

# 工程结构多刚度尺度分析与模态理论<sup>\*</sup>

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**摘要** 当前工程结构设计大多基于成熟的线性振动理论, 未综合考虑非线性、多尺度等特性的影响, 导致各类工程振动问题频发, 减振措施失效。首先, 本文以质量弹簧系统为例, 对系统刚度比和质量比等关键参数开展分析, 指出刚度比对系统模态具有显著影响; 再者, 简述大跨桥梁动力学研究现状, 从系统全局动力学角度, 根据非线性动力学分析和有限元分析, 提出工程结构多刚度尺度概念, 分析并指出多刚度尺度耦合系统的全局模态、局部模态和混合模态基于不同刚度尺度的定义。为建立桥梁全局动力学模型和理论, 桥梁非线性动力学研究奠定基础。

**关键词** 工程结构, 多刚度尺度, 全局模态, 局部模态, 混合模态

**中图分类号**: TU311.3; TU333

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## Multiple Scale Analysis and Modal Theory of Engineering Structures<sup>\*</sup>

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**Abstract** At present, the design of engineering structures is generally based on the mature linear vibration theory, without considering the influence of nonlinear and multi-scale characteristics, which leads to the frequent occurrence of various vibration problems and the failure of vibration reduction measures. Firstly, this paper takes the mass-spring system as an example, and carries out the parametric influence of the stiffness ratio and mass ratio on the system vibration, of which the result points out that the stiffness ratio has a significant influence on the system modal property. In addition, the research overview on the long-span bridge dynamics is briefly discussed. From the perspective of system global dynamics, the concept of a multi-stiffness scale of engineering structure is proposed based on nonlinear dynamics analysis and finite element analysis. The global modes, local modes and mixed modes of multi-stiffness scale coupling systems based on different stiffness scales are analyzed and pointed out. The research may lay the foundation for establishing the global dynamic model and theory of bridge and studying the nonlinear dynamics of the bridge.

**Key words** engineering structure, multi-stiffness scale, global mode, local mode, hybrid mode

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## 引言

近几十年来,大跨桥梁<sup>[1]</sup>、深海隧道<sup>[2]</sup>、海洋平台<sup>[3]</sup>等各类工程结构不断涌现,人们创造了举世瞩目的工程成就,但频繁发生的各类振动问题也一直是影响结构安全、成为阻碍工程结构实现突破的瓶颈。2020年虎门大桥<sup>[4]</sup>、西堠门大桥,2021年深圳赛格大厦<sup>[5,6]</sup>相继发生振动,引起社会各界广泛关注,并引发人们对工程结构安全性的担忧。

需要指出的是,一方面,当前各类工程结构几乎均基于成熟的线性振动理论设计,而高层/超高层建筑、大跨桥梁等细长工程结构的低阶甚至高阶非线性作用突出,因此存在局限性;另一方面,结构多尺度特性突出,包括工程结构本身的多空间、多刚度尺度耦合特性,振动响应的多时间尺度特性等,导致复杂的非线性振动,如内共振、宽频振动等,为后续振动抑制和控制带来困难。开展非线性振动研究,探明耦合、几何非线性等作用导致的结构复杂动力学机理,推动各类新结构非线性振动研究,具有重要理论意义。

多时间尺度法(Multiple time scales method)通过引入快慢不同的时间尺度,得到系统的慢变动力学,求得关于时间的慢变振幅和相位,被广泛应用于求解索结构<sup>[7-9]</sup>、梁结构<sup>[10-12]</sup>、索梁组合结构<sup>[13-23]</sup>、索浅拱组合结构<sup>[24-26]</sup>等面内非线性动力学问题。

考虑空间尺度,Srinil等<sup>[27]</sup>建立有垂度斜拉索的三维自由振动理论,对影响面内和面外自由振动特性的力学参数开展详细分析。考虑拉索空间运动,康厚军等<sup>[28]</sup>建立斜拉桥的多索-浅拱空间运动模型,探究索、浅拱面内外共振非线性动力学响应。

