

谐和与高斯白噪声联合作用下二自由度系统的 随机稳定性研究*

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摘要 主要研究了谐和与高斯白噪声共同作用下二自由度系统的随机稳定性问题.首先,通过扩维的方式将非自治系统转化为自治系统.其次,利用摄动法和双傅里叶级数展开的方法求得了系统的矩 Lyapunov 指数与最大 Lyapunov 指数的近似解析结果,并和利用 Monte Carlo 仿真得到的数值结果进行了比较验证.最后,通过对系统矩 Lyapunov 指数和最大 Lyapunov 指数解析结果的研究分析,分别讨论了次谐共振和组合共振对系统随机稳定性的影响.

关键词 随机稳定性, 矩 Lyapunov 指数, 最大 Lyapunov 指数, 次谐共振, 组合共振

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引言

稳定性是动力系统的解在半无限时间区间 $[t_0, +\infty)$ 上的一种定性性质.而随机稳定性理论旨在研究随机动力系统平稳解的稳定性,即当初始值偏离平稳态时系统解的收敛性问题;最大 Lyapunov 指数和矩 Lyapunov 指数是研究动力系统随机稳定性(样本稳定性和矩稳定性)的重要指标.

很多工程问题(例如,轴向载荷作用的细长薄板,风载荷作用的高建筑物、桥梁和飞机结构等)的振动方程均具有如下的一般形式:

$$\ddot{q}_i(t) + 2\beta_i \dot{q}_i(t) + \omega_i^2 q_i(t) + \xi(t) \omega_i \sum_{j=1}^2 k_{ij} q_j(t) = 0$$
$$i = 1, 2 \quad (1)$$

其中, q_i 是广义坐标, β_i 和 ω_i 分别是系统的阻尼系数和固有频率, $\xi(t)$ 是随机激励(噪声).

随机动力系统(1)的样本稳定性是基于最大 Lyapunov 指数来判别的,其定义为

$$\lambda = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \|\mathbf{q}(t)\| \quad (2)$$

其中, $\mathbf{q} = [q_1, \dot{q}_1, q_2, \dot{q}_2]^T$, $\|\mathbf{q}\| = (\mathbf{q}^T \mathbf{q})^{1/2}$ 表示向量范数.当 $\lambda < 0$ 时,系统(1)是概率 1 稳定(样本稳定)的;反之, $\lambda > 0$ 时,系统(1)不稳定.最大 Lyapunov

指数 λ 的零点称为系统(1)概率 1 意义上的分岔点,即 D-分岔点.对于噪声诱导的平衡态附近的运动,大偏差理论揭示了一个事实:即使是概率 1 稳定($\lambda < 0$)的随机动力系统,其响应的 p 阶矩仍可以不稳定并呈指数增长.为了更全面地研究分析随机动力系统的稳定性问题,进一步研究系统的矩 Lyapunov 指数(矩稳定性)是很有必要的.系统(1)的 p 阶矩 Lyapunov 指数定义为

$$\Lambda(p) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log E \left[\|\mathbf{q}(t)\|^p \right] \quad (3)$$

其中 $E[\cdot]$ 表示数学期望.当 $\Lambda(p) < 0$ (即: $t \rightarrow +\infty$ 时, $E[\|\mathbf{q}(t)\|^p] \rightarrow 0$) 时,系统(1)是 p 阶矩稳定的;反之, $\Lambda(p) > 0$ 时,系统(1)是 p 阶矩不稳定.根据 Arnold^[1] 的结论,最大 Lyapunov 指数只是矩 Lyapunov 指数关于 p 的一阶项系数,即

$$\lambda = \left. \frac{d}{dp} \Lambda(p) \right|_{p=0} \quad (4)$$

此外人们还发现:对于一些系统,其 P-分岔点就等于矩 Lyapunov 指数的第二个零点在 $-d$ 处的值^[2](d 为系统的维数).因此矩 Lyapunov 指数具有更重要的动力学意义,能够确定随机动力系统的 D-分岔点、P-分岔点以及矩稳定性条件,可以更完

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整和更深刻地刻画系统的随机分岔行为.

尽管矩 Lyapunov 指数在研究随机动力系统的稳定性中是非常重要的, 但由于其求解过程十分困难, 目前关于它的研究结果还比较有限, 并且大部分结果均是基于近似方法(例如, 摄动法和随机平均法等)而求得的. 对于实噪声系统, 其理论基础是 Arnold 的一个著名定理^[3]. 基于这一定理, Arnold 等^[4]首次运用摄动法得到了一组二维实噪声系统的 p 阶矩 Lyapunov 指数关于小的 p 值以及弱噪声强度的双渐近展开表达式. Namachchivaya 等^[5]利用同样的方法得到了实噪声参激的两耦合振子系统的 p 阶矩 Lyapunov 指数的渐近解析结果. 对于一组二维 Ito 随机微分系统, Khasminskii 和 Moshchuk^[6]证明: 无需紧流形的条件, Arnold 的定理同样成立. 此外, 对于有限的实数 p , 系统的 p 阶矩 Lyapunov 指数以及稳定指数存在关于弱噪声强度的渐近展式. 这个结论的重要意义在于它可以揭示大偏差现象, 因为大偏差现象只发生在 p 取有限值时. 基于 Khasminskii 的结果, Namachchivaya 等^[7, 8]在文献[5]研究的基础上, 对于同样的系统同样的随机激励, 得到了有限的 p 阶矩 Lyapunov 指数对弱噪声强度的渐近展式. Xie 等^[9-12]应用摄动法和随机平均法分别计算了几类随机系统的有限 p 阶矩 Lyapunov 指数以及稳定性指标的渐近式. 此外, 刘先斌等^[13-17]分别研究了受白噪声、实噪声参数激励的二、三、四维系统的 p 阶矩 Lyapunov 指数.

在实际工程应用中, 作用在弹性系统上的激励通常是随机力, 例如, 地震、风和海浪, 而且在许多实际状况下, 周期激励和随机激励常常是同时存在的^[18, 19]. 因此, 研究谐和与随机力联合参激下弹性系统的动力学行为有着非常重要的实际价值和意义. 对于谐和与噪声共同作用的二维系统, Xie^[20, 21]运用摄动法得到了系统 p 阶矩 Lyapunov 指数的弱噪声强度的渐近展开表达式. 近期, 胡栋梁和刘先斌等针对谐和与噪声共同作用的高维系统的 p 阶矩 Lyapunov 指数开展了研究并取得了一定的成果^[22, 23]. 本文在前期研究的基础上, 进一步研究了谐和与白噪声联合参激作用下的两自由度系统的随机稳定性以及参数共振对系统随机稳定性的影响. 首先, 应用摄动法得到了系统 p 阶矩 Lyapunov 指数对弱噪声强度的渐近展式, 基于矩 Lyapunov 指数和最大 Lyapunov 指数讨论了参数共振以及参数共振附近随机因素对系统随机稳定性的影响.

