

# 分数阶混沌系统的自适应预测同步\*

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**摘要** 结合自适应控制和预测反馈控制,提出了一种新的实现分数阶混沌系统同步的自适应预测控制方法.利用分数阶 Lyapunov 稳定性理论,导出了分数阶混沌系统同步的一些新的充分条件.与已有的结果相比,该方法无需反馈增益的先验知识,且收敛速度快和在实验中很容易实现.最后数值实验进一步验证了所提同步方法的有效性.

**关键词** 分数阶系统, 混沌同步, 自适应控制, 预测控制

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## 引言

系统的同步意味着系统的轨迹逐渐趋于一致.由于在数字通信、电力电子、生物系统、化学反应和信息处理等不同的工程领域有着广泛的应用,分数阶混沌系统的同步问题一直是众多研究者研究的热点.同时不同分数阶混沌系统的同步方法相继被提出,如滑膜方法<sup>[1,2]</sup>、脉冲方法<sup>[3,4]</sup>、active 控制方法<sup>[5,6]</sup>和模糊方法<sup>[7,8]</sup>等.

自适应方法能实时收集数据和调整控制参数,因此它常用来控制和同步分数阶混沌系统.文献[9]设计了不确定分数阶混沌系统同步的自适应控制律,导出了具有模型不确定性和外部扰动的分数阶混沌系统同步的几个充分条件.文献[10,11]分别利用自适应控制器实现了含有未知参数的分数阶 Arneodo 系统和分数阶 Liu 系统的同步.另一方面,文献[12]利用预测反馈控制方法实现了离散混沌系统的控制.文献[13-15]推广了该方法并实现了整数阶连续混沌与超混沌系统的预测反馈控制和同步.基于 T-S 模糊模型和预测反馈控制,文献[16-18]进一步研究了分数阶混沌系统的控制与同步.然而这些方法中增益矩阵往往需要求解线性矩阵不等式,这给该方法的应用带来很大的限制.

针对预测反馈控制方法存在的不足,本文提出了一种新的实现分数阶混沌系统同步的自适应预

测控制方法.基于分数阶 Lyapunov 稳定性理论,设计自适应预测控制器和控制增益的分数阶自适应律,实现了分数阶混沌系统的同步,证明了在一定条件下误差系统能渐近趋于零.数值仿真表明该方法的有效性.

## 1 分数阶微积分

Caputo 微分定义为<sup>[19]</sup>:

$$\frac{d^\alpha f(t)}{dt^\alpha} = {}_0^c D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{(n-\alpha-1)} f^{(n)}(\tau) d\tau \quad (1)$$

式中,  $n-1 < \alpha < n, n \in N, \Gamma(\cdot)$  为伽马函数:

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt \quad (2)$$

考虑分数阶混沌系统:

$$\frac{d^\alpha \mathbf{x}}{dt^\alpha} = \mathbf{f}(\mathbf{x}) \quad (3)$$

其中,  $\alpha \in (0, 1)$  是分数阶,  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \mathbf{R}^n$  是系统的状态向量,  $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x})]^T \in \mathbf{R}^n$  是可微的非线性向量函数. 设  $\Omega \in \mathbf{R}^n$  是  $\mathbf{R}^n$  中的一个有界集,我们给出下面的假设.

**假设 1:** 对于任意  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T \in \Omega$  和  $\mathbf{y} = (y_1, y_2, \dots, y_n)^T \in \Omega$ , 存在一个常数  $l > 0$  满足

$$\|f_i(\mathbf{x}) - f_i(\mathbf{y})\| \leq l \|\mathbf{x} - \mathbf{y}\| \quad (1 \leq i \leq n) \quad (4)$$

其中  $\|\cdot\|$  表示  $n$  维向量空间的欧氏范数.

