,当一个神经元向另一个神经元发送信号时,两个神经元如果在某些特定条件下保持一致就意味着这两个神经元处于同步状态.目前已经发现神经元之间可以存在多种同步状态,如完全同步(complete synchronization)、滞后同步(lag synchronization),广义同步(generalized synchronization)等^[14,15].在以往的FHN神经元系统同步研究中,上述几种同步通常被单独加以讨论.对于完全同步和滞后同步,通常采用Lyapunov函数方法^[16-18]来获取神经元的同步发生条件.而对于广义同步,则常采用构造辅助系统^[19]方法来寻找同步条件.事实上,广义同步更具有一般性,完全同步和滞后同步都可以看作是一种特殊的广义同步.将滞后同步视为广义同步加以研究可能是一种新的研究思路.

本文讨论单向耦合的FHN神经元之间的滞后同步.不同于以往研究,本文将滞后同步视为一种特殊的广义同步,采用构造辅助系统的方法来讨论耦合神经元滞后同步的发生条件.首先,基于拉普拉斯变换,将辅助系统和响应系统之间的误差系统转化为Volterra积分方程.再根据Volterra积分方程理论中的逐次逼近方法(successive approximation method)^[20]解析地得到了两个单向耦合FHN神经元的滞后同步发生条件.最后,通过数值

模拟验证了本文解析判据的有效性.

1 FHN神经元模型

本文采用的FHN神经元模型如下^[11]:

$$\begin{aligned}\dot{v} &= v - v^3/3 - \omega + I_{ext} \\ \dot{\omega} &= \gamma(v + a - b\omega)\end{aligned}\quad (1)$$

其中 ω 分别表示膜电位和恢复量. $a > 0, 0 < b < 1$ 且 $0 < \gamma \ll 1, I_{ext}$ 是外部输入电流.为方便起见,将系统(1)的平衡点移到原点,方程(1)变为

$$\begin{aligned}\dot{u}_1 &= -u_1^3/3 + cu_1^2 + du_1 - u_2 \\ \dot{u}_2 &= \gamma(u_1 - bu_2)\end{aligned}\quad (2)$$

其中 $c = -v_0, d = 1 - v_0^2, v_0$ 是方程 $-v_0^3/3 + (1 - 1/b)v_0 - a/b + I_{ext} = 0$ 的唯一实根.考虑如下两个单向耦合的FHN神经元:

$$\begin{aligned}\dot{u}_1 &= -u_1^3/3 + cu_1^2 + du_1 - u_2 \\ \dot{u}_2 &= \gamma(u_1 - bu_2) \\ \dot{u}_3 &= -u_3^3/3 + cu_3^2 + du_3 - u_4 + \\ &\quad \alpha_1[u_3 - u_1(t - \tau)] \\ \dot{u}_4 &= \gamma(u_3 - bu_4) + \alpha_2[u_4 - u_2(t - \tau)]\end{aligned}\quad (3)$$

其中 $\alpha_{1,2}$ 是耦合强度, $\tau > 0$ 表示从 $u_1(u_2)$ 到 $u_3(u_4)$ 的信号传输延迟.如果在系统(3)中满足

$$\begin{aligned}\lim_{t \rightarrow \infty} \|u_3 - u_1(t - \tau)\| &= 0 \\ \lim_{t \rightarrow \infty} \|u_4 - u_2(t - \tau)\| &= 0\end{aligned}\quad (4)$$

则两个FHN神经元之间存在滞后同步.

2 广义同步观点下的滞后同步

在系统(3)中, t 时刻 $\dot{u}_{1,2,3,4}$ 的值取决于区间 $[t - \tau, t]$ 各个时刻的 $u_{1,2,3,4}$ 值.因此,在区间 $(0, \tau)$ 中, $\dot{u}_{1,2,3,4}$ 与区间 $(-\tau, 0)$ 中的 $u_{1,2,3,4}$ 具有相同的光滑度.如果 $u_{1,2,3,4}$ 在 $t \in (-\tau, 0)$ 内是不连续的,那么在 $t \in (0, \tau)$ 内 $\dot{u}_{1,2,3,4}$ 是不连续的,但 $u_{1,2,3,4}$ 连续的.随着时间的推移,系统(3)的解变得越来越光滑.所以可以假定解 $u_{1,2,3,4}$ 是足够光滑的.假定 $u_{1,2}$ 与某函数 $F_{1,2}$ 的关系如下:

$$\begin{aligned}F_1(t, \tau) &= u_1(t - \tau) \\ F_2(t, \tau) &= u_2(t - \tau)\end{aligned}\quad (5)$$