1 数学模型

考查如下的两自由度随机动力系统

$$\begin{aligned} \ddot{q}_1 + 2\varepsilon^2\beta_1\dot{q}_1 + \omega_1^2q_1 + \varepsilon\omega_1(k_{11}q_1 + k_{12}q_2)f(t) &= 0, \\ \ddot{q}_2 + 2\varepsilon^2\beta_2\dot{q}_2 + \omega_2^2q_2 + \varepsilon\omega_2(k_{21}q_1 + k_{22}q_2)f(t) &= 0 \end{aligned} \quad (5)$$

其中, $f(t) = \varepsilon \cos\omega t + \xi(t)$, $\xi(t)$ 是高斯白噪声, 噪声强度为 $2D$. ε 是一个小量, 引入 ε 是为了便于分析问题.

通过引入变换 $q_i = x_{2i-1}$, $\dot{q}_i = \omega_i x_{2i}$, $i = 1, 2$, 方程(5)可化为

$$\begin{aligned} d\mathbf{x} &= \left(\mathbf{A}_0 \mathbf{x} - \varepsilon^2 (2\mathbf{A}_1 + \mathbf{B} \cos\omega t) \mathbf{x} \right) dt - \\ &\quad \varepsilon \sqrt{2D} \mathbf{B} \mathbf{x} \cdot dW(t) \end{aligned} \quad (6)$$

其中,

$$\begin{aligned} \mathbf{A}_0 &= \begin{bmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{bmatrix}, \\ \mathbf{A}_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ k_{11} & 0 & k_{12} & 0 \\ 0 & 0 & 0 & 0 \\ k_{21} & 0 & k_{22} & 0 \end{bmatrix} \end{aligned} \quad (7)$$

令 $\varphi = \omega t$, φ 可以被考虑为一随机过程, 其微分算子 $G = \omega \partial / \partial \varphi$, $G^* = -\omega \partial / \partial \varphi$.

引入变换

$$\begin{aligned} x_1 &= r \cos\phi_1 \cos\theta, x_2 = -r \sin\phi_1 \cos\theta, \\ x_3 &= r \cos\phi_2 \sin\theta, \\ x_4 &= -r \sin\phi_2 \sin\theta, P = \|\mathbf{x}\|^p = r^p, \\ \phi_1, \phi_2 &\in [0, 2\pi], \theta \in (0, \pi/2). \end{aligned}$$

代入(6)式并整理得

$$\begin{aligned} dP &= \varepsilon^2 p (\rho_2 + \rho_1 \cos\varphi) P dt + \varepsilon \sqrt{2D} p \rho_1 P \cdot dW(t), \\ d\theta &= \varepsilon^2 (\theta_2 + \theta_1 \cos\varphi) dt + \varepsilon \sqrt{2D} \theta_1 \cdot dW(t), \\ d\phi_1 &= (\omega_1 + \varepsilon^2 (\phi_{12} + \phi_{11} \cos\varphi)) dt + \varepsilon \sqrt{2D} \phi_{11} \cdot dW(t), \\ d\phi_2 &= (\omega_2 + \varepsilon^2 (\phi_{22} + \phi_{21} \cos\varphi)) dt + \varepsilon \sqrt{2D} \phi_{21} \cdot dW(t) \end{aligned} \quad (8)$$

其中,

$$\begin{aligned} \rho_2 &= q_2^0(\theta) + q_2^1(\theta) \cos 2\phi_1 + q_2^2(\theta) \cos 2\phi_2, \\ q_2^0(\theta) &= -\frac{1}{2} [(\beta_1 + \beta_2) + (\beta_1 - \beta_2) \cos 2\theta], \\ q_2^1(\theta) &= \frac{1}{2} \beta_1 (1 + \cos 2\theta), \\ q_2^2(\theta) &= \frac{1}{2} \beta_2 (1 - \cos 2\theta), \end{aligned}$$

$$\theta_2 = s_2^0(\theta) + s_2^{c1}(\theta) \cos 2\phi_1 + s_2^{c2}(\theta) \cos 2\phi_2,$$

$$s_2^0(\theta) = \frac{1}{2}(\beta_1 - \beta_2) \sin 2\theta,$$

$$s_2^{c1}(\theta) = -\frac{1}{2}\beta_1 \sin 2\theta, \quad s_2^{c2}(\theta) = \frac{1}{2}\beta_2 \sin 2\theta,$$

$$\phi_{12} = -\beta_1 \sin 2\phi_1,$$

$$\phi_{22} = -\beta_2 \sin 2\phi_2,$$

$$\rho_1 = q_1^{s1}(\theta) \sin 2\phi_1 + q_1^{s2}(\theta) \sin 2\phi_2 + q_1^{sc}(\theta) \sin \phi_1 \cos \phi_2 + q_1^{cs}(\theta) \cos \phi_1 \sin \phi_2,$$

$$q_1^{s1}(\theta) = \frac{1}{4}k_{11}(1 + \cos 2\theta),$$

$$q_1^{s2}(\theta) = \frac{1}{4}k_{22}(1 - \cos 2\theta),$$

$$q_1^{sc}(\theta) = \frac{1}{2}k_{12} \sin 2\theta, \quad q_1^{cs}(\theta) = \frac{1}{2}k_{21} \sin 2\theta,$$

$$\theta_1 = s_1^{s1}(\theta) \sin 2\phi_1 + s_1^{s2}(\theta) \sin 2\phi_2 + s_1^{sc}(\theta) \sin \phi_1 \cos \phi_2 + s_1^{cs}(\theta) \cos \phi_1 \sin \phi_2,$$

$$s_1^{s1}(\theta) = -\frac{1}{4}k_{11} \sin 2\theta, \quad s_1^{s2}(\theta) = \frac{1}{4}k_{22} \sin 2\theta,$$

$$s_1^{sc}(\theta) = -\frac{1}{2}k_{12}(1 - \cos 2\theta),$$

$$s_1^{cs}(\theta) = \frac{1}{2}k_{21}(1 + \cos 2\theta),$$

$$\phi_{11} = \frac{1}{2}k_{11}(1 + \cos 2\phi_1) + k_{12} \tan \theta \cos \phi_1 \cos \phi_2,$$

$$\phi_{21} = \frac{1}{2}k_{22}(1 + \cos 2\phi_2) + k_{21} \cot \theta \cos \phi_1 \cos \phi_2.$$

通过引入 Wong-Zakai 修正项, 方程(8)可以转化为 Itô 随机微分方程, 即

$$\begin{aligned} dP &= \varepsilon^2 p(\tilde{\rho}_2 + \rho_1 \cos \varphi) P dt + \varepsilon \sqrt{2D} p \rho_1 P dW(t), \\ d\theta &= \varepsilon^2 (\tilde{\theta}_2 + \theta_1 \cos \varphi) dt + \varepsilon \sqrt{2D} \theta_1 dW(t), \\ d\phi_1 &= (\omega_1 + \varepsilon^2 (\tilde{\phi}_{12} + \phi_{11} \cos \varphi)) dt + \varepsilon \sqrt{2D} \phi_{11} dW(t), \\ d\phi_2 &= (\omega_2 + \varepsilon^2 (\tilde{\phi}_{22} + \phi_{21} \cos \varphi)) dt + \varepsilon \sqrt{2D} \phi_{21} dW(t) \end{aligned} \quad (9)$$

