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事件空间中非完整力学系统的 Herglotz-d′Alembert 原理与守恒律*

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摘要 研究事件空间中非完整力学系统的 Herglotz 型守恒律. 给出事件空间中 Herglotz 型广义变分原理，引入非完整约束并采用交换关系的 Hölder 定义，导出事件空间中非完整力学系统的新型微分变分原理—Herglotz-d′Alembert 原理. 引进事件空间中的空间生成元和参数生成元，建立 Herglotz-d′Alembert 原理不变性条件的变换. 基于该原理构建了事件空间中非完整非保守力学系统的 Herglotz 型守恒定理及其逆定理. 作为特例，给出了位形空间的 Herglotz 型守恒量和事件空间中完整力学系统的 Herglotz 型守恒量. 文末还给出了一个算例.

关键词 非完整力学， Herglotz-d′Alembert 原理， 守恒律， 事件空间

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引言

事件空间将空间和时间统一在一起考虑，使得时间与质点系的广义坐标处于同等地位，这样不仅使方程更简洁，还可直接给出能量积分. 因此，无论从几何角度还是动力学角度都有重要的意义[1-5]. 守恒律研究是非完整力学研究的一个重要方面，而寻求守恒量通常可利用对称性方法[5,6]和积分因子方法[7,8]等. 除此之外，守恒律也可利用微分变分原理来研究. 例如，d′Alembert 原理[9]、Jourdain 原理[10-12]、Gauss 原理[12]、Pfaff-Birkhoff-d′Alembert 原理[13]等. Herglotz 变分原理[14,15]由于其研究为非保守力学提供了一个变分方法，近年来得到广泛关注[16-25]. 但是迄今为止几乎所有 Herglotz 变分原理及其对称性的研究都限于位形空间或相空间. 最近，文献[26]基于微分变分原理研究了非保守非完整系统的 Herglotz 型守恒律. 本文将进一步研究事件空间中非保守非完整力学系统的 Herglotz 型守恒律，导出该系统的 Herglotz-d′Alembert 原理，基于所得原理建立 Herglotz 型守恒定理及其逆定理.

1 事件空间中 Herglotz-d′Alembert 原理

研究力学系统,设系统是非保守的，其广义坐标为 $q_s (s=1,2,\cdots,n)$. 构建 $n+1$ 维事件空间,该空间点的坐标为 $x_\alpha = x_\alpha(\tau)(\alpha = 1,2,\cdots,n+1)$，其中 $x_1 = t, x_{s+1} = q_s, \tau$ 为参数. 令 $x_\alpha = x_\alpha(\tau)$ 是 C^2 类函数，使得

$$\frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau} = x'_\alpha \tag{1}$$

不同时为零，有

$$\dot{x}_\alpha = \frac{\mathrm{d}x_\alpha}{\mathrm{d}t} = \frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{x'_\alpha}{x'_1} \tag{2}$$

设 $L = L(t, q_s, \dot{q}_s, z)$ 是 Herglotz 意义下的 Lagrange 函数，则在事件空间中成为

$$\Lambda(x_\alpha, x'_\alpha, z(\tau)) = x'_1 L\left(x_\alpha, \frac{x'_2}{x'_1}, \cdots, \frac{x'_{n+1}}{x'_1}, z(\tau)\right) \tag{3}$$

定义 1 确定函数 $x_\alpha(\tau)$，$\tau \in [\tau_0, \tau_1]$，使由

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事件空间中一阶微分方程

$$z'(\tau)=\Lambda(x_\alpha,x'_\alpha,z(\tau))=x'_1L\left(x_\alpha,\frac{x'_2}{x'_1},\cdots,\frac{x'_{n+1}}{x'_1},z(\tau)\right)\tag{4}$$

确定的作用量 $z(\tau)$，在给定的边界条件

$$x_\alpha(\tau)\big|_{\tau=\tau_0}=x_{\alpha0},\ x_\alpha(\tau)\big|_{\tau=\tau_1}=x_{\alpha1}\tag{5}$$

和初始值

$$z(\tau)\big|_{\tau=\tau_0}=z_0\tag{6}$$

下取得极值，即 $z(\tau_1)\to$ extr.，这里 $x_{\alpha0},x_{\alpha1},z_0$ 为常数，$\alpha=1,2,\cdots,n+1$. 称此变分问题为事件空间中 Herglotz 广义变分原理，$z(\tau)$ 为 Hamilton-Herglotz 作用量.

