一类强非线性二阶微分方程的多模态近似解析解研究*

楼智美[1†]　王元斌[2]　俞立先[1]

（1.绍兴文理学院 物理系,绍兴　312000）（2.绍兴文理学院 数学系,绍兴　312000）

摘要　利用自治力学系统的哈密顿函数为守恒量的性质,提出一种求非线性二阶微分方程多模态近似解析解的方法,称为哈密顿函数法.首先,介绍哈密顿函数法求多模态近似解的基本理论.其次,以质点在旋转的抛物线上运动为模型建立强非线性二阶微分方程.最后,用哈密顿函数法求得在给定初始条件和参数下强非线性二阶微分方程的三模态近似解析解表达式,作出三模态近似解析解的解曲线,并与直接用 Mathematica 软件作出的解曲线进行比较,讨论三模态近似解析解的精确性.结果表明:用哈密顿函数法求得的三模态近似解析解的解曲线与直接用 Mathematica 软件作出的解曲线十分吻合.

关键词　强非线性二阶微分方程,　多模态近似解析解,　哈密顿函数法

DOI:　10.6052/1672-6553-2019-067

引言

实际问题中出现的微分方程常常是非线性的,而且有许多属于强非线性的,求解非线性微分方程的近似解析解对于研究力学系统的运动特性与规律至关重要,一直受到众多学者的关注[1-20].一些学者提出了不少有效的求解方法,如谐平衡法[1-3],能量平衡法[4-6],同伦渐近法[7-9],Adomian 分解法[10-13],哈密顿函数法[14-17]等.在非惯性转动参照系中研究力学体系的运动时,常常会遇到一类分子分母均含非线性项的强非线性微分方程[18],求解比较复杂或存在发散性问题,若与此类方程相应的哈密顿函数不显含时间,则属自治力学系统,其哈密顿函数是一守恒量.若运动具有周期性,则可以假设其近似解析解是多个谐振动的迭加,即多模态解,利用哈密顿函数是守恒量的性质,可以求得多模态近似解析解中各模态的振幅与基频,从而求得多模态近似解析解表达式,这种利用哈密顿函数是守恒量的性质求解的方法称哈密顿函数法,为讨论多模态近似解析解的精确性,可以在给定初始条件和参数下,在同一平面上作出强非线性二阶微分方程的多模态近似解析解的解曲线与直接用 Mathematica 软件作出的解曲线进行比较.本文用上述方法得到了强非线性二阶微分方程的三模态近似解析解表达式,并讨论其精确性.结果表明:强非线性二阶微分方程的三模态近似解析解的解曲线与直接用 Mathematica 软件作出的解曲线十分吻合.

1　哈密顿函数法求多模态近似解析解的基本理论

若力学系统的哈密顿函数中不显含时间 t,则其哈密顿函数 H 是一守恒量. 设某力学系统有 s 个自由度,其哈密顿函数 H 可表示成

$$H=H(q_\alpha,\dot{q}_\alpha)\quad(\alpha=1,2,\cdots,s)\tag{1}$$

其中,$q_\alpha,\dot{q}_\alpha$ 分别表示广义坐标和广义速度.设初始条件为

$$q_{\alpha0}=A_\alpha,\dot{q}_{\alpha0}=0\quad(\alpha=1,2,\cdots,s)\tag{2}$$

若力学系统作周期运动,可设其多模态近似解析解为

$$q_\alpha=\sum_{i=1}^{n}a_{\alpha i}\cos(i\omega t)\quad(\alpha=1,2,\cdots,s)\tag{3}$$

则各模态的振幅满足

$$\sum_{i=1}^{n}a_{\alpha i}=A_\alpha\quad(\alpha=1,2,\cdots,s)\tag{4}$$

哈密顿函数 H 在整个运动过程中均保持不变,则其在 $0-\dfrac{T}{4}$时间内对时间积分后仍是一不变量[15].将式(3)代入式(1),并令

2018-04-29 收到第 1 稿,2019-04-09 收到修改稿.
* 国家自然科学基金资助项目(11472177)
† 通讯作者 E-mail:louzhimei@usx.edu.cn

$$\overline{H}=\int_0^{\frac{T}{4}}H(q_\alpha,\dot{q}_\alpha)\mathrm{d}t=\frac{1}{4}HT=\frac{\pi}{2\omega}H=const \quad (5)$$

则有[15]

$$\frac{\partial}{\partial a_{\alpha i}}\left(\frac{\partial \overline{H}}{\partial(1/\omega)}\right)=0 \quad (\alpha=1,2,\cdots,s,\ i=1,2,\cdots,n.) \quad (6)$$

联立式(4)和式(6),能解得各模态的振幅 $a_{\alpha i}$ 和振动的基频 ω,将求得的 $a_{\alpha i}$,ω 代入式(3)就能得到多模态近似解析解的表达式.