其中,

$$\tilde{\rho}_2 = \rho_2 + pD\rho_1^2 + D \left[\theta_1 \frac{\partial}{\partial \theta} + \phi_{11} \frac{\partial}{\partial \phi_1} + \phi_{21} \frac{\partial}{\partial \phi_2} \right] \rho_1,$$

$$\tilde{\theta}_2 = \theta_2 + D \left[\theta_1 \frac{\partial}{\partial \theta} + \phi_{11} \frac{\partial}{\partial \phi_1} + \phi_{21} \frac{\partial}{\partial \phi_2} \right] \theta_1,$$

$$\tilde{\phi}_{12} = \phi_{12} + D \left[\theta_1 \frac{\partial}{\partial \theta} + \phi_{11} \frac{\partial}{\partial \phi_1} + \phi_{21} \frac{\partial}{\partial \phi_2} \right] \phi_{11},$$

$$\tilde{\phi}_{22} = \phi_{22} + D \left[\theta_1 \frac{\partial}{\partial \theta} + \phi_{11} \frac{\partial}{\partial \phi_1} + \phi_{21} \frac{\partial}{\partial \phi_2} \right] \phi_{21}.$$

2 矩 Lyapunov 指数的摄动近似展开

令 $\varphi = z + (\alpha_1 \phi_1 + \alpha_2 \phi_2)$, $\omega = \omega_0 + \varepsilon^2 \Delta$, 其中 $\omega_0 = \alpha_1 \omega_1 + \alpha_2 \omega_2$, Δ 为频率调谐因子, α_1 和 α_2 为常数. 由方程(9)和 $\varphi = \omega t$ 可知, 随机过程 $(\phi_1, \phi_2, \theta, z)$ 是与变量 P 无关的扩散过程, 其后向 Kolmogorov 微分算子(也就是, FPK 微分算子的伴随)定义为

$$L_\varepsilon(p) = L_0(p) + \varepsilon^2 L_1(p) \quad (10)$$

其中,

$$L_0(p) = \omega_1 \frac{\partial}{\partial \phi_1} + \omega_2 \frac{\partial}{\partial \phi_2},$$

$$\begin{aligned} L_1(p) &= p(\tilde{\rho}_2 + \rho_1 \cos \varphi) + [\tilde{\theta}_2 + \theta_1 \cos \varphi + 2pD\rho_1\theta_1] \frac{\partial}{\partial \theta} \\ &\quad + [\tilde{\phi}_{12} + \phi_{11} \cos \varphi + 2pD\rho_1\phi_{11}] \frac{\partial}{\partial \phi_1} \\ &\quad + [\tilde{\phi}_{22} + \phi_{21} \cos \varphi + 2pD\rho_1\phi_{21}] \frac{\partial}{\partial \phi_2} \\ &\quad + [\Delta - \alpha_1(\tilde{\phi}_{12} + \phi_{11} \cos \varphi + 2pD\rho_1\phi_{11}) \\ &\quad - \alpha_2(\tilde{\phi}_{22} + \phi_{21} \cos \varphi + 2pD\rho_1\phi_{21})] \frac{\partial}{\partial z} \\ &\quad + D \left[\theta_1 \frac{\partial}{\partial \theta} + \phi_{11} \frac{\partial}{\partial \phi_1} + \phi_{21} \frac{\partial}{\partial \phi_2} - \right. \\ &\quad \left. (\alpha_1 \phi_{11} + \alpha_2 \phi_{21}) \frac{\partial}{\partial z} \right]^2 \end{aligned}$$

矩 Lyapunov 指数 $\Lambda(p)$ 是算子 $L_\varepsilon(p)$ 的特征值^[1, 13], 即

$$L_\varepsilon(p)\psi(p) = \Lambda(p)\psi(p) \quad (11)$$

分别对矩 Lyapunov 指数 $\Lambda(p)$ 及其相应的特征函数 $\psi(p)$ 进行级数展开可以得到

$$\begin{aligned} \Lambda(p) &= \Lambda_0(p) + \varepsilon^2 \Lambda_1(p) + \varepsilon^4 \Lambda_2(p) + \cdots + \varepsilon^{2n} \Lambda_n(p) + \cdots, \\ \psi(p) &= \psi_0(p) + \varepsilon^2 \psi_1(p) + \varepsilon^4 \psi_2(p) + \cdots + \varepsilon^{2n} \psi_n(p) + \cdots \end{aligned} \quad (12)$$

将(12)式代入(11)式可以得到

$$\begin{aligned} \varepsilon^0: \quad &L_0\psi_0 = \Lambda_0\psi_0, \\ \varepsilon^2: \quad &L_0\psi_1 = \Lambda_0\psi_1 + \Lambda_1\psi_0 - L_1\psi_0, \\ \varepsilon^4: \quad &L_0\psi_2 = \Lambda_0\psi_2 + \Lambda_1\psi_1 + \Lambda_2\psi_0 - L_1\psi_1, \\ &\vdots \end{aligned} \quad (13)$$

2.1 零阶摄动解

根据 $\Lambda(p)$ 的定义和性质可知, $\Lambda_0(p) = 0$, 从而(13)式的第一个方程可以简化为

$$\left[\omega_1 \frac{\partial}{\partial \phi_1} + \omega_2 \frac{\partial}{\partial \phi_2} \right] \psi_0(\phi_1, \phi_2, \theta, z) = 0 \quad (14)$$

应用分离变量法对方程(14)进行求解, 令 $\psi_0(\phi_1, \phi_2, \theta, z) = \Phi_1(\phi_1)\Phi_2(\phi_2)\Theta(\theta)Z(z)$, 则方程(14)可化为

$$\frac{\dot{\Phi}_1}{\Phi_1} = c_1, \frac{\dot{\Phi}_2}{\Phi_2} = c_2 \quad (15)$$

可得, $\Phi_1(\phi_1) = C_1 e^{c_1 \phi_1}$, $\Phi_2(\phi_2) = C_2 e^{c_2 \phi_2}$; 由周期性边界条件

$$\psi_0(\phi_1 + 2\pi, \phi_2, \theta, z) = \psi_0(\phi_1, \phi_2 + 2\pi, \theta, z) = \psi_0(\phi_1, \phi_2, \theta, z),$$

可知, $c_1 = c_2 = 0$. 所以我们可以得到

$$\psi_0(\phi_1, \phi_2, \theta, z) = \psi_0(\theta, z)$$

$$L_0(p)\psi_1 = [\Lambda_1(p) - L_1(p)]\psi_0(\theta, z)$$

$$= [\Lambda_1(p) - p(\tilde{\rho}_2 + \rho_1 \cos(z + \alpha_1 \phi_1 + \alpha_2 \phi_2))] \psi_0(\theta, z)$$

$$- [\tilde{\theta}_2 + \theta_1 \cos(z + \alpha_1 \phi_1 + \alpha_2 \phi_2) + 2pD\rho_1 \theta_1] \frac{\partial \psi_0(\theta, z)}{\partial \theta}$$

$$+ \alpha_1 (\tilde{\phi}_{12} + \phi_{11} \cos(z + \alpha_1 \phi_1 + \alpha_2 \phi_2) + 2pD\rho_1 \phi_{11}) \frac{\partial \psi_0(\theta, z)}{\partial z}$$

$$+ \alpha_2 (\tilde{\phi}_{22} + \phi_{21} \cos(z + \alpha_1 \phi_1 + \alpha_2 \phi_2) + 2pD\rho_1 \phi_{21}) \frac{\partial \psi_0(\theta, z)}{\partial z}$$

$$- \Delta \frac{\partial \psi_0(\theta, z)}{\partial z} - D \left[\theta_1 \frac{\partial}{\partial \theta} - (\alpha_1 \phi_{11} + \alpha_2 \phi_{21}) \frac{\partial}{\partial z} \right]^2 \psi_0(\theta, z)$$