对 $z'(\tau)$ 求变分，有

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\delta z=\frac{\partial\Lambda}{\partial x_\alpha}\delta x_\alpha+\frac{\partial\Lambda}{\partial x'_\alpha}\delta x'_\alpha+\frac{\partial\Lambda}{\partial z}\delta z\tag{7}$$

方程(7)可视作以 δz 为变量的微分方程，可得

$$\delta z(\tau)\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)-\delta z(\tau_0)=\int_{\tau_0}^{\tau}\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\left(\frac{\partial\Lambda}{\partial x_\alpha}\delta x_\alpha+\frac{\partial\Lambda}{\partial x'_\alpha}\delta x'_\alpha\right)\mathrm{d}\tau\tag{8}$$

注意到 $z(\tau_1)\to$ extr. 以及式(6)，有

$$\delta z(\tau_0)=\delta z(\tau_1)=0\tag{9}$$

在式(8)中，取 $\tau=\tau_1$，得

$$\int_{\tau_0}^{\tau_1}\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\left(\frac{\partial\Lambda}{\partial x_\alpha}\delta x_\alpha+\frac{\partial\Lambda}{\partial x'_\alpha}\delta x'_\alpha\right)\mathrm{d}\tau=0\tag{10}$$

设系统的运动受 g 个非完整约束,在位形空间中约束方程为

$$\dot{q}_{\varepsilon+\beta}=\varphi_\beta(t,q_s,\dot{q}_\sigma),(\beta=1,2,\cdots,g;\ \sigma=1,2,\cdots,\varepsilon;\varepsilon=n-g)\tag{11}$$

事件空间中可表为

$$x'_{\varepsilon+\beta+1}=\Phi_\beta(x_\alpha,x'_\gamma)=\Phi_\beta(x_1,x_2,\cdots,x_{n+1},x'_1,x'_2,\cdots,x'_{\varepsilon+1})\tag{12}$$

其中

$$\Phi_\beta=x'_1\varphi_\beta\left(x_1,x_2,\cdots,x_{n+1},\frac{x'_2}{x'_1},\frac{x'_3}{x'_1},\cdots,\frac{x'_{\varepsilon+1}}{x'_1}\right)\tag{13}$$

虚位移满足 Appell-Chetaev 条件

$$\delta x_{\varepsilon+\beta+1}=\frac{\partial\Phi_\beta}{\partial x'_\gamma}\delta x_\gamma,\ (\beta=1,2,\cdots,g;\gamma=1,2,\cdots,\varepsilon+1;\varepsilon=n-g)\tag{14}$$

令 $\widetilde{\Lambda}$ 为 Λ，并借助约束(12)消去 $x'_{\varepsilon+\beta+1}$ 所得表达式，即

$$\widetilde{\Lambda}(x_\alpha,x'_\gamma,z(\tau))=\Lambda(x_\alpha,x'_\gamma,\Phi_\beta,z(\tau))\tag{15}$$

则有

$$\frac{\partial\widetilde{\Lambda}}{\partial x_\alpha}=\frac{\partial\Lambda}{\partial x_\alpha}+\frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\frac{\partial\Phi_\beta}{\partial x_\alpha}\tag{16}$$

$$\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}=\frac{\partial\Lambda}{\partial x'_\gamma}+\frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\frac{\partial\Phi_\beta}{\partial x'_\gamma}\tag{17}$$