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引理1<sup>[20]</sup>:设

$$V(t) = \frac{1}{2} \mathbf{y}_1^T(t) \mathbf{Q}_1 \mathbf{y}_1(t) + \frac{1}{2} \mathbf{y}_2^T(t) \mathbf{Q}_2 \mathbf{y}_2(t) \quad (5)$$

其中,  $\mathbf{y}_1(t) \in \mathbf{R}^n$  和  $\mathbf{y}_2(t) \in \mathbf{R}^n$  具有连续的一阶导数,  $\mathbf{Q}_1 \in \mathbf{R}^{n \times n}$  和  $\mathbf{Q}_2 \in \mathbf{R}^{n \times n}$  为两个正定矩阵. 若存在正定矩阵  $\mathbf{Q}_3 \in \mathbf{R}^{n \times n}$  和常数  $h_0 > 0$  使得

$$\frac{d^\alpha V(t)}{dt^\alpha} \leq -h_0 \mathbf{y}_1^T(t) \mathbf{Q}_3 \mathbf{y}_1(t) \quad (6)$$

则  $\|\mathbf{y}_1(t)\|$  和  $\|\mathbf{y}_2(t)\|$  有界, 且  $\mathbf{y}_1(t)$  渐近趋于 0 (即  $\lim_{t \rightarrow +\infty} \|\mathbf{y}_1(t)\| = 0$ ).

引理2<sup>[19]</sup>: 设  $\mathbf{x}(t) \in \mathbf{R}^n$  是一个可微函数向量, 那么对于任意  $t \geq 0$ , 且  $\alpha \in (0, 1)$ , 下列不等式成立

$$\frac{1}{2} \frac{d^\alpha (\mathbf{x}^T(t) \mathbf{P} \mathbf{x}(t))}{dt^\alpha} \leq \mathbf{x}^T(t) \mathbf{P} \frac{d^\alpha \mathbf{x}(t)}{dt^\alpha} \quad (7)$$

其中  $\mathbf{P} \in \mathbf{R}^{n \times n}$  是一个  $n \times n$  维常值正定矩阵.

## 2 控制器设计

假设系统(3)作为驱动系统, 可控响应系统为:

$$\frac{d^\alpha \mathbf{y}}{dt^\alpha} = \mathbf{f}(\mathbf{y}) + \mathbf{u} \quad (8)$$

其中,  $\mathbf{y} = (y_1, y_2, \dots, y_n)^T \in \mathbf{R}^n$  是系统的状态向量,  $\mathbf{u} = (u_1, u_2, \dots, u_n)^T \in \mathbf{R}^n$  是控制输入向量,  $\mathbf{f}(\mathbf{y}) = [f_1(\mathbf{y}), f_2(\mathbf{y}), \dots, f_n(\mathbf{y})]^T \in \mathbf{R}^n$  是可微的非线性向量函数.

定义1: 对于系统(3)和系统(8), 如果  $\lim_{t \rightarrow +\infty} \|\mathbf{y}(t) - \mathbf{x}(t)\| = 0$ , 则称系统(3)和系统(8)实现同步.

定义同步误差为  $\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{x}(t)$ , 根据系统(3)和系统(8), 误差系统可表示为:

$$\frac{d^\alpha \mathbf{e}}{dt^\alpha} = \mathbf{f}(\mathbf{y}) - \mathbf{f}(\mathbf{x}) + \mathbf{u} \quad (9)$$

显然, 系统(3)和系统(8)同步的问题转化为当  $t \rightarrow +\infty$  时误差系统(9)的零解稳定性问题. 设计如下的自适应预测控制器:

$$u_i(t) = -\beta_i [f_i(\mathbf{y}) - f_i(\mathbf{x}) + e_i] \quad (1 \leq i \leq n) \quad (10)$$

$$\frac{d^\alpha \beta_i}{dt^\alpha} = \gamma_i \{e_i [f_i(\mathbf{y}) - f_i(\mathbf{x})] + e_i^2\} \quad (1 \leq i \leq n) \quad (11)$$