2 强非线性二阶微分方程的建立

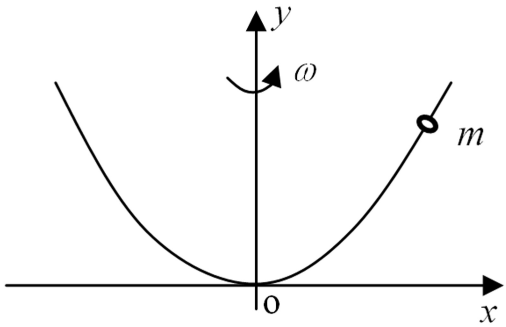

图1 匀速转动的抛物线型金属丝

Fig.1 A parabolic wire rotating at a constant speed

如图1所示,一质量为 m 的光滑小环,套在一光滑的抛物线型金属丝上,并可沿着金属丝滑动,抛物线型金属丝以角速度 ω 绕轴匀速转动.设抛物线的方程为 $x^2=4py$,则小环在 x 方向的运动微分方程为[18]

$$\ddot{x}+\frac{x\dot{x}^2}{x^2+4p^2}+\frac{bx}{x^2+4p^2}=0 \quad (7)$$

其中,

$$b=4p^2\left(\frac{g}{2p}-\omega^2\right)=4p^2(\omega_e^2-\omega^2) \quad (8)$$

$\omega_e=\sqrt{\frac{g}{2p}}$ 为小环保持相对平衡时金属丝的转动角速度,当 $\omega<\omega_e$ 时,$b>0$;当 $\omega>\omega_e$ 时,$b<0$;当 $\omega=\omega_e$ 时,$b=0$.

3 强非线性二阶微分方程的三模态近似解析解及其精确性分析

方程(7)中第二项的分子分母中均含非线性项,属于强非线性二阶微分方程.用一般的方法很难求得其解析近似解.文献[7]用同伦渐近法得到其近似解析解,但求解过程十分繁复,不宜推广应用,文献[14]用哈密顿函数法只求得其单模态近似解析解,且没有对近似解进行比较研究,现用哈密顿函数方法求得其三模态近似解析解,并通过作图与数值解进行比较,分析多模态近似解的精确性.

与方程(7)相应的哈密顿函数为[18]

$$H=\frac{1}{2}\dot{x}^2+\frac{1}{8p^2}\dot{x}^2x^2+\frac{b}{8p^2}x^2 \quad (9)$$

设系统的三模态近似解为

$$x=a_1\cos\omega t+a_2\cos3\omega t+a_3\cos5\omega t \quad (10)$$

初始条件为

$$x_0=A,\ \dot{x}_0=0 \quad (11)$$

则各模态振幅满足

$$a_1+a_2+a_3=A \quad (12)$$

将式(10)代入式(9)后并按式(5)积分,有

$$\overline{H}=\frac{1}{8}(a_1^2+9a_2^2+25a_3^2)\omega\pi+\frac{b}{32p^2}(a_1^2+a_2^2+a_3^2)\frac{\pi}{\omega}$$
$$+\frac{1}{128p^2}(a_1^4+9a_2^4+25a_3^4+20a_1^2a_2^2+52a_1^2a_3^2$$
$$+68a_2^2a_3^2-8a_1^3a_2-40a_1^2a_2a_3-80a_1a_2^2a_3)\omega\pi \quad (13)$$

将式(13)代入式(6)得

$$\frac{\partial}{\partial a_1}\left(\frac{\partial\overline{H}}{\partial(1/\omega)}\right)=-\left[\frac{1}{4}a_1+\frac{1}{32p^2}(a_1^3+10a_1a_2^2\right.$$
$$+26a_1a_3^2-6a_1^2a_2-20a_1a_2a_3-$$
$$\left.20a_2^2a_3)\right]\pi\omega^2+\frac{b}{16p^2}a_1\pi=0 \quad (14)$$

$$\frac{\partial}{\partial a_2}\left(\frac{\partial\overline{H}}{\partial(1/\omega)}\right)=-\left[\frac{9}{4}a_2+\frac{1}{32p^2}(9a_2^3+10a_1^2a_2+\right.$$
$$34a_2a_3^2-2a_1^3-10a_1^2a_3$$
$$\left.-40a_1a_2a_3)\right]\pi\omega^2+\frac{b}{16p^2}a_2\pi=0 \quad (15)$$

$$\frac{\partial}{\partial a_3}\left(\frac{\partial\overline{H}}{\partial(1/\omega)}\right)=-\left[\frac{25}{4}a_3+\frac{1}{32p^2}(25a_3^3+26a_1^2a_3\right.$$
$$+34a_2^2a_3-10a_1^2a_2$$
$$\left.-20a_1a_2^2)\right]\pi\omega^2+\frac{b}{16p^2}a_3\pi=0 \quad (16)$$