本文将以质量弹簧系统、悬索桥、斜拉桥和拱桥动力学模型为例,从系统全局动力学角度,根据非线性动力学分析和有限元分析,基于课题组前期在桥梁线性和非线性动力学方面的研究,提出工程结构多刚度尺度概念,分析并指出多刚度尺度耦合

系统的全局模态、局部模态和混合模态基于不同刚度尺度的定义,为建立桥梁全局动力学模型和理论,桥梁非线性动力学等动力学与控制研究奠定基础。同时为开展复杂工程结构的非线性振动问题提供新的思路。

## 1 多刚度尺度

### 1.1 质量弹簧系统

图1所示为两质点弹簧系统,说明多刚度尺度系统的全局和局部模态振动特性,其中 $k_1$ 、 $k_2$ 分别表示两弹簧刚度, $m_1$ 、 $m_2$ 分别表示两质点质量。质点 $m_1$ 和弹簧 $k_1$ 与地面基础直接相连,可理解为工程结构的主体部分,为大尺度刚度,可独立存在;而质点 $m_2$ 和弹簧 $k_2$ 代表工程结构的附属部分,为小尺度刚度,不能单独受力,需依附于主体部分。

根据牛顿第二定律,易得上述两质点弹簧系统的无阻尼自由振动方程,写成矩阵形式如下:

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{pmatrix} + \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0 \quad (1)$$

其中, $x_1$ 、 $x_2$ 分别表示质点 $m_1$ 、 $m_2$ 在水平向的位移。假设方程具有简谐解

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \sin(\omega t + \varphi) \quad (2)$$

将公式(2)代入运动方程(1),得到

$$\begin{pmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0 \quad (3)$$

其中, $A_1$ 、 $A_2$ 分别为两质点的振动幅值。为求解其特征值问题,令系数矩阵行列式为零,得到系统的频率方程如下:

$$k_1 k_2 - k_2 m_1 \omega^2 - k_1 m_2 \omega^2 - k_2 m_2 \omega^2 + m_1 m_2 \omega^4 = 0 \quad (4)$$

上述方程(4)的两个根即为系统的两阶频率 $\omega_1$ 、 $\omega_2$  ( $\omega_1 < \omega_2$ ),得到

$$\omega_{1,2}^2 = \frac{k_2 m_1 + k_1 m_2 + k_2 m_2 \pm \sqrt{(k_2 m_1 + k_1 m_2 + k_2 m_2)^2 - 4k_1 k_2 m_1 m_2}}{2m_1 m_2} \quad (5)$$

为探究系统刚度比 $\gamma_k$ 和质量比 $\gamma_m$ 对系统固有振动特性的影响,定义如下的关系:

$$\gamma_k = \frac{k_2}{k_1}, \quad \gamma_m = \frac{m_2}{m_1} \quad (6)$$

将求得各阶频率 $\omega_1$ 和 $\omega_2$ ,代入方程(3),得

到对应的第一阶振型幅值比

$$\frac{A_1}{A_2} = \frac{2\gamma_k \gamma_m}{\gamma_m - \gamma_k + \gamma_k \gamma_m + \sqrt{(\gamma_k + \gamma_m + \gamma_k \gamma_m)^2 - 4\gamma_k \gamma_m}} \quad (7)$$

以及第二阶振型幅值比

$$\frac{A_1}{A_2} = \frac{2\gamma_k \gamma_m}{\gamma_m - \gamma_k + \gamma_k \gamma_m - \sqrt{(\gamma_k + \gamma_m + \gamma_k \gamma_m)^2 - 4\gamma_k \gamma_m}} \quad (8)$$