$F_{1,2}$ 可以由系统(3)确定.任何函数都可以在有限区间上表示为叠加在直线上的傅里叶正弦级数,如图1所示^[21].所以函数可以近似表示为:

$$F_1(t, \tau) = p_0(t) + p_1(t)\tau +$$

$$F_2(t, \tau) = q_0(t) + q_1(t)\tau + \sum_{k=1}^{N-2} q_{k+1}(t) \sin(k\pi\tau) \quad (6)$$

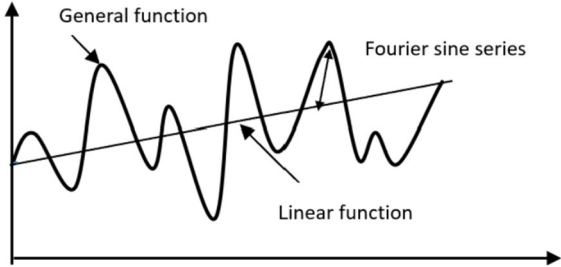


图 1 在有限区间上,任意函数=线性函数+傅里叶正弦级数
Fig.1 On a finite interval, an arbitrary function=a linear function + a Fourier sine series

其中 N 根据精度需要确定.显然,

$$\begin{aligned} u_1 &\equiv F_1(t, 0) = p_0(t) \\ u_2 &\equiv F_2(t, 0) = q_0(t) \end{aligned} \quad (7)$$

因此,系统(3)中的滞后同步可以被视为一种特殊的广义同步,即:

$$\begin{aligned} \lim_{t \rightarrow \infty} \|u_3 - F_1(t, \tau)\| &= 0 \\ \lim_{t \rightarrow \infty} \|u_4 - F_2(t, \tau)\| &= 0 \end{aligned} \quad (8)$$

3 基于辅助系统方法的同步条件研究

基于辅助系统方法^[19],系统(3)的辅助系统定义如下:

$$\begin{aligned} \dot{u}_5 &= -u_5^3/3 + cu_5^2 + du_5 - u_6 + \alpha_1[u_5 - u_1(t - \tau)] \\ \dot{u}_6 &= \gamma(u_5 - bu_6) + \alpha_2[u_6 - u_2(t - \tau)] \end{aligned} \quad (9)$$

广义同步关系(8)可以在系统(3)中实现,只要如下关系成立:

$$\lim_{t \rightarrow \infty} \|u_3 - u_5\| = 0, \lim_{t \rightarrow \infty} \|u_4 - u_6\| = 0 \quad (10)$$

令

$$\begin{aligned} e_1 &= \frac{u_3 - u_5}{2}, e_2 = \frac{u_4 - u_6}{2} \\ e_3 &= \frac{u_3 + u_5}{2}, e_4 = \frac{u_4 + u_6}{2} \end{aligned}$$

根据系统(3)和(9), $e_{1,2,3,4}$ 的控制方程可以写成:

$$\begin{aligned} \dot{e}_1 &= (d + \alpha_1)e_1 - e_2 + \omega_1 \\ \dot{e}_2 &= \gamma e_1 + (\alpha_2 - b\gamma)e_2 \\ \dot{e}_3 &= (d + \alpha_1)e_3 - e_4 - \alpha_1 F_1(t, \tau) + \omega_2 \\ \dot{e}_4 &= \gamma e_3 + (\alpha_2 - b\gamma)e_4 - \alpha_2 F_2(t, \tau) \end{aligned} \quad (11)$$

其中

$$\begin{aligned} \omega_1 &= -e_1^3/3 + (2ce_3 - e_3^2)e_1 \\ \omega_2 &= -e_3^3/3 + c(e_1^2 + e_3^2) - e_1^2 e_3 \end{aligned}$$

条件(10)等效为

$$\lim_{t \rightarrow \infty} \|e_{1,2}\| = 0 \quad (12)$$

考虑如下拉普拉斯变换:

$$\begin{aligned} \dot{e}_i(s) &= L[e_i](s) = \int_0^{+\infty} e_i(t) e^{-st} dt \\ e_i(t) &= L^{-1}[\dot{e}_i](t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \dot{e}_i(s) e^{st} ds, i=1, 2, 3, 4 \end{aligned} \quad (13)$$