由于非完整约束的存在，须考虑变分和微分运算的交换性问题. 这里采用交换关系的 Hölder 定义[27]，即假定全部变分满足交换关系

$$\delta x'_\alpha=\frac{\mathrm{d}}{\mathrm{d}\tau}\delta x_\alpha,(\alpha=1,2,\cdots,n+1)\tag{18}$$

将式(14)对 τ 求导，并考虑到关系式(18)，得

$$\delta x'_{\varepsilon+\beta+1}=\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\Phi_\beta}{\partial x'_\gamma}\delta x_\gamma+\frac{\partial\Phi_\beta}{\partial x'_\gamma}\delta x'_\gamma\tag{19}$$

式(17)两边同乘 $\delta x'_\gamma$，并将式(19)代入可得

$$\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}\delta x'_\gamma=\frac{\partial\Lambda}{\partial x'_\gamma}\delta x'_\gamma+\frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\frac{\partial\Phi_\beta}{\partial x'_\gamma}\delta x'_\gamma=\frac{\partial\Lambda}{\partial x'_\gamma}\delta x'_\gamma+\frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\left(\delta x'_{\varepsilon+\beta+1}-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\Phi_\beta}{\partial x'_\gamma}\delta x_\gamma\right)=\frac{\partial\Lambda}{\partial x'_\alpha}\delta x'_\alpha-\frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\Phi_\beta}{\partial x'_\gamma}\delta x_\gamma\tag{20}$$

将式(16)和式(20)代入方程(10)，对含 $\delta x'_\alpha$ 的项进行分部积分运算，并利用边界条件(5)可得

$$\int_{\tau_0}^{\tau_1}\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\left(\frac{\partial\widetilde{\Lambda}}{\partial x_\alpha}\delta x_\alpha-\frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\frac{\partial\Phi_\beta}{\partial x_\alpha}\delta x_\alpha+\left(\frac{\partial\Lambda}{\partial z}\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}\right)\delta x_\gamma+\frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\Phi_\beta}{\partial x'_\gamma}\delta x_\gamma\right)\mathrm{d}\tau=0\tag{21}$$

将式(14)代入式(21)，考虑 $[\tau_0,\tau_1]$ 的任意性，有

$$\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\left[\frac{\partial\widetilde{\Lambda}}{\partial x_\gamma}-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}+\frac{\partial\Lambda}{\partial z}\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}+\frac{\partial\widetilde{\Lambda}}{\partial x_{\varepsilon+\beta+1}}\frac{\partial\Phi_\beta}{\partial x'_\gamma}-\frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\left(\frac{\partial\Phi_\beta}{\partial x_\gamma}-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\Phi_\beta}{\partial x'_\gamma}+\frac{\partial\Phi_\beta}{\partial x_{\varepsilon+\rho+1}}\frac{\partial\Phi_\rho}{\partial x'_\gamma}\right)\right]\delta x_\gamma=0\tag{22}$$

式(22)可称为事件空间中非完整力学系统的 Herglotz-d'Alembert 原理. 由 δx_γ 的独立性，有

$$\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\left[\frac{\partial\widetilde{\Lambda}}{\partial x_\gamma}-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}+\frac{\partial\Lambda}{\partial z}\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}+\frac{\partial\widetilde{\Lambda}}{\partial x_{\varepsilon+\beta+1}}\frac{\partial\Phi_\beta}{\partial x'_\gamma}-\frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\left(\frac{\partial\Phi_\beta}{\partial x_\gamma}-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\Phi_\beta}{\partial x'_\gamma}+\frac{\partial\Phi_\beta}{\partial x_{\varepsilon+\rho+1}}\frac{\partial\Phi_\rho}{\partial x'_\gamma}\right)\right]=0,$$

$$(\gamma = 1,2,\cdots,\varepsilon+1) \tag{23}$$

方程(23)可称为事件空间中非完整力学系统的 Herglotz 型运动微分方程.