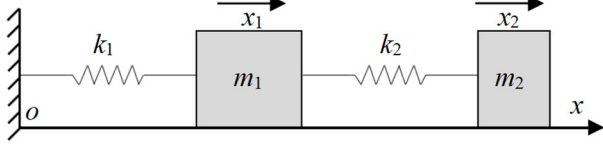


图 1 两质量弹簧系统  
Fig.1 Two mass spring system

### 1.2 参数分析

图 2 给出两质点质量相同时,即  $\gamma_m = 1$ , 系统振型受弹簧刚度比的影响.可以看到,当刚度比  $\gamma_k$  逐渐减小接近零时,即  $\gamma_k \ll 1$ , 两弹簧刚度相差较大,即结构主体部分具有大尺度刚度  $k_1$ , 附属部分具有小尺度刚度  $k_2$ , 系统多刚度尺度特性明显.此时系统第一阶振型振幅比  $A_1/A_2$  很小,表现为以质点  $m_2$  为主的振动,而质点  $m_1$  几乎不振动,这种以结构中小刚度构件振动为主导的振动模式,我们称为局部模态.系统第二阶振型振幅比  $A_1/A_2$  很大,表现为以质点  $m_1$  为主的振动,这种以结构中大刚度构件振动为主导的振动模式,我们称为全局模态.结构多刚度尺度特性越显著,局部模态和全局模态越易区分.此外,当  $\gamma_k > 1$ , 即  $k_2 > k_1$  时,结构多刚度尺度特性不显著,此时两阶振型的振幅比趋近 1 和 -1, 表现为两构件振幅相当的振动,称为混合模态.

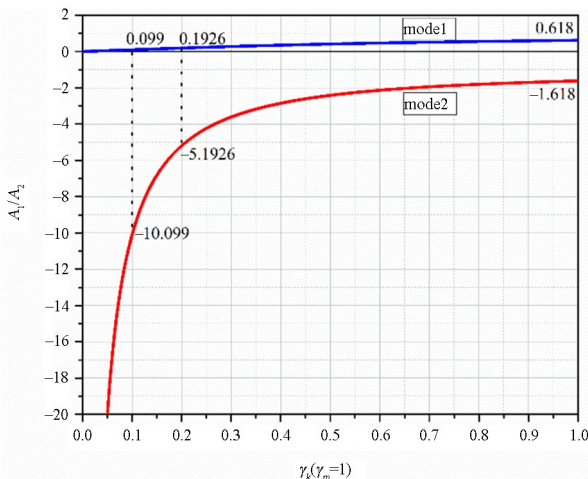


图 2 刚度比对系统振型的影响  
Fig.2 Influence of stiffness ratio on system mode

与质点刚度相比,质量比对系统振型影响不

大.图 3 给出系统振型随质量比和刚度比变化的影响.从图中可看出,系统质量比仅对第二阶振型有影响,而对第一阶振型几乎不产生影响,且仅当  $\gamma_k$  较小时,系统表现出显著的局部模态和全局模态,如图 4 所示,这是由于此时系统具有明显的多刚度尺度特性.

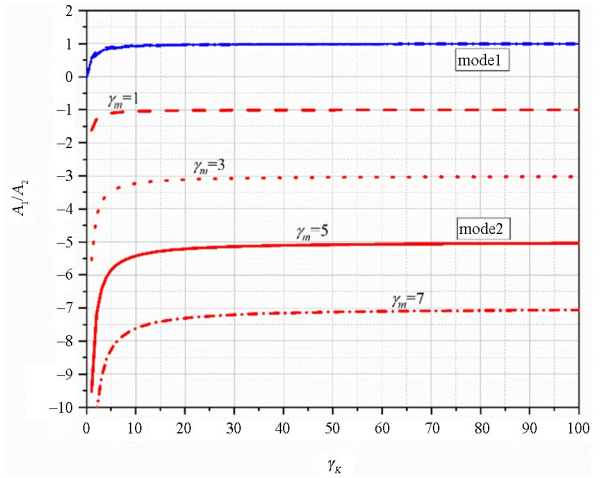


图 3 质量比对系统振型的影响  
Fig.3 Influence of mass ratio on system mode

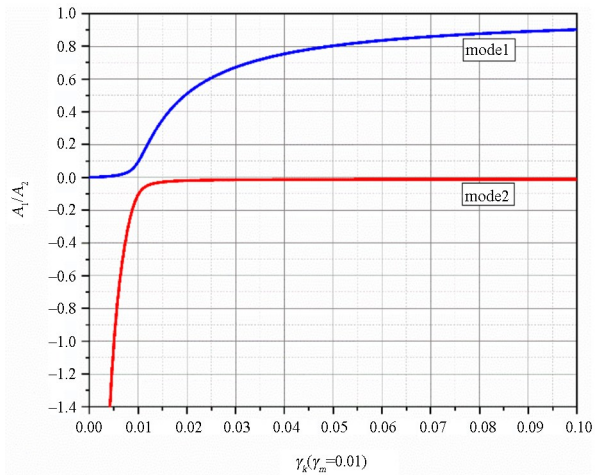


图 4 质量比  $\gamma_m = 0.01$  时,系统振型随刚度比的变化  
Fig.4 Variation of system mode with stiffness ratio when mass ratio  $\gamma_m = 0.01$