联立式(12)和式(14)~(16)四式,可以解得 a_1,a_2,a_3,ω 等4个未知量.由于式(12)和式(14)~(16)四式组成的是非线性代数方程组,比较复杂,可借助数学软件求解.讨论用哈密顿函数法得到的

近似解析解的精确性,采用的方法是将用哈密顿函数法得到的近似解析解曲线与直接用 Mathematica 软件得到的数值解曲线进行比较.为方便,现设$A=1$, $p=b=0.25$,则式(12)和式(14)~(16)可简化为

$$a_1+a_2+a_3=1 \tag{17}$$

$$a_1-(a_1+2a_1^3+20a_1a_2^2+52a_1a_3^2-12a_1^2a_2-40a_1a_2a_3-40a_2^2a_3)\omega^2=0 \tag{18}$$

$$a_2-(9a_2+18a_2^3+20a_1^2a_2+68a_2a_3^2-4a_1^3-20a_1^2a_3-80a_1a_2a_3)\omega^2=0 \tag{19}$$

$$a_3-(25a_3+50a_3^3+52a_1^2a_3+68a_2^2a_3-20a_1^2a_2-40a_1a_2^2)\omega^2=0 \tag{20}$$

由式(17)~(20),可解得

$$a_1=-0.008246354,\ a_2=0.074772601,$$
$$a_3=0.933473753,\ \omega=0.120425784. \tag{21}$$

将式(21)代入式(10),得三模态近似解析解为

$$x=-0.008246354\cos\omega t+0.074772601\cos3\omega t+0.933473753\cos5\omega t \tag{22}$$

其中,$\omega=0.120425784$.

根据式(22)可作出解曲线如图 2 中红线所示,图 2 中的黑线是根据式(7)直接用 Mathematica 软件作出的同参数下的解曲线.图 2 表明,根据哈密顿函数法得到的三模态近似解析解作出的解曲线与直接用 Mathematica 软件作出的解曲线十分吻合,特别是在第一个周期内,其振幅和周期的偏差很小,说明用哈密顿函数法得到多模态解去近似作周期运动的强非线性运动问题是一种十分有效的方法.

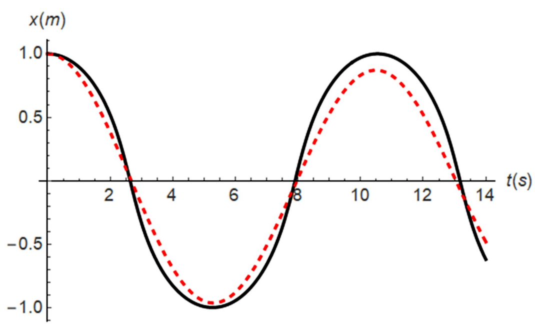

图 2　哈密顿函数法得到三模态近似解曲线—红线,数值解曲线—黑线

Fig.2　The red dashed: three modes approximate solution obtained by Hamilton function method, The solid: the numerical solution

4　结论

作周期运动的力学系统可假设其解是多个不同频率谐振动的迭加, 即多模态解.本文利用自治力学系统的哈密顿函数是一守恒量的性质,求得近似解析解各模态的振幅和振动的基频.在给定的初始条件和参数下,在同一平面上作出强非线性微分方程的三模态近似解析解曲线与直接用 Mathematica 软件作出的解曲线进行比较(如图 2),结果表明:三模态近似解析解的解曲线与直接用 Mathematica 软件作出的解曲线十分吻合.用哈密顿函数法求强非线性微分方程的近似解,思路简单清晰,物理意义明确,结果精确有效,可操作性强,值得推广应用.

参　考　文　献

1　Belendez A, Gimeno E, Alvarez M L, et al. A novel rational harmonic balance approach for periodic solutions of conservative nonlinear oscillators. *International Journal of Nonlinear Science Numeral*, 2009,10(1):13~26

2　Zhang H, Ma T W. Iterative harmonic balance for period-one rotating solution of parametric pendulum. *Nonlinear Dynamics*, 2012,70(4):2433~2444

3　Ju P J. Global residue harmonic balance method for a non-linear oscillator with discontinuity. *Applied Mathematical Modeling*, 2015,39(21):6738~6742

4　He J H. Preliminary report on the energy balance for non-linear oscillations. *Mechanics Research Communications*, 2002,29(2-3):107~111

5　Ganji D D, Ranjbar M N, Akbarzade M. Comparison of energybalance period for arising nonlinear oscillator equations (He′s energy balance period for nonlinear oscillators with and without discontinuities). *Acta Applicandae Mathematicae*, 2009,108(2):353~362