2 Herglotz-d′Alembert 原理不变性条件

事件空间坐标 x_α 的等参数变分可定义为

$$\delta x_\alpha = \bar{x}_\alpha(\bar{\tau}) - x_\alpha(\tau),\ \bar{\tau} = \tau \tag{24}$$

非等参数变分为

$$\Delta x_\alpha = \bar{x}_\alpha(\bar{\tau}) - x_\alpha(\tau), \bar{\tau} = \tau + \Delta\tau \tag{25}$$

考虑事件空间可变路径 $\bar{x}_\alpha(\tau+\Delta\tau)$ 无限接近真实路径 $x_\alpha(\tau)$，因此变分 $\Delta\tau$ 是一个足够小的量，将 $\bar{x}_\alpha(\tau+\Delta\tau)$ 展开，保留一阶小量,可得

$$\bar{x}_\alpha(\tau+\Delta\tau) = \bar{x}_\alpha(\tau) + x'_\alpha(\tau)\Delta\tau \tag{26}$$

因此有

$$\delta x_\alpha = \bar{x}_\alpha(\tau+\Delta\tau) - x_\alpha(\tau) - \bar{x}_\alpha(\tau+\Delta\tau) + \bar{x}_\alpha(\tau) = \Delta x_\alpha - x'_\alpha\Delta\tau \tag{27}$$

引进 F_s 和 f 作为事件空间中的空间生成元和参数生成元

$$\Delta x_\alpha = \varepsilon F_\alpha(x,x'),\ \Delta\tau = \varepsilon f(x,x') \tag{28}$$

于是有

$$\delta x_\alpha = \varepsilon[F_\alpha(x,x') - x'_\alpha f(x,x')] \tag{29}$$

将式(29)代入式(22)，整理可得

$$\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\Bigg\{\Bigg[\frac{\partial\widetilde{\Lambda}}{\partial x_\gamma} + \frac{\partial\widetilde{\Lambda}}{\partial x_{\varepsilon+\beta+1}}\frac{\partial\Phi_\beta}{\partial x'_\gamma} - \frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\left(\frac{\partial\Phi_\beta}{\partial x_\gamma} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\Phi_\beta}{\partial x'_\gamma} + \frac{\partial\Phi_\beta}{\partial x_{\varepsilon+\rho+1}}\frac{\partial\Phi_\rho}{\partial x'_\gamma}\right)\Bigg](F_\gamma - x'_\gamma f) + \frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}(F'_\gamma - x''_\gamma f - x'_\gamma f')\Bigg\}\varepsilon - \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}(F_\gamma - x'_\gamma f)\right]\varepsilon = 0 \tag{30}$$

由条件(14)和式(29)，生成元应满足条件

$$F_{\varepsilon+\beta+1} - x'_{\varepsilon+\beta+1}f = \frac{\partial\Phi_\beta}{\partial x'_\gamma}(F_\gamma - x'_\gamma f) \tag{31}$$

在式(30)中加上和减去 $\varepsilon\frac{\mathrm{d}}{\mathrm{d}\tau}\left[G_N\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\right]$ 项,其中 $G_N = G_N(x_\alpha, x'_\gamma, z)$ 称为规范函数，并利用式(31)，以及

$$\frac{\mathrm{d}\widetilde{\Lambda}}{\mathrm{d}\tau} = \frac{\partial\widetilde{\Lambda}}{\partial x_\gamma}x'_\gamma + \frac{\partial\widetilde{\Lambda}}{\partial x_{\varepsilon+\beta+1}}x'_{\varepsilon+\beta+1} + \frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}x''_\gamma + \frac{\partial\widetilde{\Lambda}}{\partial z}\Lambda \tag{32}$$