对系统(11)中等式两边取拉普拉斯变换,有

$$\begin{aligned} (s - d - \alpha_1) \dot{e}_1 + \dot{e}_2 - e_{10} - \dot{\omega}_1 &= 0 \\ \gamma \dot{e}_1 - (s - \alpha_2 + b\gamma) \dot{e}_2 + e_{20} &= 0 \\ (s - d - \alpha_1) \dot{e}_3 + \dot{e}_4 - e_{30} + \alpha_1 \dot{F}_1(s, \tau) - \dot{\omega}_2 &= 0 \\ \gamma \dot{e}_3 - (s + b\gamma - \alpha_2) \dot{e}_4 + e_{40} - \alpha_2 \dot{F}_2(s, \tau) &= 0 \end{aligned} \quad (14)$$

其中, $e_{i0}, i=1, 2, 3, 4$ 是系统(11)中给定的初值. 另外

$$\begin{aligned} \dot{\omega}_i(s) &= \int_0^{+\infty} \omega_i(t) e^{-st} dt, i=1, 2, \\ \dot{F}_j(s, \tau) &= \int_0^{+\infty} F_j(t, \tau) e^{-st} dt, j=1, 2. \end{aligned}$$

求解系统(14)中的前两个方程,有

$$\begin{aligned} \dot{e}_1 &= \frac{(s + \xi_3)e_{10} - e_{20}}{s^2 + \xi_1 s + \xi_2} + \frac{(s + \xi_3) \dot{\omega}_1}{s^2 + \xi_1 s + \xi_2} \\ \dot{e}_2 &= \frac{(s + \xi_4)e_{20} + \gamma e_{10}}{s^2 + \xi_1 s + \xi_2} + \frac{\gamma \dot{\omega}_1}{s^2 + \xi_1 s + \xi_2} \end{aligned} \quad (15)$$

其中

$$\begin{aligned} \xi_1 &= b\gamma - d - \alpha_1 - \alpha_2 \\ \xi_2 &= [1 - b(d + \alpha_1)]\gamma + \alpha_2(d + \alpha_1) \\ \xi_3 &= b\gamma - \alpha_2 \\ \xi_4 &= -d - \alpha_1 \end{aligned}$$

对(15)中的等式两端取拉普拉斯逆变换,利用卷积定理,有

$$\begin{aligned} e_1 &= e_{10} \eta_1(t) - e_{20} \eta_3(t) + \int_0^t \eta_1(t - \tau) \omega_1 d\tau \\ e_2 &= e_{20} \eta_2(t) + \gamma e_{10} \eta_3(t) + \gamma \int_0^t \eta_3(t - \tau) \omega_1 d\tau \end{aligned} \quad (16)$$

其中

$$\eta_1(t) = L^{-1} \left[\frac{s + \xi_3}{s^2 + \xi_1 s + \xi_2} \right] =$$

$$\frac{1}{s_2 - s_1} [(\xi_3 - s_1)e^{-s_1 t} - (\xi_3 - s_2)e^{-s_2 t}]$$

$$\eta_2(t) = L^{-1} \left[\frac{s + \xi_4}{s^2 + \xi_1 s + \xi_2} \right] =$$

$$\frac{1}{s_2 - s_1} [(\xi_4 - s_1)e^{-s_1 t} - (\xi_4 - s_2)e^{-s_2 t}]$$

$$\eta_3(t) = L^{-1} \left[\frac{1}{s^2 + \xi_1 s + \xi_2} \right] =$$

$$\frac{1}{s_2 - s_1} (e^{-s_1 t} - e^{-s_2 t})$$

s_1, s_2 是方程 $s^2 + \xi_1 s + \xi_2 = 0$ 的两个根. 显然, $\eta_{1,2,3} \rightarrow 0$ 是条件(12)成立的必要条件. 根据 Routh-Hurwitz 判据, 要保证 $\eta_{1,2,3} \rightarrow 0$ 成立应满足:

$$\xi_1 > 0, \xi_2 > 0 \quad (17)$$

在上述条件下, 当公式(16)变为

$$e_1 = \int_0^t \eta_1(t-\epsilon) \omega_1 d\epsilon, e_2 = \gamma \int_0^t \eta_3(t-\epsilon) \omega_1 d\epsilon \quad (18)$$

方程(18)是一个 Volterra 积分方程, 可使用逐次逼近法^[20]求解. 根据该方法, $e_{1,2} = 0$ 分别是(18)

中两个方程的唯一解. 这意味着当条件(17)满足的时候滞后同步会在系统(3)中发生.