6　Ebrahimi K H, Ganji D D. A study on the motion of a rigid rod rocking back and cubic-quintic Duffing oscillators by using He′s energy balance method. *International Journal of Nonlinear Science*, 2010,10(4):447~451

7　Marinca V, Herisanu N. Determination of periodic solutions for the motion of a particle on a rotating parabola by means of the optimal homotopy asymptotic mehtod. *Journal of Sound and Vibration*, 2010,329(9):1450~1459

8　He J H . Homotopy perturbation method for bifurcation of nonlinear problems. *International Journal of Nonlinear Science and Numerical Simulation*, 2005,6(2):207~208

9　Ramos J I. Series approach to the Lane-Emden equation and comparison with the homotopy perturbation method. *Chaos, Solitons and Fractals*, 2008,38(2):400~408

10　方锦清,姚伟光. 逆算符方法求解非线性动力学方程及其一些应用实例. 物理学报, 1993,42(9):1375~1384 (Fang J Q, Yao W G. Inverse operator method for solutions of nonlinear dynamical equations and some typical applications. *Acta Physica Sinica*, 1993,42(9):1375~1384(in Chinese))

11 Duan J S. Convenient analytic recurrence algorithms for the Adomian polynomials. *App1ied Mathematics & Computation*, 2011,217(13):6337~6348

12 Duan J S, Rach R. The degenerate from of the Adomian polynomials in the power series method for nonlinear ordinary differential equations. *Journal of Mathematics System Science*, 2015,5(10):411~428

13 Rach R, Wazwaz A M, Duan J S. A reliable modification of the Adomian decomposition method for higher-order nonlinear differential equations. *Kybernetes*, 2013,42(2):282~308

14 Kargar A, Akbarzade M. Determination of periodic solutions for the motion of a particle on a rotating parabola by means of the He's Hamiltonian approach. *Advance Studies Theoretical Physics*, 2012,6(10):473~476

15 Durmaz S, Altay Demirbag S, Kaya M O. High order hamiltonian approach to nonlinear oscillators. *International Journal of Nonlinear Sciences & Numerical Simulation*, 2010,11(8):565~570

16 Yildirim A, Saadatnia Z, Askari H. Application of the Hamiltonian approach to nonlinear oscillators with rational and irrational elastic terms. *Mathematical and Computer Modeling*, 2011,54:697~703

17 Hermann M, Saravi M, Khah H E. Analytical study of nonlinear oscillatory systems using the Hamiltonian approach technique. *Journal of Theoretical Applied Physics*, 2014,8(3):133

18 周衍柏. 理论力学(第三版). 北京:高等教育出版社, 2009 (Zhou Y B. Theoretical mechanics(the third edition). Beijing:Higher Education Press, 2009 (in Chinese))

19 李京颍,丁光涛. Mathews-Lakshmanan 振子方程的分析力学解法. 动力学与控制学报, 2018,16(1):1~5 (Li J Y, Ding G T. Analytical mechanics methods for solving Mathews-Lakshmanan oscillator. *Journal of Dynamics and Control*, 2018,16(1):1~5 (in Chinese))

20 都琳,侯平兰. Duffing 方程的辛精细积分方法研究. 动力学与控制学报, 2017,15(1):1~5 (Du L, Hou P L. Symplectic precise integration method for Duffing equation. *Journal of Dynamics and Control*, 2017,15(1):1~5 (in Chinese))

STUDY ON MULTIMODE APPROXIMATE ANALYTICAL SOLUTION OF A CLASS OF STRONGLY NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS *

Lou Zhimei[1†] Wang Yuanbin[2] Yu Lixian[1]

(1.*Department of Physics*, *Shaoxing University*, *Shaoxing* 312000, *China*)

(2.*Department of Mathematics*, *Shaoxing University*, *Shaoxing* 312000, *China*)

Abstract Hamiltonian is a conservation quantity of autonomous mechanical system, and the multimode approximate analytical solutions of a nonlinear second order differential equation can be obtained by using characteristics of the conservation quantity, which is called Hamiltonian method. First, the basic theory of Hamiltonian method was introduced. Second, a strong nonlinear second order differential equation for the motion of a particle on a rotating parabola was established. Finally, the three-mode approximate analytical solutions of the strong nonlinear second order differential equation were obtained under given initial conditions and parameters. The approximate solution using Hamiltonian method was verified by the numerical solution using Mathematics software, which showed that the approximate solution is in good agreement with the numerical one.

Key words strongly nonlinear second order differential equation, multimode approximate analytical solutions, Hamiltonian method

Received 29 April 2018, revised 9 April 2019.

* The project supported by the National Natural Science Foundation of China(11472177).

† Corresponding author E-mail: louzhimei@ usx.edu.cn