得到

$$\varepsilon\Bigg\{\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\Bigg[\frac{\partial\widetilde{\Lambda}}{\partial x_\alpha}F_\alpha + \frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}F'_\gamma + \left(\widetilde{\Lambda} - x'_\gamma\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}\right)f' - \frac{\partial\Lambda}{\partial z}G_N + G'_N - \frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\left(\frac{\partial\Phi_\beta}{\partial x_\gamma} - \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\Phi_\beta}{\partial x'_\gamma} + \frac{\partial\Phi_\beta}{\partial x_{\varepsilon+\rho+1}}\frac{\partial\Phi_\rho}{\partial x'_\gamma}\right)\times (F_\gamma - x'_\gamma f)\Bigg] - \frac{\mathrm{d}}{\mathrm{d}\tau}\Bigg[\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\times \left(\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}F_\gamma + \left(\widetilde{\Lambda} - x'_\gamma\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}\right)f + G_N\right)\Bigg]\Bigg\} = 0 \tag{33}$$

式(33)可称为事件空间中非完整力学系统 Herglotz-d′Alembert 原理不变性条件的变换.

3 Herglotz 型守恒定理

由 Herglotz-d′Alembert 原理不变性条件式(33)，可得到如下定理

定理 1　如果事件空间中的空间生成元 F_α，参数生成元 f，以及规范函数 G_N 满足条件

$$\frac{\partial\widetilde{\Lambda}}{\partial x_\alpha}F_\alpha + \frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}F'_\gamma + \left(\widetilde{\Lambda} - x'_\gamma\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}\right)f' - \frac{\partial\Lambda}{\partial z}G_N + G'_N - \frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\left(\frac{\partial\Phi_\beta}{\partial x_\gamma} - \frac{d}{\mathrm{d}\tau}\frac{\partial\Phi_\beta}{\partial x'_\gamma} + \frac{\partial\Phi_\beta}{\partial x_{\varepsilon+\rho+1}}\frac{\partial\Phi_\rho}{\partial x'_\gamma}\right) = 0 \tag{34}$$

和限制方程

$$F_{\varepsilon+\beta+1} - x'_{\varepsilon+\beta+1}f = \frac{\partial\Phi_\beta}{\partial x'_\gamma}(F_\gamma - x'_\gamma f) \tag{35}$$

则

$$I_N = \exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\left(\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}F_\gamma + \left(\widetilde{\Lambda} - x'_\gamma\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}\right)f + G_N\right) = \text{const.} \tag{36}$$

是非完整系统(23)的 Herglotz 型守恒量.

称定理 1 为事件空间中非完整力学系统的 Herglotz 型守恒定理.

当取 $\tau = t$ 时，定理 1 给出通常位形空间的结果，即有如下推论：

推论 1　如果空间和时间的生成元 F_s, f 以及规范函数 $G(t, q_s, \dot{q}_\sigma, z)$ 满足条件

$$\frac{\partial\widetilde{L}}{\partial q_s}F_s + \frac{\partial\widetilde{L}}{\partial\dot{q}_\sigma}\dot{F}_\sigma + \left(\widetilde{L} - \dot{q}_\sigma\frac{\partial\widetilde{L}}{\partial\dot{q}_\sigma}\right)\dot{f} + \frac{\partial\widetilde{L}}{\partial t}f - \frac{\partial L}{\partial z}G + \dot{G} - \frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial\varphi_\beta}{\partial q_\sigma} - \frac{\mathrm{d}}{dt}\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma} + \frac{\partial\varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial\varphi_\gamma}{\partial\dot{q}_\sigma}\right)(F_\sigma - \dot{q}_\sigma f) = 0 \tag{37}$$

以及限制方程

$$\frac{\partial\varphi_\beta}{\partial\dot{q}_\sigma}(F_\sigma-\dot{q}_\sigma f)+\varphi_\beta f-F_{\varepsilon+\beta}=0 \quad (38)$$

则

$$I=\exp\left(-\int_a^t\frac{\partial L}{\partial z}\mathrm{d}\theta\right)\left[\frac{\partial\widetilde{L}}{\partial\dot{q}_\sigma}F_\sigma+\left(\widetilde{L}-\dot{q}_\sigma\frac{\partial\widetilde{L}}{\partial\dot{q}_\sigma}\right)f+G\right]=\mathrm{const.} \quad (39)$$

是非完整力学系统的 Herglotz 型守恒量.

