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非保守非线性刚-热-弹耦合动力学的 Lagrange 方程*

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摘要 如何将 Lagrange 方程应用于弹性动力学，一直是国内外学术界关注的理论和应用研究课题.在这类问题获得基本解决之后，Lagrange 方程应用于耦合动力学的理论难题又摆在我们的面前.本文采用 Lagrange-Hamilton 体系，成功地将 Lagrange 方程应用于非保守非线性刚-热-弹耦合动力学.进而应用非保守非线性刚-热-弹耦合动力学的 Lagrange 方程推导出非保守非线性刚-热-弹耦合动力学的控制方程.讨论了应用耦合动力学的 Lagrange 方程解决实际工程技术问题的途径.

关键词 耦合动力学， Lagrange 方程， 非保守系统， 非线性系统， Lagrange-Hamilton 体系

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引言

1788 年 Lagrange 出版的不朽名著《Mecanique Analytique》是世界上最早的一本分析力学的著作[1]，完成了分析力学的 Lagrange 体系. 1834 年 W.R. Hamilton 建立了 Hamilton 原理和正则方程，把分析力学推进一步[2]，完成了分析力学的 Hamilton 体系.1894 年 Hertz 首次将约束和系统分成完整的和非完整的两大类[3]，到 1899 年 Appell 在《理性力学》中提出 Appell 方程为止[4]，基本上已完成了线性非完整约束的理论.

作为对以上开创性工作的继续，20 世纪分析力学对非线性、非完整、非定常、变质量等力学系统作了进一步研究，对于运动的稳定性问题作了广泛的研究.相应的也出现了多部分析力学专著[5-18].文献[19,20] 总结了我国学者对分析力学研究的贡献，文献[21]是我国第一部分析力学专著，文献[22]是我国第一部非完整系统分析力学专著.

我们高兴地看到，进入 21 世纪的不长的历史时期内，理论分析力学和应用分析力学并驾齐驱，出现一派欣欣向荣的景象[23-29]，我国学者也出版了多部分析力学专著[30-32].

航空、航天、航海、机器人、核设施、交通运输和机械制造的高速发展，促使力学向着学科杂交和非线性方向迈进[32-35]，即向着耦合动力学方向迈进.这类耦合动力学，尚无现成的基本方程供我们利用，而且仅仅通过力的分析和位移分析来建立其基本方程相当困难.分析力学的特点是对能量与功的分析代替对力(及其相应的位移)与力矩(及其相应的角位移)的分析，根据问题的物理背景，应用功能转换原理和能量守恒定律，正确建立耦合运动的动能和势能，应用能量法来研究问题是一条可行的途径.应用能量法来研究问题可以从两个方面着手：(1)应用耦合动力学的 Hamilton 型变分原理，通过对耦合动力学的变分原理求驻值，得到耦合动力学的基本微分方程，通过求解微分方程求得各类耦合运动动力学问题的合理解；也可以，从各类耦合运动动力学的 Hamilton 型变分原理出发，应用变分直接方法—Ritz 方法或者有限元素法，直接求得各类耦合运动动力学问题的近似解.这种方法能够方便地应用电子计算机进行计算.(2)应用耦合动力学的 Lagrange 方程，推导耦合动力学的基本微分方程，通过求解微分方程求得各类耦合运动动力学问题的合理解.可以采用两种方法建立耦合动力学的 Lagrange 方程：一是借助 Lagrange-Hamilton 体系，对耦合动力学的 Hamilton 型变分原理求驻值，得到耦合动力学的 Lagrange 方程；二是将前面建立耦合运动的动能和势能代入一般的 Lagrange 方程

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中,从而获得耦合动力学的 Lagrange 方程;不赘述.研究表明,也可以从各类耦合运动动力学的 Lagrange 方程出发,应用变分直接方法—Ritz 方法或者有限元素法,直接求得各类耦合运动动力学问题的近似解.这种方法也能够方便地应用电子计算机进行计算.文献[36]的工作表明,分析动力学的这个研究方向有着广阔的发展前景.

如何将 Lagrange 方程应用于弹性动力学的问题,一直是各国学者关注的理论和应用研究课题.由于这类问题的难度较大,进展比较缓慢.我国学者的研究是卓有成效的,注意到将分析力学从质点刚体力学扩展到弹性力学、从离散系统扩展到连续系统的问题,锲而不舍地研究将 Lagrange 方程应用于弹性动力学[21,30-32,36-41].国外学者的研究也层出不穷,近期研究内容涉及 Euler - Bernoulli 梁理论[42],不同截面形式柱状结构的稳定性问题[43],各向异性的壳结构[44],广义位移和柯西应力问题[45],声振系统[46],流致振动分析[47].国际知名学者 Goldstein 的著作《Classical Mechanics》[11],从第一版到第三版都将此作为一个专题,研究将 Lagrange 方程应用于弹性动力学的问题,为解决这个理论难题做出重要贡献,可以代表部分国际学者对这一领域的研究的历史和现状.本文作者应用变导的概念和运算法则,研究了 Lagrange 方程中的求导的性质,进而将 Lagrange 方程应用于线性弹性动力学[48]和非线性弹性动力学[41].应用这种方法也可以将 Lagrange 方程应用于流体力学和电动力学等学科[36].

在 Lagrange 方程应用于弹性力学问题获得基本解决之后,Lagrange 方程应用于耦合动力学的理论难题又摆在我们的面前.适应航天、航空和航海工程技术的需要,本文作者研究了 Lagrange 方程应用于线性刚-弹耦合动力学[48],进而研究了 Lagrange 方程应用于非线性刚-弹耦合动力学[49].