4 数值验证

对系统(3)和(9)进行数值模拟验证条件(17)的有效性. 取 $a = 0.6, b = 0.7, \gamma = 0.03, I_{ext} = 0.6$ 和 $\alpha_1 = -0.8$. 根据条件(17), 系统(3)中会出现滞后同步, 只要下列关系成立

$$\alpha_2 < b\gamma - d - \alpha_1 = 0.0728$$

$$\alpha_2 < [b(d + \alpha_1) - 1]\gamma / (d + \alpha_1) = 0.6004$$

注意到条件(17)不包含 τ , 这意味着对任意时滞量同步条件 $\alpha_2 < 0.0728$ 都有效. 在系统(3)和(9)中取初始条件, $u_1(t) = 0.1, u_2(t) = 0.2, t \in (-\tau, 0], (u_3(0), u_4(0)) = (0.12, 0.21), (u_5(0), u_6(0)) = (0.15, 0.22)$. 图2中给出了不同 α_2 和 τ 值时响应系统和辅助系统之间的同步误差 $e_1 = u_3 - u_5, e_2 = u_2 - u_4$ 的变化趋势, 以及 $u_{1,3}$ 和 $u_{2,4}$ 的时程图.

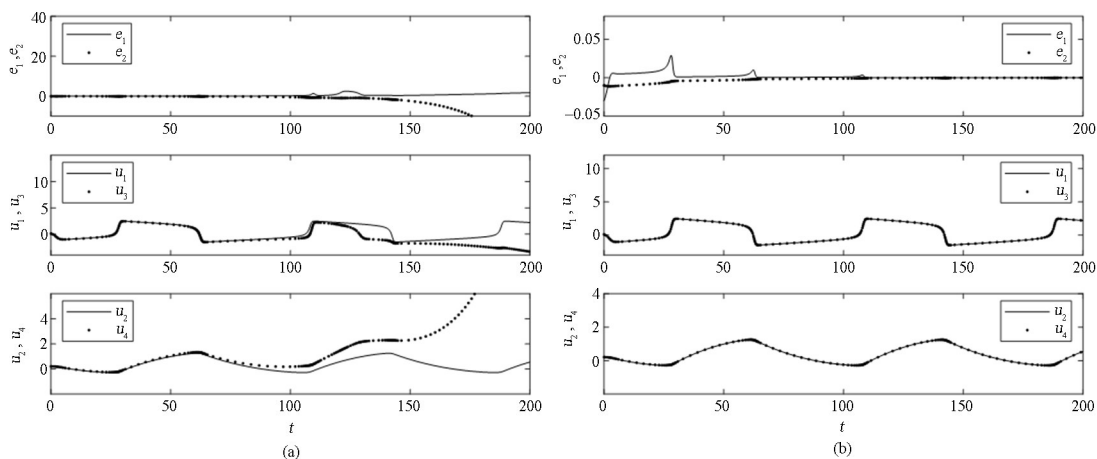


图2 系统(3)和(9)中不同 α_2 和 τ 值时同步误差 e_1, e_2 的趋势变化图, 及 $u_{1,3}$ 和 $u_{2,4}$ 的时程图,

其中(a) $\alpha_2 = 0.08, \tau = 0$ (b) $\alpha_2 = 0.02, \tau = 0$ (c) $\alpha_2 = 0.02, \tau = 5$ (d) $\alpha_2 = 0.02, \tau = 20$

Fig.2 Synchronization errors e_1, e_2 and functional relations between u_1 and u_3 , and between u_2 and u_4 in systems (3) and (9) with (a) $\alpha_2 = 0.08, \tau = 0$ (b) $\alpha_2 = 0.02, \tau = 0$ (c) $\alpha_2 = 0.02, \tau = 5$ (d) $\alpha_2 = 0.02, \tau = 20$

由图2可知, 条件(17)与数值结果非常吻合, 由此证明了条件(17)的有效性.

5 结论

本文研究了两个单向耦合 FHN 神经元的滞后同步问题. 与以往采用的方法不同, 本文将神经元之间的滞后同步视为一种特殊的广义同步, 令构造出的辅助系统与响应系统完全同步来获得原系统的滞后同步条件. 通过拉普拉斯变换方法可以将辅助系统与响应系统之间的误差系统由微分方程

表示转化为 Volterra 积分方程表示, 再利用积分方程理论中的逐次逼近方法得到误差系统原点稳定的解析条件. 该解析条件与滞后同步中的滞后量大小无关, 直接数值模拟验证了本文同步判据的有效性. 本文为研究神经元的滞后同步提供了一种新的思路.

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