推论1已由文献[26]给出.

当系统不存在非完整约束(12)，定理1给出事件空间中完整非保守力学系统的结果，即有如下推论：

推论2 如果事件空间中的空间生成元 F_α，参数生成元 f，以及规范函数 G_{N} 满足条件

$$\frac{\partial\Lambda}{\partial x_\alpha}F_\alpha+\frac{\partial\Lambda}{\partial x'_\alpha}F'_\alpha+\left(\Lambda-x'_\alpha\frac{\partial\Lambda}{\partial x'_\alpha}\right)f'-\frac{\partial\Lambda}{\partial z}G_{\mathrm{N}}+G'_{\mathrm{N}}=0 \quad (40)$$

则

$$I_{\mathrm{N}}=\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\left(\frac{\partial\Lambda}{\partial x'_\alpha}F_\alpha+\left(\Lambda-x'_\alpha\frac{\partial\Lambda}{\partial x'_\alpha}\right)f+G_{\mathrm{N}}\right)=\mathrm{const.} \quad (41)$$

是事件空间中完整非保守力学系统的 Herglotz 型守恒量.

推论2称为事件空间中完整非保守力学系统的 Herglotz 型守恒定理.

4 Herglotz 型守恒定理之逆定理

设非完整系统(23)存在守恒量

$$I(x_\alpha,x'_\gamma,z)=\mathrm{const.} \quad (42)$$

将式(42)对参数 τ 求导数，得

$$\frac{\mathrm{d}I}{\mathrm{d}\tau}=\frac{\partial I}{\partial x_\gamma}x'_\gamma+\frac{\partial I}{\partial x_{\varepsilon+\beta+1}}\Phi_\beta+\frac{\partial I}{\partial x'_\gamma}x''_\gamma+\frac{\partial I}{\partial z}\Lambda=0 \quad (43)$$

将方程(29)代入式(22)，得

$$\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\left[\frac{\partial\widetilde{\Lambda}}{\partial x_\gamma}-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}+\frac{\partial\Lambda}{\partial z}\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}+\frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial\Phi_\beta}{\partial x'_\gamma}+\frac{\partial\widetilde{\Lambda}}{\partial x_{\varepsilon+\beta+1}}\frac{\partial\Phi_\beta}{\partial x'_\gamma}-\frac{\partial\Lambda}{\partial x'_{\varepsilon+\beta+1}}\left(\frac{\partial\Phi_\beta}{\partial x_\gamma}+\frac{\partial\Phi_\beta}{\partial x_{\varepsilon+\rho+1}}\frac{\partial\Phi_\rho}{\partial x'_\gamma}\right)\right]\times(F_\gamma-x'_\gamma f)=0 \quad (44)$$

比较式(43)与式(44)中项 x''_γ 的系数，可得

$$\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\left(\frac{\partial^2\widetilde{\Lambda}}{\partial x'_\lambda\partial x'_\gamma}-\frac{\partial^2\Phi_\beta}{\partial x'_\lambda\partial x'_\gamma}\right)\times(F_\lambda-x'_\lambda f)=\frac{\partial I}{\partial x'_\gamma} \quad (45)$$

再令守恒量(42)与式(36)相等，即

$$\exp\left(-\int_{\tau_0}^{\tau}\frac{\partial\Lambda}{\partial z}\mathrm{d}\varphi\right)\left(\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}F_\gamma+\left(\widetilde{\Lambda}-x'_\gamma\frac{\partial\widetilde{\Lambda}}{\partial x'_\gamma}\right)f+G_{\mathrm{N}}\right)=I \quad (46)$$

这样，由式(45)和式(46)，在已知 $I(x_\alpha,x'_\gamma,z)=\mathrm{const.}$ 下，可找到相应的无限小变换. 于是有

定理2 对于事件空间中非完整系统(23)，如果已知守恒量(42)，则可由式(45)和式(46)求得变换的生成元 F_γ、f 和规范函数 G_{N}.