本文研究 Lagrange 方程应用于非保守非线性刚-热-弹耦合动力学的问题.采用 Lagrange-Hamilton 体系:对于保守系统,Lagrange 方程是 Hamilton 变分原理的驻值条件,对于非保守系统,Lagrange 方程是 Hamilton 型拟变分原理的拟驻值条件[32,36].成功地将 Lagrange 方程应用于非保守非线性刚-热-弹耦合动力学.进而应用非保守非线性刚-热-弹耦合动力学的 Lagrange 方程推导出非保守非线性刚-热-弹耦合动力学的控制方程.讨论了应用耦合动力学的 Lagrange 方程解决实际工程技术问题的途径,并且给出一个算例.

1 刚-热-弹耦合系统的 Lagrange 方程

假设作用在物体质心的主矢和主矩,以及作用在物体上的体积力和面积力,均为既有保守力又有非保守力,采用实体张量符号,可以将非保守非线性刚-热-弹耦合系统的一类变量的 Hamilton 拟变分原理表示为:

$$\delta\pi_H-\delta Q=0 \tag{1}$$

式中:

$$\begin{aligned}\boldsymbol{\pi}_H=\int_{t_0}^{t_1}\Big[\iiint_V\Big(\frac{1}{2}\rho\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}+\rho\Big(\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\boldsymbol{x}\Big)\cdot\\\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}+\frac{1}{2}\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\Big)\mathrm{d}V+\frac{1}{2}\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\cdot\boldsymbol{J}\cdot\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}-\boldsymbol{\pi}^*+\\(\boldsymbol{F}+\boldsymbol{F}_N)\cdot\boldsymbol{X}^c+(\boldsymbol{M}+\boldsymbol{M}_N)\cdot\boldsymbol{\theta}\Big]\mathrm{d}t\end{aligned} \tag{2}$$

$$\begin{aligned}\boldsymbol{\pi}^*=\int_{t_0}^{t_1}\Big\{\iiint_V\Big[A\Big(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla\Big)-\\\frac{\partial A\Big(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla\Big)}{\partial\Big(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla\Big)}\cdot\boldsymbol{I}\alpha\Delta T-\\(\boldsymbol{f}+\boldsymbol{f}_N)\cdot\boldsymbol{u}\Big]\mathrm{d}V-\iint_{S_\sigma}(\boldsymbol{T}+\boldsymbol{T}_N)\cdot\boldsymbol{u}\mathrm{d}S\Big\}\mathrm{d}t\end{aligned} \tag{3}$$

$$\begin{aligned}\delta Q=\int_{t_0}^{t_1}(\boldsymbol{X}^c\cdot\delta\boldsymbol{F}_N+\boldsymbol{\theta}\cdot\delta\boldsymbol{M}_N)\mathrm{d}t+\\\int_{t_0}^{t_1}\Big(\iiint_{V\mathrm{q}}\boldsymbol{u}\cdot\delta\boldsymbol{f}_N\mathrm{d}V+\iint_{S_f}\boldsymbol{u}\cdot\delta\boldsymbol{T}_N\mathrm{d}S\Big)\mathrm{d}t\end{aligned} \tag{4}$$

其边界条件为:

$$\boldsymbol{u}-\bar{\boldsymbol{u}}=\boldsymbol{0}\quad(\text{在 } S_u \text{ 上}) \tag{5}$$

其中,$\boldsymbol{u}$ 为位移,$\bar{\boldsymbol{u}}$ 为边界位移,V 为体积,S_σ 为应力边界面,S_u 为位移边界面,A 为应变能函数,t 为时间,ρ 为物质密度,∇为 Hamilton 算子;$\boldsymbol{F}$ 和 $\boldsymbol{F}_N$ 分别为保守和非保守主矢,$\boldsymbol{M}$ 和 $\boldsymbol{M}_N$ 分别为保守和非保守主矩,$\boldsymbol{f}$ 为保守体积力,$\boldsymbol{f}_N$ 为非保守体积力,$\boldsymbol{T}$ 为保守面积力,$\boldsymbol{T}_N$ 为非保守面积力;$\boldsymbol{X}^c$ 为刚体质心的矢径,$\boldsymbol{J}$ 为对质心的转动惯量(设为常量),$\boldsymbol{\theta}$ 为刚体的转角;$\boldsymbol{I}$ 为单位张量,$\boldsymbol{n}$ 为边界面外法线;α 为热膨胀系数,ΔT 为温度增量.

可见,一类变量的非保守非线性刚-热-弹耦合系统的动能为:

$$T=\iiint_V \frac{1}{2}\rho \frac{d\boldsymbol{X}^c}{dt}\cdot\frac{d\boldsymbol{X}^c}{dt}dV+\frac{1}{2}\frac{d\boldsymbol{\theta}}{dt}\cdot\boldsymbol{J}\cdot\frac{d\boldsymbol{\theta}}{dt}+\iiint_V[\rho(\frac{d\boldsymbol{X}^c}{dt}+\frac{d\boldsymbol{\theta}}{dt}\times\boldsymbol{x})\cdot\frac{d\boldsymbol{u}}{dt}+\frac{1}{2}\rho\frac{d\boldsymbol{u}}{dt}\cdot\frac{d\boldsymbol{u}}{dt}]dV \tag{6}$$

非保守非线性刚-热-弹耦合系统的势能为:

$$U=-\boldsymbol{F}\cdot\boldsymbol{X}^c-\boldsymbol{M}\cdot\boldsymbol{\theta}+\iiint_V[A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)-\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)}\cdot\boldsymbol{I}\alpha\Delta T-\boldsymbol{f}\cdot\boldsymbol{u}]dV-\iint_{S_\sigma}\boldsymbol{T}\cdot\boldsymbol{u}dS \tag{7}$$