## 2 工程实例

### 2.1 悬索桥

随着悬索桥跨径不断增大,其主缆、吊杆和桥面梁长度相应增加,在风雨、地震、车辆等荷载作用下易发生大幅振动.经有限元建模计算,某悬索桥吊杆横向弯曲刚度约  $10^5 \text{ N} \cdot \text{m}^2$  量级,主缆和桥面梁的横向弯曲刚度约  $10^{10} \text{ N} \cdot \text{m}^2$  量级,而桥塔刚度更大,一般不发生振动.因此悬索桥也是具有显



著多刚度尺度的系统,其空间构型如图5所示.由于对悬索桥全局动力学振动本质缺乏足够认识,导致施加如抗风索、调液阻尼器、液压阻尼器等减振措施后仍难以控制振动<sup>[29]</sup>.回忆等和徐亮等<sup>[30, 31]</sup>对建立的悬索桥全桥模型开展模态分析,分析指出主梁和主缆的抗弯刚度对悬索桥模态特性具有重要影响,并且分别得到主缆和主梁主导的模态.此外,悬索桥吊杆通常具有较大的轴向刚度,影响全桥面内抗弯刚度.王志骞和康厚军等<sup>[32]</sup>建立悬索桥的带有离散弹簧的索梁模型,利用传递矩阵方法研究其面内固有振动特性,分析指出离散弹簧、主缆和主梁刚度影响桥梁主缆和主梁的振动特性,揭示出可能发生的全局振动特性和局部振动特性.

对于悬索桥,悬索和桥面梁是受力主体,且为大尺度刚度构件,将以悬索和桥面梁振动为主的振型称为全局模态,如图6(b)所示;吊杆是附属构件,且其横向刚度较小,为小尺度刚度构件,将以吊杆横向振动为主的振型称为局部模态,如图6(a)所示.



图5 某双塔悬索桥

Fig.5 A suspension bridge with two towers

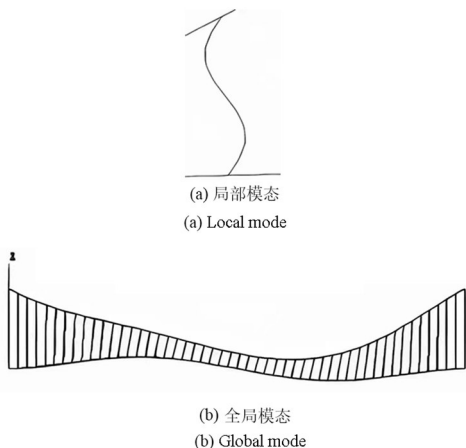


图6 有限元得到的悬索桥模态

Fig.6 Mode of suspension bridge obtained by finite element model

## 2.2 斜拉桥

斜拉桥,如图7所示,因其造型优美、受力合理等优点在土木工程中得到广泛应用.随着跨径的不断增大,结构整体刚度降低,阻尼减小,斜拉索、桥面梁振动问题突出,吸引了众多学者展开相关研究.一方面,斜拉索因其自重轻,易在风雨等外荷载下发生大幅振动,继而引起疲劳甚至破坏;另一方面,受斜拉桥主梁和桥塔振动的影响,振动能量在耦合处向拉索传递,导致拉索发生大幅振动,并可引起连接件的疲劳断裂.Irvine等<sup>[33]</sup>,Benedettini等和Rega等<sup>[34, 35]</sup>,Srinil最早开始建立悬索的线性和非线性振动理论.考虑边界耦合作用,郭铁丁等<sup>[36, 37]</sup>运用边界共振调谐技术建立拉索的边界耦合非线性动力学理论.彭剑等<sup>[38]</sup>、苏潇阳等<sup>[39]</sup>分别建立斜拉索-磁流变阻尼器(MR)和索-调谐质量阻尼器(TMD)模型,开展斜拉索的非线性振动抑制研究.