方程(19)的可解性条件为

$$\begin{aligned} & \langle [\Lambda_1(p) - L_1(p)] \psi_0(\vartheta, z), \Psi_0(\theta, z) \rangle \\ &= \frac{1}{4\pi^2} \int_0^{\pi/2} \int_0^{2\pi} \mathcal{F}(\theta, z) \int_0^{2\pi} \int_0^{2\pi} [\Lambda_1(p) - L_1(p)] \psi_0(\theta, z) \\ & d\phi_1 d\phi_2 dz d\theta = 0 \end{aligned} \quad (20)$$

对于任意的函数 $\mathcal{F}(\theta, z)$, (20)式恒成立, 因此, 我们可以得到

$$\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} [\Lambda_1(p) - L_1(p)] \psi_0(\theta, z) d\phi_1 d\phi_2 = 0 \quad (21)$$

$$\mu_z = -\frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \left(\alpha_1 \left(\begin{aligned} & \tilde{\phi}_{12} + \phi_{11} \cos(z + \alpha_1 \phi_1 + \alpha_2 \phi_2) \\ & + 2pD\rho_1 \phi_{11} \end{aligned} \right) + \alpha_2 \left(\begin{aligned} & \tilde{\phi}_{22} + \phi_{21} \cos(z + \alpha_1 \phi_1 + \alpha_2 \phi_2) \\ & + 2pD\rho_1 \phi_{21} \end{aligned} \right) - \Delta \right) d\phi_1 d\phi_2,$$

$$\tilde{Q} = \frac{p}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \rho_1 \cos(z + \alpha_1 \phi_1 + \alpha_2 \phi_2) d\phi_1 d\phi_2,$$

$$\sigma_{\theta\theta}^2(\theta) = \tilde{a} \cos^2 2\theta + \tilde{b} \cos 2\theta + \tilde{c},$$

$$\mu_\theta = \mu_1 + p\mu_2,$$

$$\mu_1(\theta) = \sigma_{\theta\theta}^2(\theta) \cot 2\theta - (\Delta_1 - \Delta_2) \sin 2\theta,$$

$$\theta \in (0, \pi/2), z \in [0, 2\pi] \quad (16)$$

方程(14)的伴随方程为

$$L_0^* \Psi_0 = 0 \quad (17)$$

利用同样的方法可以求得

$$\Psi_0(\theta, z) = \frac{\mathcal{F}(\theta, z)}{4\pi^2}, \quad \theta \in \left(0, \frac{\pi}{2}\right), z \in [0, 2\pi] \quad (18)$$

式中 $\mathcal{F}(\theta, z)$ 为任意函数.

2.2 二阶摄动解

把(16)式代入(13)式的第二个方程得

$$\begin{aligned} & \tilde{L}(p)\psi_0 = \left(\sigma_{\theta\theta}^2 \frac{\partial^2 \psi_0}{\partial \theta^2} + \sigma_z^2 \frac{\partial^2 \psi_0}{\partial z^2} \right) + (\mu_\theta + \tilde{\mu}_\theta) \frac{\partial \psi_0}{\partial \theta} \\ & + \mu_z \frac{\partial \psi_0}{\partial z} + (Q + \tilde{Q}) \psi_0 = \Lambda_1(p) \psi_0 \end{aligned} \quad (19)$$

经过一系列积分计算, 可解性条件(21)可以简化为

$$\begin{aligned} \tilde{L}(p)\psi_0 &= \left(\sigma_{\theta\theta}^2 \frac{\partial^2 \psi_0}{\partial \theta^2} + \sigma_z^2 \frac{\partial^2 \psi_0}{\partial z^2} \right) + (\mu_\theta + \tilde{\mu}_\theta) \frac{\partial \psi_0}{\partial \theta} \\ & + \mu_z \frac{\partial \psi_0}{\partial z} + (Q + \tilde{Q}) \psi_0 = \Lambda_1(p) \psi_0 \end{aligned} \quad (22)$$

其中:

$$\sigma_z^2 = \frac{D}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} (\alpha_1 \phi_{11} + \alpha_2 \phi_{21})^2 d\phi_1 d\phi_2,$$

$$\tilde{\mu}_\theta = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \theta_1 \cos(z + \alpha_1 \phi_1 + \alpha_2 \phi_2) d\phi_1 d\phi_2,$$

$$\mu_2(\theta) = \frac{1}{2} (2\tilde{a} \cos 2\theta + (\tilde{b} - \tilde{d})) \sin 2\theta,$$

$$Q = pQ_1 + \frac{1}{2} p^2 Q_2,$$

$$Q_1(\theta) = \sigma_{\theta\theta}^2(\theta) + (\Delta_1 - \Delta_2) \cos 2\theta + (\Delta_1 + \Delta_2),$$

$$Q_2(\theta) = -\tilde{a} \cos^2 2\theta + \tilde{d} \cos 2\theta + \tilde{c},$$

$$\tilde{a} = \frac{D}{16} (2(k_{12}^2 + k_{21}^2) - (k_{11}^2 + k_{22}^2)),$$

$$\tilde{b} = \frac{D}{4} (k_{21}^2 - k_{12}^2),$$

$$\tilde{c} = \frac{D}{16} (2(k_{12}^2 + k_{21}^2) + (k_{11}^2 + k_{22}^2)),$$

$$\tilde{d} = \frac{D}{8} (k_{11}^2 - k_{22}^2),$$

$$\Delta_i = -\frac{1}{2}\beta_i + \frac{D}{8}k_{ii}^2, \quad i = 1, 2$$

p 阶矩 Lyapunov 指数的二阶摄动解 $\Lambda_1(p)$ 为 (22) 式的最大特征值, $\psi_0(\theta, z)$ 是相应的特征函数.