非保守非线性刚-热-弹耦合系统的拟势能为:

$$U_{qs}=-\boldsymbol{F}_N\cdot\boldsymbol{X}^c-\boldsymbol{M}_N\cdot\boldsymbol{\theta}-\iiint_{Vq}\boldsymbol{f}_N\cdot\boldsymbol{u}dV-\iint_{S_\sigma}\boldsymbol{T}_N\cdot\boldsymbol{u}dS \tag{8}$$

非保守非线性刚-热-弹耦合系统的余虚功为:

$$\delta Q=\int_{t_0}^{t_1}[(\boldsymbol{X}^c\cdot\delta\boldsymbol{F}_N+\boldsymbol{\theta}\cdot\delta\boldsymbol{M}_N)+\iiint_{Vq}\boldsymbol{u}\cdot\delta\boldsymbol{f}_N dV+\iint_{S_f}\boldsymbol{u}\cdot\delta\boldsymbol{T}_N dS]dt \tag{9}$$

根据如上的论述,式(1)可以变换为:

$$\begin{aligned}\delta\pi_H-\delta Q&=\delta\int_{t_0}^{t_1}(T-U)dt-\delta\int_{t_0}^{t_1}\delta U_{qs}dt-\delta Q\\&=\int_{t_0}^{t_1}(\frac{\partial T}{\partial\dot{\boldsymbol{X}}^c}\cdot\delta\dot{\boldsymbol{X}}^c-\frac{\partial U}{\partial\boldsymbol{X}^c}\cdot\delta\boldsymbol{X}^c)dt+\int_{t_0}^{t_1}(\frac{\partial T}{\partial\dot{\boldsymbol{\theta}}}\cdot\delta\dot{\boldsymbol{\theta}}-\frac{\partial U}{\partial\boldsymbol{\theta}}\cdot\delta\boldsymbol{\theta})dt+\\&\int_{t_0}^{t_1}(\frac{\partial T}{\partial\dot{\boldsymbol{u}}}\cdot\delta\dot{\boldsymbol{u}})-\frac{\partial U}{\partial\boldsymbol{u}}\cdot\delta\boldsymbol{u})-\int_{t_0}^{t_1}\delta U_{qs}dt-\delta Q=0\end{aligned} \tag{10}$$

进行分部积分,可得:

$$\int_{t_0}^{t_1}\frac{\partial T}{\partial\dot{\boldsymbol{X}}^c}\cdot\delta\dot{\boldsymbol{X}}^c dt=\frac{\partial T}{\partial\dot{\boldsymbol{X}}^c}\cdot\delta\boldsymbol{X}^c\Big|_{t_0}^{t_1}-\int_{t_0}^{t_1}\frac{d}{dt}\frac{\partial T}{\partial\dot{\boldsymbol{X}}^c}\cdot\delta\boldsymbol{X}^c dt \tag{11}$$

$$\int_{t_0}^{t_1}\frac{\partial T}{\partial\dot{\boldsymbol{\theta}}}\cdot\delta\dot{\boldsymbol{\theta}}dt=\frac{\partial T}{\partial\dot{\boldsymbol{\theta}}}\cdot\delta\boldsymbol{\theta}\Big|_{t_0}^{t_1}-\int_{t_0}^{t_1}\frac{d}{dt}\frac{\partial T}{\partial\dot{\boldsymbol{\theta}}}\cdot\delta\boldsymbol{\theta}dt \tag{12}$$

$$\int_{t_0}^{t_1}\frac{\partial T}{\partial\dot{\boldsymbol{u}}}\cdot\delta\dot{\boldsymbol{u}}dt=\frac{\partial T}{\partial\dot{\boldsymbol{u}}}\cdot\delta\boldsymbol{u}\Big|_{t_0}^{t_1}-\int_{t_0}^{t_1}\frac{d}{dt}\frac{\partial T}{\partial\dot{\boldsymbol{u}}}\cdot\delta\boldsymbol{u}dt \tag{13}$$

将式(11)、(12)和(13)代入式(10),按惯例在时域边界 $t=t_0$ 和 $t=t_1$ 处取 $\delta\boldsymbol{u}=0,\delta\boldsymbol{X}^c=0,\delta\boldsymbol{\theta}=0$,可得:

$$\begin{aligned}\delta\pi_H-\delta Q&=\int_{t_0}^{t_1}(\delta T-\delta U)dt-\int_{t_0}^{t_1}\delta U_{qs}dt-\delta Q\\&=\int_{t_0}^{t_1}(-\frac{d}{dt}\frac{\partial T}{\partial\dot{\boldsymbol{X}}^c}\cdot\delta\boldsymbol{X}^c-\frac{\partial U}{\partial\boldsymbol{X}^c}\cdot\delta\boldsymbol{X}^c)dt+\\&\int_{t_0}^{t_1}(-\frac{d}{dt}\frac{\partial T}{\partial\dot{\boldsymbol{\theta}}}\cdot\delta\boldsymbol{\theta}-\frac{\partial U}{\partial\boldsymbol{\theta}}\cdot\delta\boldsymbol{\theta})dt+\\&\int_{t_0}^{t_1}(-\frac{d}{dt}\frac{\partial T}{\partial\dot{\boldsymbol{u}}}\cdot\delta\boldsymbol{u}-\frac{\partial U}{\partial\boldsymbol{u}^q}\cdot\delta\boldsymbol{u})dt-\\&\int_{t_0}^{t_1}\delta U_{qs}dt-\delta Q=0\end{aligned} \tag{14}$$

因为:

$$\int_{t_0}^{t_1}-\delta U_{qs}dt-\delta Q=\int_{t_0}^{t_1}[(\boldsymbol{F}_N\cdot\delta\boldsymbol{X}^c+\boldsymbol{M}_N\cdot\delta\boldsymbol{\theta}+\iiint_V\boldsymbol{f}_N\cdot\delta\boldsymbol{u}dV+\iint_{S_\sigma}\boldsymbol{T}_N\cdot\delta\boldsymbol{u}dS]dt \tag{15}$$