图7 某多塔斜拉桥

Fig.7 A cable-stayed bridge with multiple towers

此外,Fujino等<sup>[14]</sup>、Gattulli等<sup>[15, 16]</sup>、康厚军等<sup>[17, 18]</sup>等建立斜拉桥施工阶段的斜拉索-梁模型,探究其线性自由振动特性以及系统模态内共振与索梁内共振动力学行为,并开展系统参数分析.从工程角度,斜拉桥的振动分为两类:局部振动和全局振动.局部振动是指拉索的振动,认为拉索的两锚固端是固定的.全局振动是指桥塔和主梁的振动,认为拉索不发生振动,而仅提供张力.实际工程中,两种振动存在密切联系,当局部和全局振动频率接近时,两种振动之间可实现能量传递,全桥表现为混合振动,即拉索与主梁产生内共振.

再者,为探究斜拉桥运营中,不同拉索之间、索与桥面梁之间的耦合共振动力学,康厚军等<sup>[24-26]</sup>建立斜拉桥的多索-浅拱动力学理论模型,利用多尺

度摄动分析方法系统探讨了1:1,1:2,1:3等系统内共振非线性动力学行为,并开展相关试验研究<sup>[40,41]</sup>.

从全局动力学角度,基于斜拉桥中拉索对桥面梁的弹性支承作用,康厚军等和苏潇阳等<sup>[42-44]</sup>对漂浮式、无背索和混合体系多塔斜拉桥开展全桥竖弯刚度评估分析,建立多梁离散弹簧模型及其动力学理论,利用传递矩阵法进行求解,研究结果对斜拉桥全桥竖弯刚度评估具有重要理论意义.基于某斜拉桥的有限元建模与计算得到,其桥塔横向刚度为 $10^{15} \text{ N} \cdot \text{m}^2$ 量级,梁横向刚度为 $10^{11} \text{ N} \cdot \text{m}^2$ 量级,而拉索横向刚度仅为 $10^3 \text{ N} \cdot \text{m}^2$ 量级,因此是典型的多刚度尺度系统.依据本文提出的多刚度尺度模态理论,将以小尺度刚度构件拉索振动为主的振型称为局部模态,如图8(a)所示;将以大尺度刚度构件梁振动为主的振型称为全局模态,如图8(b)所示.此外,将拉索和梁均参与的振动称为混合模态.桥塔的抗弯刚度比梁的刚度更大出几个量级,为更大尺度刚度尺度构件,理论分析和场地监测表明其振动较小,一般不予考虑.

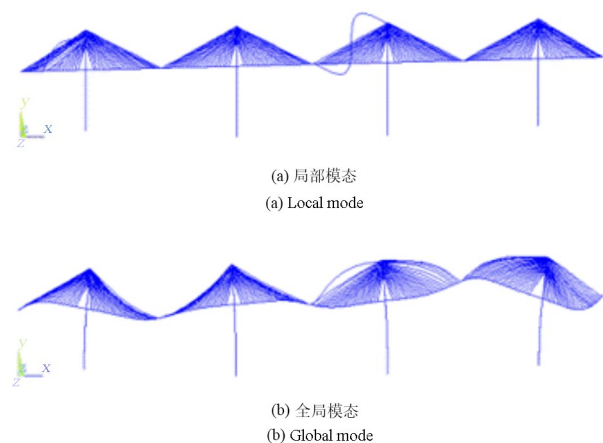


图8 有限元得到的斜拉桥模态

Fig.8 Mode of cable-stayed bridge obtained by finite element model

### 2.3 拱桥

拱桥以其优美的造型,深受人们喜爱,拱桥建造也正朝着大跨径发展.目前,由中国工程院院士郑皆连主导设计的广西天峨龙滩特大桥主拱肋已完成合龙,跨径达600 m,建成后将是世界最大跨径的钢管混凝土拱桥.随着跨径的不断增长,在施工与运营阶段,其拱肋、吊杆和桥面梁的振动问题日益突出.