当 ε 充分小时, 由于 $\Lambda_0(p) = 0$, 随机系统的 p 阶矩 Lyapunov 指数 $\Lambda(p)$ 主要取决于二阶项 $\Lambda_1(p)$, 其他的更高阶项对矩 Lyapunov 指数影响甚微, 可以忽略. 因此, 系统的 p 阶矩 Lyapunov 指数的二阶近似解为

$$\Lambda(p) \cong \varepsilon^2 \Lambda_1(p).$$

所以, 求解系统矩 Lyapunov 指数的近似解就

$$\hat{C}_{rsmn} = \frac{(1 + \operatorname{sgn}(m))(1 + \operatorname{sgn}(n))}{\pi^2} \int_0^{\pi/2} \int_0^{2\pi} \tilde{L}(p) (\cos rz \cos 2s\theta) \times \cos mz \cos 2n\theta dz d\theta,$$

$$\bar{C}_{rsmn} = \frac{(1 + \operatorname{sgn}(m))(1 + \operatorname{sgn}(n))}{\pi^2} \int_0^{\pi/2} \int_0^{2\pi} \tilde{L}(p) (\sin rz \cos 2s\theta) \times \cos mz \cos 2n\theta dz d\theta,$$

$$\hat{D}_{rsmn} = \frac{(1 + \operatorname{sgn}(m))(1 + \operatorname{sgn}(n))}{\pi^2} \int_0^{\pi/2} \int_0^{2\pi} \tilde{L}(p) (\cos rz \cos 2s\theta) \times \sin mz \cos 2n\theta dz d\theta,$$

$$\bar{D}_{rsmn} = \frac{(1 + \operatorname{sgn}(m))(1 + \operatorname{sgn}(n))}{\pi^2} \int_0^{\pi/2} \int_0^{2\pi} \tilde{L}(p) (\sin rz \cos 2s\theta) \times \sin mz \cos 2n\theta dz d\theta.$$

欲使方程(24)存在非平凡解, 则其相应的系数矩阵的行列式应为零. 故而, 计算 $\Lambda_1(p)$ 就转化为求解其系数矩阵的最大特征值. 在本文中, 我们采用截断的方式对方程(24)进行近似求解. 考虑截断的系统为

$$\sum_{r=0}^M \sum_{s=0}^N \begin{pmatrix} \hat{C}_{rsmn} & \bar{C}_{rsmn} \\ \hat{D}_{rsmn} & \bar{D}_{rsmn} \end{pmatrix} \begin{pmatrix} u_{rs} \\ v_{rs} \end{pmatrix} = \Lambda_1(p) \begin{pmatrix} u_{mn} \\ v_{mn} \end{pmatrix} \quad (25)$$

通过对截断的方程(25)进行求解, 我们可以得到方程(24)的近似解.

令 $M = N$

$$\mathbf{U} = \begin{bmatrix} u_{00} & u_{01} & \cdots & u_{0N} \\ u_{10} & u_{11} & \cdots & u_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{N0} & u_{N1} & \vdots & u_{NN} \end{bmatrix} = \begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ U_N \end{bmatrix},$$

$$\mathbf{U}_m = [u_{m0} \quad u_{m1} \quad \cdots \quad u_{mN}], \quad m = 0, 1, \dots, N$$

是求(22)式的最大特征值问题. 相应的最大 Lyapunov 指数可以表示为

$$\lambda = \left. \frac{d}{dp} \Lambda(p) \right|_{p=0} \cong \varepsilon^2 \left. \frac{d}{dp} \Lambda_1(p) \right|_{p=0}.$$

2.3 求解特征值问题

由 Bolotin^[24]、Wedig^[25] 和 Namachchivay^[7, 8] 知, $\psi_0(\theta, z)$ 可以展开为正交的双 Fourier 级数的形式, 即

$$\psi_0(\theta, z) = \sum_{n=0}^{\infty} u_{0n} \cos 2n\theta + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (u_{mn} \cos mz + v_{mn} \sin mz) \cos 2n\theta \quad (23)$$

将(23)式代入(22)式, 然后分别对 z 和 θ 进行积分可以得到

$$\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \begin{pmatrix} \hat{C}_{rsmn} & \bar{C}_{rsmn} \\ \hat{D}_{rsmn} & \bar{D}_{rsmn} \end{pmatrix} \begin{pmatrix} u_{rs} \\ v_{rs} \end{pmatrix} = \Lambda_1(p) \begin{pmatrix} u_{mn} \\ v_{mn} \end{pmatrix} \quad (24)$$

其中:

$$\mathbf{V} = \begin{bmatrix} v_{10} & v_{11} & \cdots & v_{1N} \\ v_{20} & v_{21} & \cdots & v_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ v_{N0} & v_{N1} & \vdots & v_{NN} \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_N \end{bmatrix},$$

$$\mathbf{V}_m = [v_{m0} \quad v_{m1} \quad \cdots \quad v_{mN}], \quad m = 1, 2, \dots, N$$

方程(25)可以表示为

$$\mathbf{C}\mathbf{y} = \Lambda_1(p)\mathbf{y} \quad (26)$$

其中,

$$\mathbf{y} = [U_0, U_1, \dots, U_N, V_1, V_2, \dots, V_N]^T \quad (27)$$

因此, 通过求解系数矩阵 \mathbf{C} 的最大特征值, 我们可以得到随机系统 p 阶矩 Lyapunov 指数 $\Lambda(p)$ 的二阶近似 $\Lambda_1(p)$. 我们构建一系列的子矩阵, 通过计算其特征值, 可以得到系数矩阵 \mathbf{C} 最大特征值的一系列近似解; 当 $N \rightarrow \infty$ 时, 其相应的特征值即是我们需要的.

3 参数共振及随机稳定性分析

3.1 数值仿真

我们将通过 Monte Carlo 数值仿真的方法求得系统(5) p 阶矩 Lyapunov 指数 $\Lambda(p)$ 的数值解, 进而对其近似的解析结果进行验证. 令 $y_1 = q_1, y_2 = \dot{q}_1, y_3 = q_2, y_4 = \dot{q}_2$, 系统(5)可化为

$$dY = \left(\tilde{A}_0 + \varepsilon^2 (\tilde{A}_1 + \tilde{B} \cos \omega t) \right) Y dt + \varepsilon \sqrt{2D} \tilde{B} Y \cdot dW(t) \quad (28)$$

其中:

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \tilde{A}_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_1^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_2^2 & 0 \end{bmatrix},$$

$$\tilde{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\beta_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\beta_2 \end{bmatrix},$$

$$\tilde{B} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\omega_1 k_{11} & 0 & -\omega_1 k_{12} & 0 \\ 0 & 0 & 0 & 0 \\ -\omega_2 k_{21} & 0 & -\omega_2 k_{22} & 0 \end{bmatrix}.$$

由于方程(28)比较特殊, 其相应的 Wong-Zakai 修正项恰好等于零, 因此其相应的 Itô 随机微分方程为

$$dY = \left(\tilde{A}_0 + \varepsilon^2 (\tilde{A}_1 + \tilde{B} \cos \omega t) \right) Y dt + \varepsilon \sqrt{2D} \tilde{B} Y dW(t) \quad (29)$$

根据 Xie 和 Huang^[26]可知, 直接对 Itô 随机微分方程(29)进行 Monte Carlo 数值仿真即可求得系统(5) p 阶矩 Lyapunov 指数 $\Lambda(p)$ 的数值解. 在数值仿真的过程中, 系统参数的选择为 $\omega_1 = 3, \omega_2 = 2, k_{11} = 0.2, k_{12} = -0.4, k_{21} = 0.4, k_{22} = 0.4, \beta_1 = \beta_2 = 0.1$.