进而可得:

$$\begin{aligned}\delta\pi_H-\delta Q&=\delta\int_{t_0}^{t_1}(T-U-U_{qs})dt-\delta Q\\&=\int_{t_0}^{t_1}-(\frac{d}{dt}\frac{\partial T}{\partial\dot{\boldsymbol{X}}^c}+\frac{\partial U}{\partial\boldsymbol{X}^c}-\boldsymbol{F}_N)\cdot\delta\boldsymbol{X}^c dt\\&\int_{t_0}^{t_1}-(\frac{d}{dt}\frac{\partial T}{\partial\dot{\boldsymbol{\theta}}}+\frac{\partial U}{\partial\boldsymbol{\theta}}-\boldsymbol{M}_N)\cdot\delta\boldsymbol{\theta}dt\\&\int_{t_0}^{t_1}-\{\frac{d}{dt}\frac{\partial T}{\partial\dot{\boldsymbol{u}}}+\frac{\partial U}{\partial\boldsymbol{u}}-\iiint_V\boldsymbol{f}_N dV-\iint_{S_\sigma}\boldsymbol{T}_N dS\}\cdot\delta\boldsymbol{u}dt=0\end{aligned} \tag{16}$$

考虑到 $\delta\boldsymbol{u},\delta\boldsymbol{X}^c$ 和 $\delta\boldsymbol{\theta}$ 的任意性,由上式可得一类变量的非保守非线性刚-热-弹耦合动力学的 Lagrange 方程组:

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\boldsymbol{X}}^c}+\frac{\partial U}{\partial\boldsymbol{X}^c}-\boldsymbol{F}_N=\boldsymbol{0} \tag{17}$$

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\boldsymbol{\theta}}}+\frac{\partial U}{\partial\boldsymbol{\theta}}-\boldsymbol{M}_N=\boldsymbol{0} \tag{18}$$

$$\frac{d}{dt}\frac{\partial T}{\partial\dot{\boldsymbol{u}}}+\frac{\partial U}{\partial\boldsymbol{u}}-\iiint_V\boldsymbol{f}_N dV-\iint_{S_\sigma}\boldsymbol{T}_N dS=\boldsymbol{0} \tag{19}$$

2　应用 Lagrange 方程建立非保守非线性刚-热-弹耦合系统的控制方程

以下,应用非保守非线性刚-热-弹耦合动力学

Lagrange 方程推导其控制方程.为此,需要推导计算 Lagrange 方程中的有关动能和有关和势能的各项.

首先,推导计算 Lagrange 方程中的有关动能的各项.考虑到 $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\boldsymbol{u}}}$ 中的 $\dot{\boldsymbol{u}}=\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}$, $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}$ 中的 $\dot{\boldsymbol{\theta}}=\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}$, $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\boldsymbol{X}}^c}$ 中的 $\dot{\boldsymbol{X}}^c=\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}$,则有:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\boldsymbol{u}}}=&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\boldsymbol{u}}}\{\iiint_V \frac{1}{2}\rho\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}\mathrm{d}V+\\
&\frac{1}{2}\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\cdot\boldsymbol{J}\cdot\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}+\\
&\iiint_V[\rho(\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\boldsymbol{x})\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}+\\
&\frac{1}{2}\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}]\mathrm{d}V\}\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\boldsymbol{u}}}\iiint_V[\rho(\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\boldsymbol{x})\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}+\\
&\frac{1}{2}\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}]\mathrm{d}V\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\iiint_V\rho(\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\boldsymbol{x}+\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t})\mathrm{d}V\\
=&\iiint_V\rho[\frac{\mathrm{d}^2\boldsymbol{X}^c}{\mathrm{d}t^2}+\frac{\mathrm{d}^2\boldsymbol{\theta}}{\mathrm{d}t^2}\times\boldsymbol{x}+\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times(\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\boldsymbol{x})+\frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2}]\mathrm{d}V
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\boldsymbol{\theta}}}=&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\boldsymbol{\theta}}}\{\iiint_V \frac{1}{2}\rho\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}\mathrm{d}V+\frac{1}{2}\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\cdot\boldsymbol{J}\cdot\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}+\\
&\iiint_V[\rho(\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\boldsymbol{x})\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}+\\
&\frac{1}{2}\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}]\mathrm{d}V\}\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\boldsymbol{\theta}}}[\iiint_V\rho(\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\boldsymbol{x})\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\mathrm{d}V+\\
&\frac{1}{2}\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\cdot\boldsymbol{J}\cdot\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}]\\
=&-\iiint_V\rho\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\times\boldsymbol{x})\mathrm{d}V+\boldsymbol{J}\cdot\frac{\mathrm{d}^2\boldsymbol{\theta}}{\mathrm{d}t^2}
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \dot{\boldsymbol{X}}^c}=&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\boldsymbol{X}}^c}\{\iiint_V \frac{1}{2}\rho\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}\mathrm{d}V+\\
&\frac{1}{2}\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\cdot\boldsymbol{J}\cdot\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}+\\
&\iiint_V[\rho(\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\boldsymbol{x})\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}+\\
&\frac{1}{2}\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}]\mathrm{d}V\}\\
=&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\boldsymbol{X}}^c}[\iiint_V\rho(\frac{1}{2}\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}+\\
&\frac{\mathrm{d}\boldsymbol{X}^c}{\mathrm{d}t}\cdot\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t})\mathrm{d}V]\\
=&\iiint_V\rho(\frac{\mathrm{d}^2\boldsymbol{X}^c}{\mathrm{d}t^2}+\frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2})\mathrm{d}V
\end{aligned}
\tag{22}
$$