定理2称为事件空间中非完整系统 Herglotz 型守恒定理的逆定理.

5 算例

例 设力学系统的 Herglotz 型 Lagrange 函数为

$$L=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2)-z \quad (47)$$

非完整约束为

$$\dot{q}_2-t\dot{q}_1=0 \quad (48)$$

泛函 z 满足微分方程

$$\dot{z}=\frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2)-z \quad (49)$$

令 $x_1=t,x_2=q_1,x_3=q_2$，则事件空间中 Herglotz 型 Lagrange 函数为

$$\Lambda=x'_1\left[\frac{1}{2}\left(\frac{x'_2}{x'_1}\right)^2+\frac{1}{2}\left(\frac{x'_3}{x'_1}\right)^2-z\right] \quad (50)$$

方程(49)成为

$$z'=x'_1\left[\frac{1}{2}\left(\frac{x'_2}{x'_1}\right)^2+\frac{1}{2}\left(\frac{x'_3}{x'_1}\right)^2-z\right] \quad (51)$$

由约束方程(48)

$$x'_3=x_1x'_2\triangleq\Phi_1 \quad (52)$$

将约束方程(52)嵌入式(50)，得

$$\widetilde{\Lambda}=\frac{1}{2}(1+x_1^2)\frac{(x'_2)^2}{x'_1}-zx'_1 \quad (53)$$

方程(23)给出

$$2x_1\frac{(x'_2)^2}{x'_1}+\frac{(1+x_1^2)x'_2}{(x'_1)^3}(x'_1x''_2-x'_2x''_1)+z'+(1+x_1^2)\frac{(x'_2)^2}{2x'_1}+x'_1z-\frac{x'_3x'_2}{x'_1}=0,$$

$$2x_1x'_2+\frac{1+x_1^2}{(x'_1)^2}(x'_1x''_2-x'_2x''_1)+(1+x_1^2)x'_2-x'_3=0 \quad (54)$$

将方程(51)和约束(52)代入方程(54)，易知两个方程彼此不独立. 方程(34)和方程(35)给出

$$x_1\frac{(x'_2)^2}{x'_1}F_1-[\frac{1}{2}(1+x_1^2)\left(\frac{x'_2}{x'_1}\right)^2+z]F'_1+(1+x_1^2)\frac{x'_2}{x'_1}F'_2-\frac{x'_3x'_2}{x'_1}F_1+x'_3F_2+G_Nx'_1+G'_N=0,$$

$$x_1(F_2-x'_2f)+x_1x'_2f-F_3=0 \tag{55}$$

考虑到约束(52)，方程组(55)有解

$$f=0,F_1=0,F_2=\frac{x'_1e^{-x_1+a}}{(1+x_1^2)x'_2},$$

$$F_3=\frac{x_1x'_1e^{-x_1+a}}{(1+x_1^2)x'_2},$$

$$G_N=e^{-x_1+a}[\ln\frac{x'_2}{x'_1}+x_1+\frac{1}{2}\ln(1+x_1^2)-1] \tag{56}$$

其中 $a=x_1(\tau_0)$ 为常数. 由定理 1，得到守恒量

$$I_N=\ln\frac{x'_2}{x'_1}+x_1+\frac{1}{2}\ln(1+x_1^2)=\text{const.} \tag{57}$$

其次，若已知守恒量(57)，由式(45)和(46)，有

$$e^{x_1-a}\frac{(1+x_1^2)x'_2}{(x'_1)^2}[\frac{x'_2}{x'_1}(F_1-x'_1f)-(F_2-x'_2f)]=-\frac{1}{x'_1},$$

$$e^{x_1-a}\frac{1+x_1^2}{x'_1}[-\frac{x'_2}{x'_1}(F_1-x'_1f)+(F_2-x'_2f)]=\frac{1}{x'_2},$$

$$e^{x_1-a}\{[-(1+x_1^2)\frac{(x'_2)^2}{(x'_1)^2}-z]F_1+(1+x_1^2)\frac{x'_2}{x'_1}F_2+G_N\}=\ln\frac{x'_2}{x'_1}+x_1+\frac{1}{2}\ln(1+x_1^2) \tag{58}$$