其中  $\gamma_i > 0$  ( $i = 1, 2, \dots, n$ ) 为常数. 把方程(10)代入方程(9)得:

$$\frac{d^\alpha e_i}{dt^\alpha} = (1 - \beta_i) [f_i(\mathbf{y}) - f_i(\mathbf{x})] - \beta_i e_i \quad (1 \leq i \leq n) \quad (12)$$

评注1: 系统(12)可看成如下两个耦合相同分

数阶混沌系统的误差系统:

$$\frac{d^\alpha x_i}{dt^\alpha} = (1 - \beta_i) f_i(\mathbf{x}) - \frac{1}{2} \beta_i (x_i - y_i) \quad (1 \leq i \leq n) \quad (13)$$

$$\frac{d^\alpha y_i}{dt^\alpha} = (1 - \beta_i) f_i(\mathbf{y}) - \frac{1}{2} \beta_i (y_i - x_i) \quad (1 \leq i \leq n) \quad (14)$$

因此, 本文所提的方法本质上是对两个初始弱耦合分数阶混沌系统通过减少它们的速度和增加它们的耦合强度来取得同步.

评注2: 系统(12)在  $\beta_i = 0$  时, 它是一个混沌系统, 在  $\beta_i = 1$  时, 它是一个渐近稳定系统. 因此本文所提的自适应预测同步方法是通过自适应选取耦合强度  $\beta_i$  使误差系统(12)从混沌态向稳定态转变, 从而实现分数阶混沌系统的同步.

定理1: 对于给定的系统状态的初始条件且假设1的条件成立, 所设计的控制律(10)和自适应律(11)可以实现驱动系统(3)和响应系统(8)渐近同步, 并且同步误差渐近趋于零.

证明: 构造正定 Lyapunov 函数如下:

$$V(t) = \frac{1}{2} \sum_{i=1}^n e_i^2 + \frac{1}{2} \sum_{i=1}^n \frac{1}{\gamma_i} (\beta_i - L_i)^2 \quad (15)$$

其中,  $L_i$  是常数且满足  $\sum_{j=1}^n |1 - L_j| l < L_i$  ( $1 \leq i \leq n$ ).

根据引理2,  $V(t)$  沿系统(12)的分数阶导数满足

$$\begin{aligned} \frac{d^\alpha V(t)}{dt^\alpha} &\leq \sum_{i=1}^n e_i \frac{d^\alpha e_i}{dt^\alpha} + \sum_{i=1}^n \frac{1}{\gamma_i} (\beta_i - L_i) \frac{d^\alpha \beta_i}{dt^\alpha} \\ &= \sum_{i=1}^n e_i(t) \{ (1 - \beta_i) [f_i(\mathbf{y}) - f_i(\mathbf{x})] - \beta_i e_i(t) \} \\ &\quad + \sum_{i=1}^n (\beta_i - L_i) \{ e_i [f_i(\mathbf{y}) - f_i(\mathbf{x})] + e_i^2 \} \\ &= \sum_{i=1}^n \{ (1 - L_i) e_i [f_i(\mathbf{y}) - f_i(\mathbf{x})] - L_i e_i^2 \} \\ &\leq \sum_{i=1}^n (1 - L_i) |e_i| l \|e\| - \sum_{i=1}^n L_i e_i^2 \\ &\leq \sum_{i=1}^n |1 - L_i| l \|e\|^2 - \sum_{i=1}^n L_i e_i^2 \\ &= \sum_{i=1}^n \left[ \sum_{j=1}^n |1 - L_j| l - L_i \right] e_i^2 \leq 0 \end{aligned} \quad (16)$$

由引理1可知同步误差渐近趋于零, 即驱动系统(3)渐近同步响应系统(8).