然后,推导计算 Lagrange 方程中的有关势能的各项:

$$
\begin{aligned}
\frac{\partial U}{\partial \boldsymbol{X}^c}=&\frac{\partial}{\partial \boldsymbol{X}^c}\{-\boldsymbol{F}\cdot\boldsymbol{X}^c-\boldsymbol{M}\cdot\boldsymbol{\theta}+\\
&\iiint_V[A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)-\\
&\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)}\cdot\boldsymbol{I}\alpha\Delta T-\\
&\boldsymbol{f}\cdot\boldsymbol{u}]\mathrm{d}V-\iint_{S_\sigma}\boldsymbol{T}\cdot\boldsymbol{u}\mathrm{d}S\}\\
=&\frac{\partial}{\partial \boldsymbol{X}^c}(-\boldsymbol{F}\cdot\boldsymbol{X}^c)=-\boldsymbol{F}
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\frac{\partial U}{\partial \boldsymbol{\theta}}=&\frac{\partial}{\partial \boldsymbol{\theta}}\{-\boldsymbol{F}\cdot\boldsymbol{X}^c-\boldsymbol{M}\cdot\boldsymbol{\theta}+\\
&\iiint_V[A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)-\\
&\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)}\cdot\boldsymbol{I}\alpha\Delta T-\\
&\boldsymbol{f}\cdot\boldsymbol{u}]\mathrm{d}V-\iint_{S_\sigma}T\cdot\boldsymbol{u}\mathrm{d}S\}\\
=&\frac{\partial}{\partial \boldsymbol{\theta}}(-\boldsymbol{M}\cdot\boldsymbol{\theta})=-\boldsymbol{M}
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\frac{\partial U}{\partial \boldsymbol{u}}=&\frac{\partial}{\partial \boldsymbol{u}}\{-\boldsymbol{F}\cdot\boldsymbol{X}^c-\boldsymbol{M}\cdot\boldsymbol{\theta}+\\
&\iiint_V[A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)-\\
&\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)}\cdot\boldsymbol{I}\alpha\Delta T-\\
&\boldsymbol{f}\cdot\boldsymbol{u}]\mathrm{d}V-\iint_{S_\sigma}\boldsymbol{T}\cdot\boldsymbol{u}\mathrm{d}S\}\\
=&\frac{\partial}{\partial \boldsymbol{u}}\{\iiint_V[A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\nabla)-
\end{aligned}
$$

$$\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}\cdot\boldsymbol{I}\alpha\Delta T-\boldsymbol{f}\cdot\boldsymbol{u}]\mathrm{d}V-\iint_{S_\sigma}\boldsymbol{T}\cdot\boldsymbol{u}\mathrm{d}S\} \tag{25}$$

应用 Green 定理,并考虑到边界条件(5),可得:

$$\begin{aligned}&\frac{\partial}{\partial\boldsymbol{u}}\iiint_V[A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)-\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}\cdot\boldsymbol{I}\alpha\Delta T]\mathrm{d}V\\&=\iint_{S_\sigma}(\boldsymbol{I}+\boldsymbol{u}\,\nabla)\cdot[\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}-\frac{\partial^2A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)^2}\cdot\boldsymbol{I}\alpha\Delta T]\cdot\boldsymbol{n}\mathrm{d}S-\\&\iiint_V(\boldsymbol{I}+\boldsymbol{u}\,\nabla)\cdot[\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}-\frac{\partial^2A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)^2}\cdot\boldsymbol{I}\alpha\Delta T]\cdot\nabla\mathrm{d}V\end{aligned} \tag{26}$$

进而可得:

$$\begin{aligned}&\frac{\partial}{\partial\boldsymbol{u}}\{\iiint_V[A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)-\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}\cdot\boldsymbol{I}\alpha\Delta T-\boldsymbol{f}\cdot\boldsymbol{u}]\mathrm{d}V-\iint_{S_\sigma}\boldsymbol{T}\cdot\boldsymbol{u}\mathrm{d}S\}\\&=-\iiint_V\{[(\boldsymbol{I}+\boldsymbol{u}^e\,\nabla)\cdot\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}-\frac{\partial^2A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)^2}\cdot\\&\boldsymbol{I}\alpha\Delta T]\cdot\nabla+\boldsymbol{f}\}\mathrm{d}V+\iint_{S_\sigma}\{(\boldsymbol{I}+\boldsymbol{u}\,\nabla)\cdot[\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}-\frac{\partial^2A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)^2}\cdot\\&\boldsymbol{I}\alpha\Delta T]\cdot\boldsymbol{n}-\boldsymbol{T}\}\mathrm{d}S\end{aligned} \tag{27}$$

将推导计算有关动能和有关势能的各项的结果代入 Lagrange 方程中,可得:

$$\iiint_V\rho(\frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2}+\frac{\mathrm{d}^2\boldsymbol{X}^c}{\mathrm{d}t^2})\mathrm{d}V-\boldsymbol{F}-\boldsymbol{F}_N=\boldsymbol{0} \tag{28}$$

$$-\iiint_{V^e}\rho\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\times\boldsymbol{x})\mathrm{d}V+\boldsymbol{J}\cdot\frac{\mathrm{d}^2\boldsymbol{\theta}}{\mathrm{d}t^2}-\boldsymbol{M}-\boldsymbol{M}_N=\boldsymbol{0} \tag{29}$$

$$\begin{aligned}&\iiint_V\rho[\frac{\mathrm{d}^2\boldsymbol{X}^c}{\mathrm{d}t^2}+\frac{\mathrm{d}^2\boldsymbol{\theta}}{\mathrm{d}t^2}\times\boldsymbol{x}+\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times(\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\boldsymbol{x})+\frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2}]\mathrm{d}V-\\&\iiint_V\{(\boldsymbol{I}+\boldsymbol{u}\,\nabla)\cdot[\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}-\frac{\partial^2A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)^2}\cdot\boldsymbol{I}\alpha\Delta T]\cdot\\&\nabla+\boldsymbol{f}+\boldsymbol{f}_N\}\mathrm{d}V+\iint_{S_\sigma}\{(\boldsymbol{I}+\boldsymbol{u}\,\nabla)\cdot[\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}-\frac{\partial^2A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)^2}\cdot\boldsymbol{I}\alpha\Delta T]\cdot\\&\boldsymbol{n}-\boldsymbol{T}-\boldsymbol{T}_N\}\mathrm{d}S=\boldsymbol{0}\end{aligned} \tag{30}$$