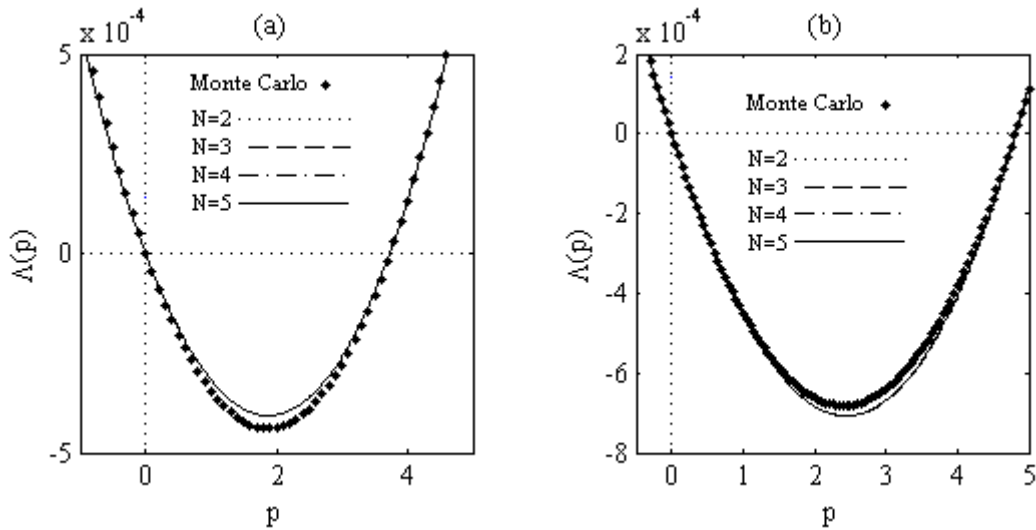


图 1 矩 Lyapunov 指数 $D = 0.5, \Delta = 0$; (a) $\omega_0 = 2\omega_1$; (b) $\omega_0 = \omega_1 + \omega_2$
 Fig.1 Moment Lyapunov exponent $D = 0.5, \Delta = 0$; (a) $\omega_0 = 2\omega_1$; (b) $\omega_0 = \omega_1 + \omega_2$

系统的 p 阶矩 Lyapunov 指数 $\Lambda(p)$ 解析结果的 3-6 阶近似值以及数值结果关于 p 的变化曲线, 如图 1 所示. 由图 1 可知, 随着 N 的增大, 系统 p 阶矩 Lyapunov 指数 $\Lambda(p)$ 的近似解析结果是收敛的. 比较图 1 中的解析结果与数值结果可以发现, 利用正则摄动法获得的近似解析结果和 Monte Carlo 仿真得到的数值结果相差很小. 因此, 本文通过正则摄动法求得的近似解析结果是可靠的. 下文我们将基于 p 阶矩 Lyapunov 指数 $\Lambda(p)$ 解析解的 5 阶 ($N = 4$)

近似值来讨论参数共振对系统随机稳定性的影响.

3.2 参数共振

当 $\omega = 2\omega_1$ (即 $\alpha_1 = 2, \alpha_2 = 0, \Delta = 0$) 时, 系统将会发生次谐共振现象. 图 2 和图 3 分别画出了次谐共振附近, 调频参数 Δ 对系统 p 阶矩 Lyapunov 指数 $\Lambda(p)$ 和最大 Lyapunov 指数 λ 的影响. 从图 3 可以看出, 当调频参数 $|\Delta|$ 比较小时, 系统将发生共振 (即次谐共振), 且共振使得系统的样本稳定性大幅度地减弱; 最大 Lyapunov 指数由 $\lambda < 0$ 变为 $\lambda > 0$,

即系统发生了D-分岔,出现了概率1失稳现象.由图2可知, $\Delta \geq 0$ 时,随着调频参数 Δ 逐渐地增大,系统的稳定性指标越来越大,也就是其矩稳定性在不断地增强.综合图2和图3可知,次谐共振大幅度地降低了系统的随机稳定性,容易导致失稳现象出现;随着调频参数 $|\Delta|$ (也就是,偏离共振的程度)的不断增大,系统越来越稳定.

当 $\omega = \omega_1 + \omega_2$ (即 $\alpha_1 = 1, \alpha_2 = 1, \Delta = 0$) 时,系统将会发生组合共振现象.图4和图5分别画出了组合共振附近,调频参数 Δ 对系统 p 阶矩 Lyapunov 指数 $\Lambda(p)$ 和最大 Lyapunov 指数 λ 的影响.从图5可以看出,当调频参数 $|\Delta|$ 比较小时,系统将发生共振(即组合共振),组合共振的发生不但没有减弱系统的样本稳定性,而且还使得系统的样本稳定性大幅度地增强;最大 Lyapunov 指数由 $\lambda > 0$ 变为 $\lambda < 0$,即系统由不稳定状态变为了稳定状态.由图4可知, $\Delta \geq 0$ 时,随着调频参数 Δ 逐渐地增大,系统的稳定性指标越来越小,也就是其矩稳定性在不断地减弱.综合图4和图5可知,组合共振不但没有减弱系统的随机稳定性,而且还使得系统的随机稳定性大幅度地增强;随着调频参数 $|\Delta|$ (也就是,偏离共振的程度)的不断减小,系统越来越稳定.

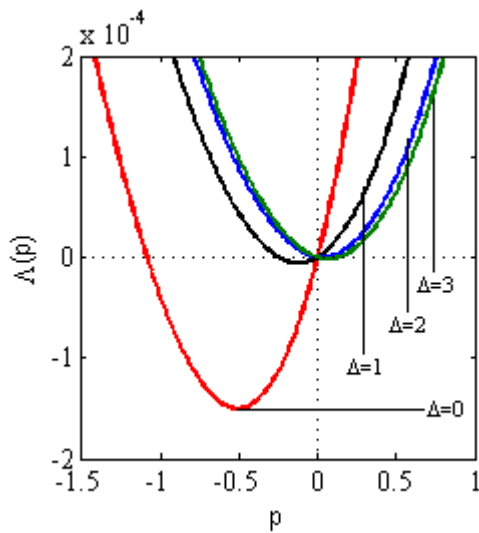


图2 Δ 对矩Lyapunov指数的影响 $k_{11} = 0.8, k_{12} = -0.8,$
 $k_{21} = 0.2, k_{22} = 0.2, D = 0.5$

Fig.2 Effect of parameter Δ on the moment Lyapunov exponent
 $k_{11} = 0.8, k_{12} = -0.8, k_{21} = 0.2, k_{22} = 0.2, D = 0.5$

图6和图7分别讨论了系统出现次谐共振和组合共振时,参激系数对系统最大 Lyapunov 指数的影响.