## 3 数值仿真

选取分数阶 Chua 混沌系统作为驱动系统:

$$\begin{cases} \frac{d^\alpha x_1}{dt^\alpha} = a[x_2 - x_1 - f(x_1)] \\ \frac{d^\alpha x_2}{dt^\alpha} = x_1 - x_2 + x_3 \\ \frac{d^\alpha x_3}{dt^\alpha} = -bx_2 - cx_3 \end{cases} \quad (17)$$

式中  $f(x_1) = m_1 x_1 + \frac{1}{2}(m_0 - m_1)(|x_1 + 1| - |x_1 - 1|)$ . 当  $a=10.725, b=10.593, c=0.268, m_0 = -1.1726, m_1 = -0.7872, \alpha = 0.95$ , 初始值  $[x_1(0), x_2(0), x_3(0)] = [0.6, -0.7, -0.7]$  时, 图1为该分数阶Chua系统的混沌吸引子.

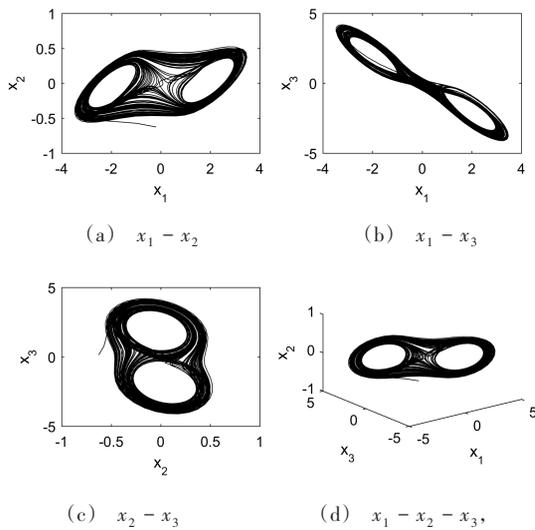


图1 分数阶Chua系统的吸引子

Fig.1 Attractors of fractional-order Chua system

响应系统表示为:

$$\begin{cases} \frac{d^\alpha y_1}{dt^\alpha} = a[y_2 - y_1 - f(y_1)] + u_1 \\ \frac{d^\alpha y_2}{dt^\alpha} = y_1 - y_2 + y_3 + u_2 \\ \frac{d^\alpha y_3}{dt^\alpha} = -by_2 - cy_3 + u_3 \end{cases} \quad (18)$$

式中,  $f(y_1) = m_1 y_1 + \frac{1}{2}(m_0 - m_1)(|y_1 + 1| - |y_1 - 1|)$ , 其它参数取值同上. 自适应预测控制器设计为

$$\begin{cases} u_1 = -\beta_1[a(y_2 - y_1 - f(y_1)) - a(x_2 - x_1 - f(x_1)) + e_1] \\ u_2 = -\beta_2(y_1 - y_2 + y_3 - x_1 + x_2 - x_3 + e_2) \\ u_3 = -\beta_3(-by_2 - cy_3 + bx_2 + cx_3 + e_3) \end{cases} \quad (19)$$

控制器增益的分数阶自适应律为:

$$\begin{cases} \frac{d^\alpha \beta_1}{dt^\alpha} = e_1[a(y_2 - y_1 - f(y_1)) - a(x_2 - x_1 - f(x_1))] + e_1^2 \\ \frac{d^\alpha \beta_2}{dt^\alpha} = e_2(y_1 - y_2 + y_3 - x_1 + x_2 - x_3) + e_2^2 \\ \frac{d^\alpha \beta_3}{dt^\alpha} = e_3(-by_2 - cy_3 + bx_2 + cx_3) + e_3^2 \end{cases} \quad (20)$$

响应系统的初值  $[y_1(0), y_2(0), y_3(0)] = [0.4, -0.6, 0.6]$ , 控制器增益的初值  $[\beta_1(0), \beta_2(0), \beta_3(0)] = [0.2, 0.2, 0.1]$ , 数值仿真结果如下: 图2显示驱动系统与响应系统的状态响应曲线, 图3显示同步误差渐近趋于零. 图4显示当  $t \rightarrow +\infty$  时, 增益  $\beta_i (1 \leq i \leq 4)$  趋于一个常数.