康厚军等<sup>[45,46]</sup>运用传递矩阵法建立了单跨拱桥桥面内自由振动理论,并对其自振特性和稳定性等进行参数分析,研究得到以主梁和拱肋振动主导的模态.基于拱桥斜拉扣挂的施工工艺,易壮鹏等<sup>[47]</sup>和吕建根等<sup>[48]</sup>建立索-拱组合模型,探究系统发生外部主共振和索拱内共振的非线性动力学响应,受系统参数影响,不同模态之间和索拱之间存在能量传递.近期,康厚军等<sup>[49]</sup>采用传递矩阵法求解拱桥悬臂施工与合龙状态下的面内自振特性,建立了拱桥施工过程中面内竖弯刚度的评估方法,对影响系统模态的斜拉索和拱肋参数开展系统参数分析.



图9 某单跨拱桥

Fig.9 A single span arch bridge

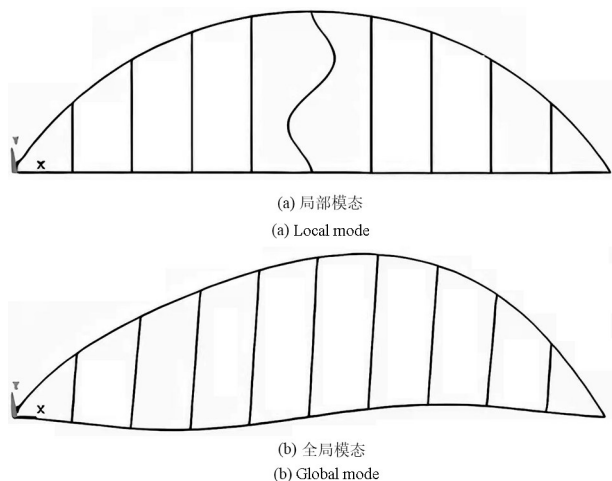


图10 有限元得到的拱桥模态

Fig.10 Mode of arch bridge obtained by finite element model

经有限元建模计算,某拱桥主拱肋面内横向抗弯刚度为 $10^9 \text{ N} \cdot \text{m}^2$ 量级,主梁横向刚度为 $10^8 \text{ N} \cdot \text{m}^2$ 量级,吊杆横向刚度为 $10^5 \text{ N} \cdot \text{m}^2$ 量级,因此拱桥也是具有显著多刚度尺度的系统.虽然目前对拱桥动力学的研究众多,但其各类振动问题时有发生,对拱桥的全局动力学缺乏足够的认识.基于课题组的前期研究和对组合结构动力学的理解,将以小尺度刚度构件吊杆振动为主的振型称为局部模态,如

图 10(a)所示;将以大尺度刚度构件拱肋和主梁振动为主的振型称为全局模态,如图 10(b)所示;将吊杆、拱肋和主梁同时参与的振动称为混合模态。

### 3 结论

工程结构朝着大跨径发展,受材料、几何和边界耦合非线性等的影响,其振动问题为学界和工程界关注的重要问题,本文指出:一方面,当前非线性动力学设计大多基于成熟的线性振动理论,忽略了高阶非线性作用,在一些情况下无法准确揭示系统内在的动力学本质;另一方面,当前研究局限于低维非线性系统模型,缺乏对如大跨桥梁等高维非线性系统全局动力学、模态共振动力学、耦合动力学的足够认识。

本文以质量弹簧系统、悬索桥、斜拉桥和拱桥动力学模型为例,从系统全局动力学角度,根据非线性动力学和有限元分析,提出工程结构多刚度尺度概念,给出多刚度尺度耦合系统的全局模态、局部模态和混合模态基于不同刚度尺度的定义,分析表明这类工程结构具有复杂非线性动力学特性,特定情况下可表现出具有危害性的复杂大幅振动,因此开展基于多刚度尺度的非线性动力学研究具有重要意义。为建立桥梁全局动力学模型和理论,桥梁非线性动力学、振动与控制等研究奠定基础。

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