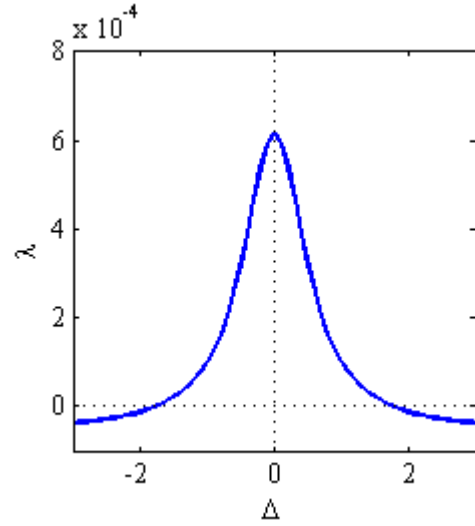


图3 Δ 对最大Lyapunov指数的影响 $k_{11} = 0.8, k_{12} = -0.8,$
 $k_{21} = 0.2, k_{22} = 0.2, D = 0.5$

Fig.3 Effect of parameter Δ on the Largest Lyapunov exponent
 $k_{11} = 0.8, k_{12} = -0.8, k_{21} = 0.2, k_{22} = 0.2, D = 0.5$

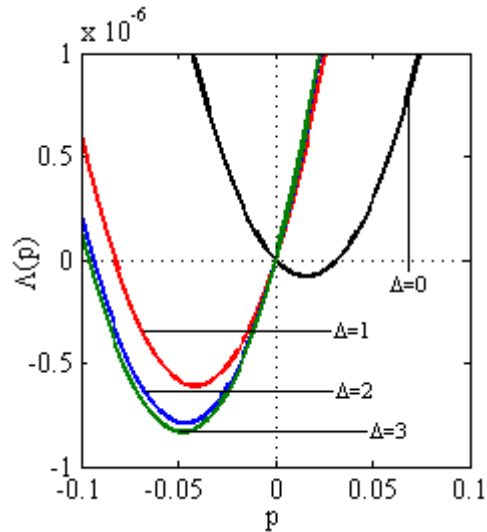


图4 Δ 对矩Lyapunov指数的影响 $k_{11} = 0.8, k_{12} = -0.8,$
 $k_{21} = 0.2, k_{22} = 0.2, D = 0.5$

Fig.4 Effect of parameter Δ on the moment Lyapunov exponent
 $k_{11} = 0.8, k_{12} = -0.8, k_{21} = 0.2, k_{22} = 0.2, D = 0.5$

从图6可以看出,随着参激系数 k_{12}, k_{21} 和 k_{22} 的变化,系统的最大 Lyapunov 指数并没有发生明显的改变;次谐共振发生时,系统的最大 Lyapunov 指数有着急剧的增长.次谐共振发生时,随着参激系数 k_{11} 的变化,系统的最大 Lyapunov 指数虽然有着不同的改变,但次谐共振的发生均使系统的最大 Lyapunov 指数有着显著的增长.总之,随着参激系数 k_{11}, k_{12}, k_{21} 和 k_{22} 的变化,并没有使次谐共振对系统随机稳定性的作用效果有着本质的改变,均使系统的随机稳定性有着显著的减弱.

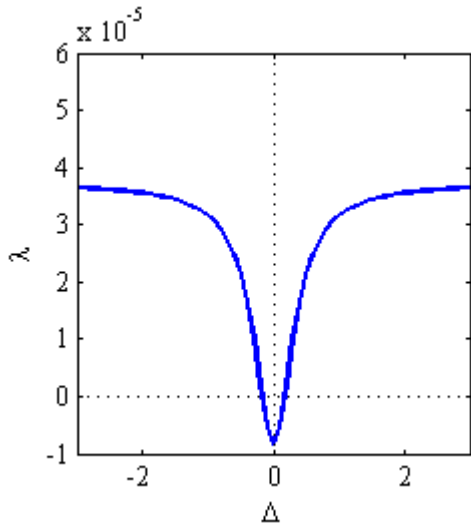


图 5 Δ 对最大 Lyapunov 指数的影响 $k_{11} = 0.8, k_{12} = -0.8,$
 $k_{21} = 0.2, k_{22} = 0.2, D = 0.5$

Fig.5 Effect of parameter Δ on the Largest Lyapunov exponent
 $k_{11} = 0.8, k_{12} = -0.8, k_{21} = 0.2, k_{22} = 0.2, D = 0.5$

而从图 7 可以发现, 随着参激系数 k_{11} 和 k_{22} 的变化, 并没有使组合共振对系统随机稳定性的作用效果有着本质的改变, 组合共振的出现均起到增强系统的随机稳定性的效果; 而随着参激系数 k_{12} 和 k_{21} 的变化, 组合共振对系统随机稳定性的作用效果有着本质的改变 (例如: 从图 7(b) 可以看出, 当 $k_{12} < 0$ 时, 组合共振的出现起到增强系统的随机稳定性的效果; 而当 $k_{12} > 0$ 时, 组合共振的出现则有着减弱系统的随机稳定性的效果)。

对比图 6 和图 7 可知, 次谐共振的出现均使系统的随机稳定性有着显著的减弱, 在实际工程中应尽量避免次谐共振的发生; 而随着不同参数的变化, 组合共振的出现对系统的随机稳定性有着不同的作用效果, 故可以通过调控参激系数来有效地控制组合共振对系统随机稳定性的影响。

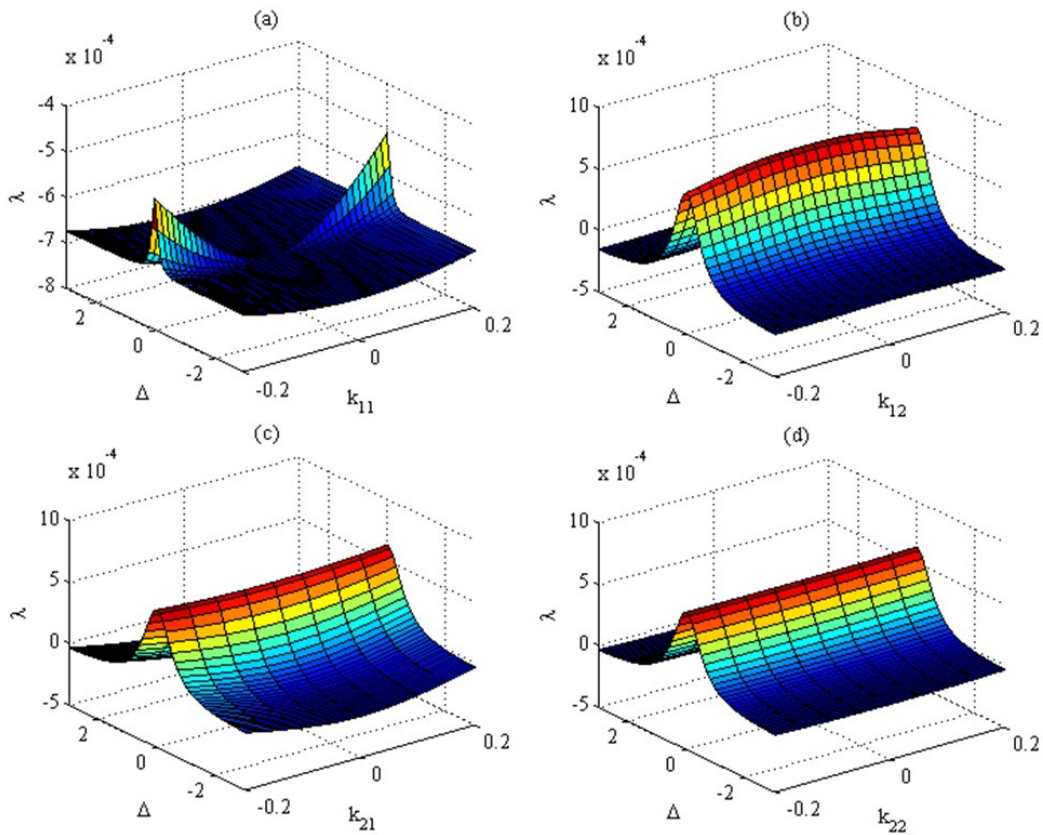


图 6 次谐共振时, 参激系数 k_{11}, k_{12}, k_{21} 和 k_{22} 对最大 Lyapunov 指数的影响:

- (a) $k_{12} = -0.8, k_{21} = 0.2, k_{22} = 0.2$
- (b) $k_{11} = 0.8, k_{21} = 0.2, k_{22} = 0.2$
- (c) $k_{11} = 0.8, k_{12} = -0.8, k_{22} = 0.2$
- (d) $k_{11} = 0.8, k_{12} = -0.8, k_{21} = 0.2$

Fig.6 Effect of parameters k_{11}, k_{12}, k_{21} and k_{22} on the Largest Lyapunov exponent as subharmonic resonance

- (a) $k_{12} = -0.8, k_{21} = 0.2, k_{22} = 0.2$
- (b) $k_{11} = 0.8, k_{21} = 0.2, k_{22} = 0.2$
- (c) $k_{11} = 0.8, k_{12} = -0.8, k_{22} = 0.2$
- (d) $k_{11} = 0.8, k_{12} = -0.8, k_{21} = 0.2$

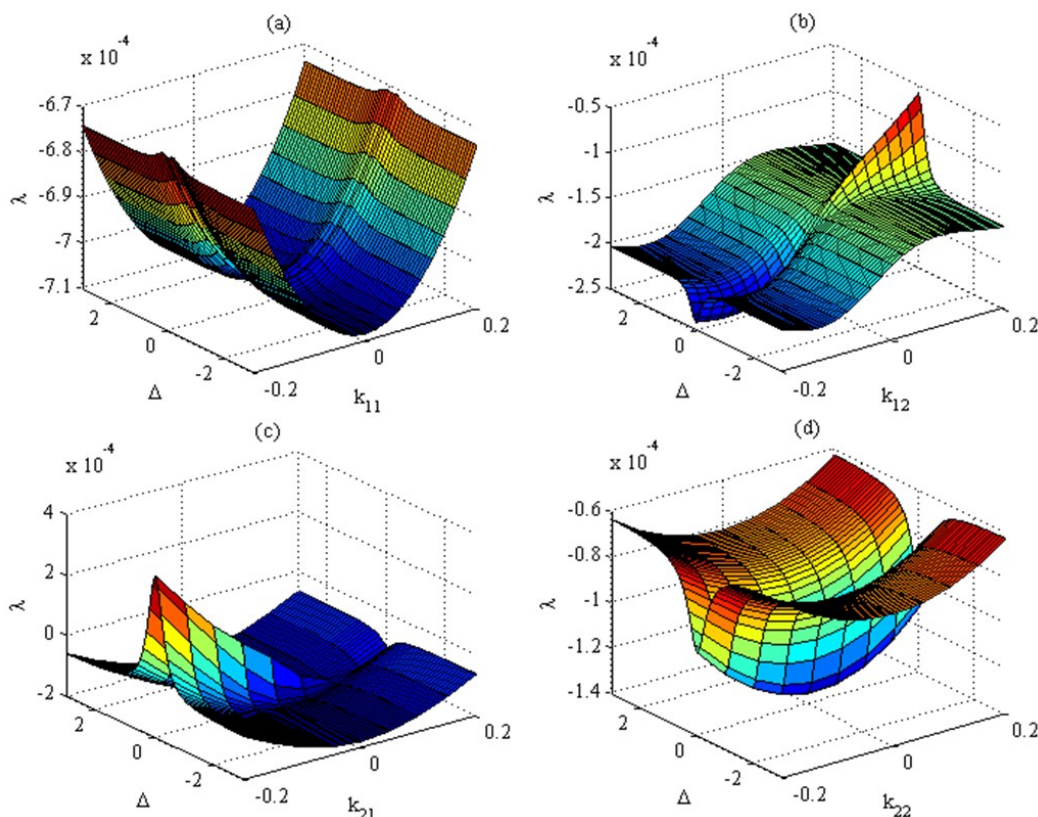


图7 组合共振时,参激系数 k_{11} 、 k_{12} 、 k_{21} 和 k_{22} 对最大Lyapunov指数的影响:

- (a) $k_{12} = -0.8, k_{21} = 0.2, k_{22} = 0.2$ (b) $k_{11} = 0.8, k_{21} = 0.2, k_{22} = 0.2$
 (c) $k_{11} = 0.8, k_{12} = -0.8, k_{22} = 0.2$ (d) $k_{11} = 0.8, k_{12} = -0.8, k_{21} = 0.2$

Fig.7 Effect of parameters k_{11} , k_{12} , k_{21} and k_{22} on the Largest Lyapunov exponent as combination additive resonance

- (a) $k_{12} = -0.8, k_{21} = 0.2, k_{22} = 0.2$ (b) $k_{11} = 0.8, k_{21} = 0.2, k_{22} = 0.2$
 (c) $k_{11} = 0.8, k_{12} = -0.8, k_{22} = 0.2$ (d) $k_{11} = 0.8, k_{12} = -0.8, k_{21} = 0.2$

4 结论

本文基于矩Lyapunov指数研究了谐和与高斯白噪声联合作用下两自由度系统的参数共振以及随机稳定性问题.利用摄动法和双傅里叶级数正展开的方法计算得到了系统的 p 阶矩Lyapunov指数 $\Lambda(p)$ 与最大Lyapunov指数 λ 的近似解析结果;通过对系统的 p 阶矩Lyapunov指数 $\Lambda(p)$ 与最大Lyapunov指数 λ 的分析,分别讨论了次谐共振和组合共振对系统随机稳定性的影响.研究发现:

1)次谐共振的出现均使系统的随机稳定性有着显著的减弱,在实际工程中应尽量避免次谐共振的发生.

2)组合共振的发生可以使得系统的随机稳定性有着大幅度的增强;但随着不同参数的变化,组合共振的出现对系统的随机稳定性有着不同的作用效果(即:组合共振的发生不仅可以增强系统的稳定性,有时还会减弱系统的稳定性),故可以通过

调控参激系数来有效地控制组合共振对系统随机稳定性的影响.

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STOCHASTIC STABILITY OF A TWO DEGREES-OF-FREEDOM SYSTEM UNDER COMBINED HARMONIC AND GAUSSIAN WHITE NOISE EXCITATION *

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Abstract The stochastic stability of a two degrees-of-freedom system under combined harmonic and Gaussian white noise excitation were investigated. Firstly, the non-autonomous system was transformed into an autonomous system by increasing the dimension. Secondly, using both singular perturbation and double Fourier series, the approximate analytical solutions of the moment Lyapunov exponents and the largest Lyapunov exponents were obtained, which agree well with the results obtained by the Monte Carlo simulation. Finally, based on the moment Lyapunov exponents and the largest Lyapunov exponents, the effects of subharmonic resonance and combination additive resonance on the stochastic stability of the two degrees-of-freedom system were discussed.

Key words stochastic stability, moment Lyapunov exponent, largest Lyapunov exponent, subharmonic resonance, combination additive resonance

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