将式(30)写成微分形式,应用可变函数选值的理

论[50],可得:

$$\iiint_V \rho\left(\frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2}+\frac{\mathrm{d}^2\boldsymbol{X}^c}{\mathrm{d}t^2}\right)\mathrm{d}V-\boldsymbol{F}-\boldsymbol{F}_N=\boldsymbol{0} \quad (31)$$

$$-\iiint_{V^e} \rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t}\times\boldsymbol{x}\right)\mathrm{d}V+\boldsymbol{J}\cdot\frac{\mathrm{d}^2\boldsymbol{\theta}}{\mathrm{d}t^2}-\boldsymbol{M}-\boldsymbol{M}_N=\boldsymbol{0} \quad (32)$$

$$\rho\left[\frac{\mathrm{d}^2\boldsymbol{X}^c}{\mathrm{d}t^2}+\frac{\mathrm{d}^2\boldsymbol{\theta}}{\mathrm{d}t^2}\times\boldsymbol{x}+\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\left(\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\boldsymbol{x}\right)+\frac{\mathrm{d}^2\boldsymbol{u}}{\mathrm{d}t^2}-\right.$$
$$(\boldsymbol{I}+\boldsymbol{u}\,\nabla)\cdot\left[\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}-\right.$$
$$\left.\frac{\partial^2 A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)^2}\cdot\boldsymbol{I}\alpha\Delta T\right]\cdot\nabla-$$
$$\boldsymbol{f}-\boldsymbol{f}_N=0\ (\text{在 } V \text{ 中}) \quad (33)$$

$$(\boldsymbol{I}+\boldsymbol{u}\,\nabla)\cdot\left[\frac{\partial A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}-\right.$$
$$\left.\frac{\partial^2 A(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)}{\partial(\frac{1}{2}\nabla\boldsymbol{u}+\frac{1}{2}\boldsymbol{u}\,\nabla+\frac{1}{2}\nabla\boldsymbol{u}\cdot\boldsymbol{u}\,\nabla)^2}\cdot\boldsymbol{I}\alpha\Delta T\right]\cdot\boldsymbol{n}-$$
$$\boldsymbol{T}-\boldsymbol{T}_N=\boldsymbol{0}\ (\text{在 } S_\sigma \text{ 上}) \quad (34)$$

这就是一类变量的非保守非线性刚-热-弹耦合动力学的控制方程.

应用类似方法,可得两类变量的非保守非线性刚-热-弹耦合动力学的 Lagrange 方程组为:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\boldsymbol{v}^c}+\frac{\partial U}{\partial\boldsymbol{X}^c}-\boldsymbol{F}_N=\boldsymbol{0} \quad (35)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\boldsymbol{\omega}}+\frac{\partial U}{\partial\boldsymbol{\theta}}-\boldsymbol{M}_N=\boldsymbol{0} \quad (36)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\boldsymbol{v}}+\frac{\partial U}{\partial\boldsymbol{u}}-\iiint_V\boldsymbol{f}_N\mathrm{d}V-\iint_{S_\sigma}\boldsymbol{T}_N\mathrm{d}S=\boldsymbol{0} \quad (37)$$

将相关的推导结果代入 Lagrange 方程中,可得:

$$\iiint_V \rho\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{v}^c}{\mathrm{d}t}\right)\mathrm{d}V-\boldsymbol{F}-\boldsymbol{F}_N=\boldsymbol{0} \quad (38)$$

$$-\iiint_V \rho\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}\times\boldsymbol{x})\mathrm{d}V+\boldsymbol{J}\cdot\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}-\boldsymbol{M}-\boldsymbol{M}_N=\boldsymbol{0} \quad (39)$$

$$\iiint_V \rho\left[\frac{\mathrm{d}\boldsymbol{v}^c}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\times\boldsymbol{x}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{x})+\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right]\mathrm{d}V-$$
$$\iiint_{V^e}\left\{(\boldsymbol{I}+\boldsymbol{u}\,\nabla)\cdot\left[\frac{\partial A(\boldsymbol{E})}{\partial\boldsymbol{E}}-\frac{\partial^2 A(\boldsymbol{E})}{\partial\boldsymbol{E}^2}\right]\cdot\nabla+\boldsymbol{f}+\right.$$
$$\left.\boldsymbol{f}_N\right\}\mathrm{d}V+\iint_{S_\sigma}\left\{(\boldsymbol{I}+\boldsymbol{u}\,\nabla)\cdot\left[\frac{\partial A(\boldsymbol{E})}{\partial\boldsymbol{E}}-\frac{\partial^2 A(\boldsymbol{E})}{\partial\boldsymbol{E}^2}\right]\cdot\right.$$
$$\left.\boldsymbol{n}-\boldsymbol{T}-\boldsymbol{T}_N\right\}\mathrm{d}S=\boldsymbol{0} \quad (40)$$

式中,$\boldsymbol{v}^c$ 为刚体平动速度,$\boldsymbol{\omega}$ 为刚体转动角速度,$\boldsymbol{v}$ 为弹性体变形速度,$\boldsymbol{E}$ 为非线性弹性体变形的应变.