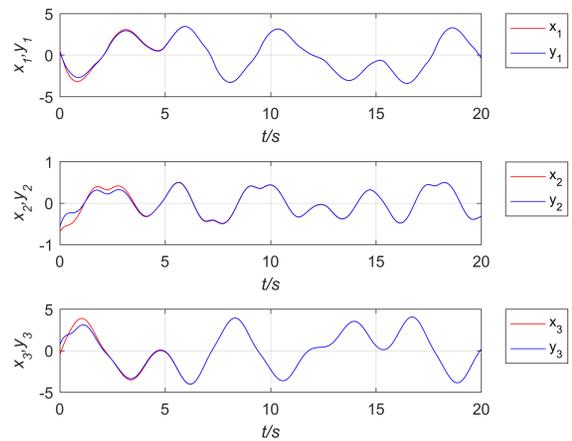


图2 驱动系统与响应系统的状态响应曲线

Fig.2 State responses of drive system and response system

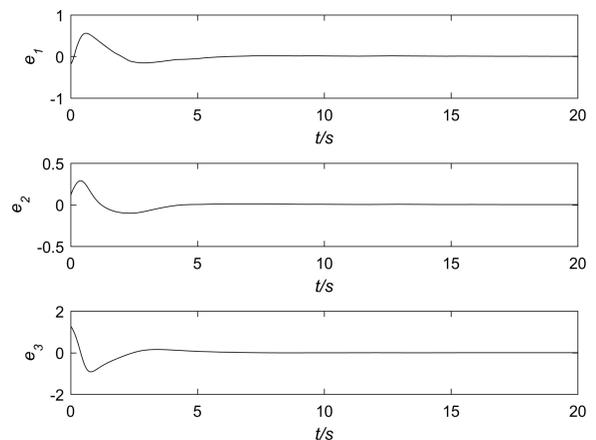


图3 同步误差状态曲线

Fig.3 Synchronous error state curve

## 4 结论

基于分数阶微积分理论和分数阶 Lyapunov 稳

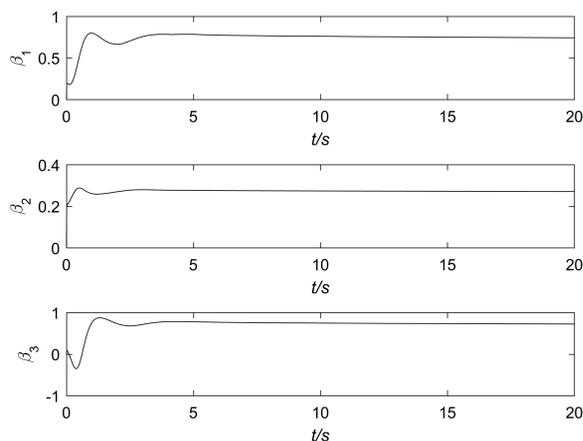


图4 控制器增益的变化曲线

Fig.4 The variation curve of controller gain

定性理论,设计自适应预测控制器和控制增益的分数阶自适应律,实现了分数阶混沌系统的同步,证明了在一定条件下误差系统能渐近趋于零.本文的方法无需反馈增益的先验知识,且收敛速度快和在实验中很容易实现.分数阶Chua系统的数值实验进一步验证了所提同步方法的有效性.

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## ADAPTIVE PREDICTIVE SYNCHRONIZATION OF FRACTIONAL ORDER CHAOTIC SYSTEMS \*

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**Abstract** A new adaptive predictive control method is proposed to realize synchronization of fractional chaotic systems by combining adaptive control and predictive feedback control. Based on theory of fractional order Lyapunov stability, some new sufficient conditions for synchronization of fractional order chaotic systems are derived. Compared with the existing results, this method requires no prior knowledge of feedback gain, and has a fast convergence speed, which is also easy to implement in experiments. Finally, numerical experiments further demonstrate the effectiveness of the proposed synchronization method.

**Key words** fractional-order systems, chaotic synchronization, adaptive control, predictive control

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