方程(58)中前两个方程彼此不独立，因此实际上方程(58)含 2 个独立方程和 3 个未知量，其解不唯一. 如取

$$G_N=e^{-x_1+a}[\ln\frac{x'_2}{x'_1}+x_1+\frac{1}{2}\ln(1+x_1^2)-1] \tag{59}$$

则有解

$$F_1=0,F_2=\frac{x'_1e^{-x_1+a}}{(1+x_1^2)x'_2} \tag{60}$$

此时参数生成元 f 可以取任意函数. 由限制方程 $x_1(F_2-x'_2f)+x_1x'_2f-F_3=0$ 解得 $F_3=\frac{x_1x'_1e^{-x_1+a}}{(1+x_1^2)x'_2}$.

如取 $G_N=0$，则有解

$$F_1=-\frac{1}{z}e^{-x_1+a}[\ln\frac{x'_2}{x'_1}+x_1+\frac{1}{2}\ln(1+x_1^2)-1],$$

$$F_2=\frac{x'_1e^{-x_1+a}}{(1+x_1^2)x'_2}-\frac{x'_2}{x'_1z}e^{-x_1+a}[\ln\frac{x'_2}{x'_1}+x_1+\frac{1}{2}\ln(1+x_1^2)-1],$$

$$F_3=\frac{x_1x'_1e^{-x_1+a}}{(1+x_1^2)x'_2}-\frac{x_1x'_2}{x'_1z}e^{-x_1+a}[\ln\frac{x'_2}{x'_1}+x_1+\frac{1}{2}\ln(1+x_1^2)-1] \tag{61}$$

6　结论

事件空间将空间和时间统一在一起，在事件空间中研究质点系的运动不仅在几何上而且从动力学角度都有重要意义. 守恒律也可以通过微分变分原理来构建. 本文研究了事件空间中非完整力学系统的守恒律. 主要工作：一是基于变分运算和微分运算交换关系的 Hölder 定义导出事件空间中非完整力学系统的 Herglotz-d′Alembert 原理(式(22))；二是引进事件空间中空间生成元和参数生成元，建立 Herglotz-d′Alembert 原理不变性条件的变换(式(33))；三是基于所得的 Herglotz-d′Alembert 原理构建了事件空间非完整力学系统的 Herglotz 型守恒定理(定理 1 和定理 2). 如果在位形空间，该定理给出文献[26]的结果(推论 1)；如果系统是完整的，由该定理可以得到完整非保守力学系统的 Herglotz 型守恒定理(推论 2). 因此，本文的结论更具一般性，它不仅可以处理保守和非保守过程，还可适用于完整和非完整系统.

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HERGLOTZ-D′ALEMBERT PRINCIPLE AND CONSERVATION LAW FOR NONHOLONOMIC MECHANICAL SYSTEMS IN EVENT SPACE*

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Abstract Herglotz conservation laws of nonholonomic mechanical systems in event space are studied. The Herglotz generalized variational principle in event space is given, and the Herglotz-d′Alembert principle, a new differential variational principle for nonholonomic mechanical systems in event space, is derived by introducing nonholonomic constraints and using the Hölder definition of commutative relation. The transformation of the invariance condition of Herglotz-d′Alembert principle is established by introducing space generators and parameter generators in the event space. Herglotz conservation theorem and its inverse for nonholonomic nonconservative mechanical systems in event space are constructed based on this principle. As particular cases, the Herglotz conservation laws in configuration space and the Herglotz conservation laws for holonomic mechanical system in event space are given. An example is given at the end of the paper to illustrate the application of Herglotz conservation laws.

Key words nonholonomic mechanics, Herglotz-d′Alembert principle, conservation law, event space

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