进而推导出两类变量的非保守非线性刚-热-弹耦合动力学的控制方程:

$$\iiint_V \rho\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{v}^c}{\mathrm{d}t}\right)\mathrm{d}V-\boldsymbol{F}-\boldsymbol{F}_N=\boldsymbol{0} \quad (41)$$

$$-\iiint_V \rho\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{v}\times\boldsymbol{x})\mathrm{d}V+\boldsymbol{J}\cdot\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}-\boldsymbol{M}-\boldsymbol{M}_N=\boldsymbol{0} \quad (42)$$

$$\rho\left[\frac{\mathrm{d}\boldsymbol{v}^c}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\times\boldsymbol{x}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{x})+\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right]-$$
$$(\boldsymbol{I}+\boldsymbol{u}\,\nabla)\cdot\left[\frac{\partial A(\boldsymbol{E})}{\partial\boldsymbol{E}}-\frac{\partial^2 A(\boldsymbol{E})}{\partial\boldsymbol{E}^2}\right]\cdot\nabla-\boldsymbol{f}-\boldsymbol{f}_N=\boldsymbol{0} \quad (43)$$

$$(\boldsymbol{I}+\boldsymbol{u}\,\nabla)\cdot\left[\frac{\partial A(\boldsymbol{E})}{\partial\boldsymbol{E}}-\frac{\partial^2 A(\boldsymbol{E})}{\partial\boldsymbol{E}^2}\right]\cdot\boldsymbol{n}-\boldsymbol{T}-\boldsymbol{T}_N=\boldsymbol{0} \quad (44)$$

3 应用举例

设有高速飞行器的非保守非线性刚-热-弹耦合动力学模型,处于常线速度而有横向转动扰动的飞行状态.如图 1 所示,其舱段内的两个固定支座(不可移动的支座)支撑的为 Bernoulli 梁.设梁截面面积为 A,单位体积的质量为 ρ,梁的长度为 L.当高速飞行器受到绕纵轴的转动扰动时,外载荷为由于转动扰动导致的惯性力和重力,其合力为载荷集度 q,并且,经受温度场 $\Delta T(x,y)=ax+by+c$ 的作用.

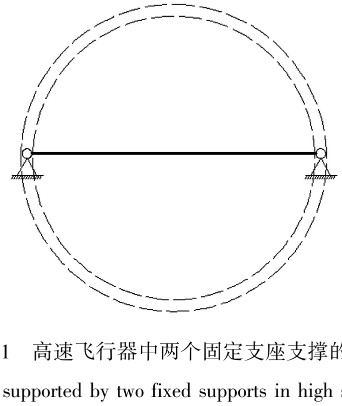

图 1 高速飞行器中两个固定支座支撑的梁

Fig.1 Beam supported by two fixed supports in high speed vehicles

我们采用 Lagrange 方程来研究梁的非保守非线性刚-热-弹耦合动力学效应，可以采用两种途径：一种是借助本文建立的非保守非线性刚-热-弹耦合动力学 Lagrange 方程，应用变分直接方法（包括有限元素法），求得问题的近似的解析解或者数值解；另一种是参照前面建立的 Lagrange 方程，推导问题的控制方程，进而求得问题的解析解或者数值解.以下，采用后一种思路来分析解决问题.

梁的复合受力状态有关的公式由控制方程(31)-(34)来提供.

假设梁的剖面是对称的，形心坐标即为主坐标，将以上方程用梁的有关参数表示，忽略 Coriolis 惯性力的影响，可得控制方程为：

$$\rho A\frac{\partial^2 w}{\partial t^2}+\frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 w}{\partial x^2})+\frac{\partial^2 w}{\partial x^2}[\int_A E\alpha y\Delta T\mathrm{d}A-N_{cf}]=q\ (\text{在 } V \text{ 域中}) \tag{45}$$

力学边界条件为：

$$EI\frac{\partial^2 w}{\partial x^2}=0 \quad (\text{在 } x=0, x=L \text{ 处}) \tag{46}$$

其中：w 为横向位移，E 为弹性模量，I 为轴惯矩，V 为梁的体积.

(1) 因为假设高速飞行器处于常线速度飞行状态，可知牵连惯性力 $\boldsymbol{f}^e=-\rho\frac{\mathrm{d}^2\boldsymbol{X}^c}{\mathrm{d}t^2}=0$.

(2) 切向惯性力 $\boldsymbol{f}^t=-\rho\frac{\mathrm{d}^2\boldsymbol{\theta}}{\mathrm{d}t^2}\times\boldsymbol{x}$.还有一项 $-\rho\frac{\mathrm{d}^2\boldsymbol{\theta}}{\mathrm{d}t^2}\times\boldsymbol{u}$，我们可以做两种理解，一种理解为相对于梁的横向位移 w 的切向加速运动，另一种理解为切向惯性力，我们采用后者.切向惯性力导致的载荷集度 q_t. 另外一部分载荷集度为重力导致的 q_g，公式中的 $q=q_t+q_g$.

(3) 法向惯性力 $\boldsymbol{f}^{cf}=-\rho\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times(\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\boldsymbol{x})$ 导致轴力 N_{cf}.引用文献[32]的结果：

$$N_{cf}=\int_0^{L-x^p}(x^a+x^p+l)\,\omega_z^2A\rho\,\mathrm{d}l \tag{47}$$

这里请注意，文献[32]中 N_{cf} 为拉力时为正，本算例中由于温度变化导致的轴力：

$$N_T=\int_A E\alpha y\Delta T\mathrm{d}A \tag{48}$$

假设压力时为正.因此，式(45)中，N_{cf} 前出现一个负号.

还有一项 $-\rho\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times(\frac{\mathrm{d}\boldsymbol{\theta}}{\mathrm{d}t}\times\boldsymbol{u})$，我们可以做两种理解，一种理解为相对于梁的横向位移 w 的向心加速运动，另一种理解为离心惯性力，我们采用后者.由于一般说来 $w\ll x$，在后面的分析中，我们忽略了这一项的影响.

(4) $-\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2}$ 可以理解为相对惯性力，也可以理解为弹性运动加速度与质量密度的乘积，我们采用后者.这便是控制方程中 $\rho A\frac{\partial^2 w}{\partial t^2}$ 项的来源.

(5) 位移边界条件：

$$w|_{x=0}=0 \tag{49}$$

$$w|_{x=L}=0 \tag{50}$$

式(45)是固定支座支持的 Bernoulli 梁结构的一般的动力学方程，去掉外载荷项 q，得方程：

$$\rho A\frac{\partial^2 w}{\partial t^2}+\frac{\partial^2}{\partial x^2}(EI\frac{\partial^2 w}{\partial x^2})+\frac{\partial^2 w}{\partial x^2}[\int_A E\alpha y\Delta T\mathrm{d}A-N_{cf}]=0 \tag{51}$$

这是结构特性方程，我们可以应用结构特性方程来研究非保守非线性刚-热-弹耦合系统的振动角频率，不失一般性.

将方程(51)中含 $[\int_A E\alpha y\Delta T\mathrm{d}A-N_{cf}]$ 的项变换为：

$$\frac{\partial^2 w}{\partial x^2}[\int_A E\alpha y\Delta T\mathrm{d}A-N_{cf}]\approx\beta EI\frac{\partial^2}{\partial x^2}(\frac{\partial^2 w}{\partial x^2}) \tag{52}$$

假设系统的一阶振动的解为：

$$w=\sin\frac{\pi x}{L}(A\cos\omega t+B\sin\omega t) \tag{53}$$

将式(53)代入式(52)，可得：

$$[\int_A E\alpha y\Delta T\mathrm{d}A-N_{cf}](\frac{\pi}{L})^2\approx\beta EI(\frac{\pi}{L})^4 \tag{54}$$

解得：

$$\beta=-\frac{[\int_A E\alpha y\Delta T\mathrm{d}A-N_{cf}]}{EI(\frac{\pi}{L})^2}=-\frac{[\int_A E\alpha y\Delta T\mathrm{d}A-N_{cf}]}{EI}(\frac{L}{\pi})^2 \tag{55}$$

进而可得：

$$A\rho\frac{\partial^2 w}{\partial t^2}+[EI-\int_A E\alpha y\Delta T\mathrm{d}A(\frac{L}{\pi})^2+$$

$$N_{cf}(\frac{L}{\pi})^2]\frac{\partial^2}{\partial x^2}(\frac{\partial^2 w}{\partial x^2})=0 \tag{56}$$

将式(53)代入(56),经计算可得:

$$-A\rho\omega^2+[EI-\int_A E\alpha y\Delta T\mathrm{d}A(\frac{L}{\pi})^2+ N_{cf}(\frac{L}{\pi})^2](\frac{\pi}{L})^4=0 \tag{57}$$

进而可以求得梁的振动角频率为:

$$\omega=(\frac{\pi}{L})^2\sqrt{\frac{EI-\int_A E\alpha y\Delta T\mathrm{d}A\,(\frac{L}{\pi})^2+(\frac{L}{\pi})^2N_{cf}}{A\rho}} \tag{58}$$

可见,由于热效应项 $\int_A E\alpha y\Delta T\mathrm{d}A$ 和惯性效应项 N_{cf} 的存在,相当于结构的剖面刚度由 EI 变换为 $EI-\int_A E\alpha y\Delta T dA\,(\frac{L}{\pi})^2+N_{cf}(\frac{L}{\pi})^2$.对应着 $N_{cf}(\frac{L}{\pi})^2>0$ 相当于梁被刚化了;$N_{cf}(\frac{L}{\pi})^2<0$ 相当于梁被柔化了;可以推知:如果 $\int_A E\alpha y\Delta T\mathrm{d}A\,(\frac{L}{\pi})^2>0$,则相当于梁被柔化了;如果 $\int_A E\alpha y\Delta T\mathrm{d}A\,(\frac{L}{\pi})^2<0$,则相当于梁被刚化了.因此,这里既存在动力刚化或柔化问题,也存在热力刚化或柔化问题.

4 结论

首先,建立非保守非线性刚-热-弹耦合系统的Hamilton拟变分原理,从该拟变分原理出发,采用Lagrange-Hamilton体系,成功地建立了非保守非线性刚-热-弹耦合系统的Lagrange方程,进而,应用Lagrange方程推导出非保守非线性刚-热-弹耦合动力学的控制方程.讨论了应用耦合动力学的Lagrange方程解决实际工程技术问题的途径,并且,对比研究了动力刚(柔)化和热力刚(柔)化问题.

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THE LAGRANGE EQUATION OF NON-CONSERVATIVE NON-LINEAR RIGID-THERMOS-ELASTIC COUPLING DYNAMICS *

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Abstract How to apply Lagrange equation to elastodynamics has always been a theoretical research topic at home and abroad. After solving such problems, the theoretical difficulties of Lagrange equation applied to the coupling dynamics are placed before us. In this paper, the Lagrange equation was successfully applied to the non-conservative non-linear rigid-thermos-elastic coupling dynamics by using the Lagrange-Hamilton system. Then the governing equations of non-conservative non-linear rigid-thermos-elastic coupling dynamics were derived by using the Lagrange equation of non-conservative non-linear rigid-thermos-elastic coupling dynamics. Finally, the way to solve practical engineering problems by using Lagrange equation of coupled dynamics was discussed.

Key words coupling dynamics, Lagrange equation, non-conservative system, nonlinear system, Lagrange-Hamilton system

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