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广义分数阶受迫 Birkhoff 方程*

宋传静†

(苏州科技大学 数理学院, 苏州 215009)

摘要 研究受迫 Birkhoff 系统的分数阶变分问题,建立具有这两种分数阶微分算子的广义分数阶受迫 Birkhoff 方程.然后,给出具有这两种分数阶微分算子的分数阶 Hamilton 方程和分数阶 Lagrange 方程.最后,讨论广义分数阶 Lotka-生化振子模型和广义分数阶 Hojman-Urrutia 模型.

关键词 分数阶微分算子, 变量微积分, 分数阶受迫 Birkhoff 方程

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引言

为处理系统中的耗散力,Riewe[1] 在 1996 年引入了分数阶导数.此后分数阶方面的研究主要基于左(右) Riemann-Liouville 分数阶导数[2,3]、左(右) Caputo 分数阶导数[4-6]、Riesz-Riemann-Liouville 分数阶导数[7,8]、Riesz-Caputo 分数阶导数[9] 及联合分数阶导数[10,11].2010 年,Agrawal[12] 给出了一些新的分数阶算子,称为分数阶微分算子,上面提到的四种分数阶导数均为其特例.

Galiullan[13] 曾指出,Birkhoff 力学是现代分析力学的重要研究方向之一.Birkhoff 力学比 Hamilton 力学更广泛[14],且已经被应用于很多领域[15].最近,Zhang[16] 基于 Agrawal 提出的分数阶微分算子研究了分数阶 Birkhoff 系统的变分问题,建立了相应的运动微分方程.

受迫 Birkhoff 系统的应用比 Birkhoff 系统更广泛[17].首先,受迫 Birkhoff 方程有广义力项,有助于处理控制问题.其次,对于具有耗散力项的力学系统,用受迫 Birkhoff 系统表示也更为方便.对于受迫 Birkhoff 系统的研究,已在积分因子[18]、对称性[19,20]、降阶法[21]、平衡稳定性[22]、梯度表示[23]、奇点分析[24] 等方面取得重要进展.

最近,基于 Riemann-Liouville 分数阶导数,分数阶受迫 Birkhoff 系统的 Noether 对称性与守恒量[25] 及对称性的摄动与绝热不变量[26] 已有研究.本文拟基于 Agrawal 的分数阶微分算子研究受迫 Birkhoff 系统,建立广义分数阶受迫 Birkhoff 方程.

本文结构安排如下:首先,列出分数阶导数的定义及其性质;然后,建立具有分数阶微分算子的广义分数阶受迫 Birkhoff 方程,并对其特例进行讨论;最后,给出两个分数阶模型说明结果的应用.

1 分数阶导数及其性质

这里简单回顾分数阶导数的定义及相关性质,详细请参考文献[12,16].

函数 $f(t)$ 的左(右) Riemann-Liouville 分数阶导数、左(右) Caputo 分数阶导数、Riesz-Riemann-Liouville 分数阶导数和 Riesz-Caputo 分数阶导数分别为

$$ {}_{t_1}^{\mathrm{RL}}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{\mathrm{d}t}\right)^n\int_{t_1}^{t}(t-\xi)^{n-\alpha-1}f(\xi)\mathrm{d}\xi \tag{1}$$

$$ {}_{t}^{\mathrm{RL}}D_{t_2}^{\beta}f(t)=\frac{1}{\Gamma(n-\beta)}\left(-\frac{d}{\mathrm{d}t}\right)^n\int_{t}^{t_2}(\xi-t)^{n-\beta-1}f(\xi)\mathrm{d}\xi \tag{2}$$

$$ {}_{t_1}^{\mathrm{C}}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_1}^{t}(t-\xi)^{n-\alpha-1}\left(\frac{d}{\mathrm{d}\xi}\right)^n f(\xi)\mathrm{d}\xi \tag{3}$$

$$ {}_{t}^{\mathrm{C}}D_{t_2}^{\beta}f(t)=\frac{1}{\Gamma(n-\beta)}\int_{t}^{t_2}(\xi-t)^{n-\beta-1}\left(-\frac{d}{\mathrm{d}\xi}\right)^n f(\xi)\mathrm{d}\xi \tag{4}$$

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† 通讯作者 E-mail:songchuanjingsun@126.com

$$ {}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}f(t)=\frac{1}{2\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_{t_1}^{t_2}|t-\xi|^{n-\alpha-1}f(\xi)\mathrm{d}\xi \tag{5}$$

$$ {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}f(t)=\frac{1}{2\Gamma(n-\alpha)}\int_{t_1}^{t_2}|t-\xi|^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(\xi)\mathrm{d}\xi \tag{6}$$

其中,$\Gamma(*)$是 Gamma 函数,α、β 是分数阶导数的阶数,$n-1\leqslant\alpha,\beta<n$,$n$ 为整数.

广义积分算子 K_P^{α} 定义为

$$\begin{aligned} K_{<t_1,t,t_2,p,q>}^{\alpha}f(t)&=p\int_{t_1}^{t}k_{\alpha}(t,\tau)f(\tau)\mathrm{d}\tau+\\ &\quad q\int_{t}^{t_2}k_{\alpha}(\tau,t)f(\tau)\mathrm{d}\tau\\ &=K_P^{\alpha}f(t) \end{aligned}\tag{7}$$

其中,$t_1<t<t_2$,$P=<t_1,t,t_2,p,q>$称为参数集,p、q 为实数,α 为阶数,$k_{\alpha}(t,\tau)$和 $k_{\alpha}(\tau,t)$称为依赖于参数 α 的核.

由算子 K_P^{α} 的定义可得

$$K_P^{\alpha}(f_1(t)+f_2(t))=K_P^{\alpha}f_1(t)+K_P^{\alpha}f_2(t) \tag{8}$$

$$K_P^{\alpha}f(t)=pK_{P_1}^{\alpha}f(t)+qK_{P_2}^{\alpha}f(t) \tag{9}$$

其中,$P=<t_1,t,t_2,p,q>$,$P_1=<t_1,t,t_2,1,0>$,$P_2=<t_1,t,t_2,0,1>$.

分数阶微分算子 A_P^{α} 和 B_P^{α} 定义为

$$A_{<t_1,t,t_2,p,q>}^{\alpha}f(t)=D^nK_P^{n-\alpha}f(t)=A_P^{\alpha}f(t) \tag{10}$$

$$B_{<t_1,t,t_2,p,q>}^{\alpha}f(t)=K_P^{n-\alpha}D^nf(t)=B_P^{\alpha}f(t) \tag{11}$$

其中,D 是经典的求导算子,$n-1<\alpha<n$,n 为整数,$P=<t_1,t,t_2,p,q>$.

显然,

$$A_P^{\alpha}(f_1(t)+f_2(t))=A_P^{\alpha}f_1(t)+A_P^{\alpha}f_2(t) \tag{12}$$

$$B_P^{\alpha}(f_1(t)+f_2(t))=B_P^{\alpha}f_1(t)+B_P^{\alpha}f_2(t) \tag{13}$$

算子 A_P^{α} 和 B_P^{α} 的分部积分公式分别为

$$\int_{t_1}^{t_2}g(t)A_P^{\alpha}f(t)\mathrm{d}t=(-1)^n\int_{t_1}^{t_2}f(t)B_{P^*}^{\alpha}g(t)\mathrm{d}t+\sum_{j=0}^{n-1}(-D)^jg(t)A_P^{\alpha-1-j}f(t)\Big|_{t_1}^{t_2} \tag{14}$$

$$\int_{t_1}^{t_2}g(t)B_P^{\alpha}f(t)\mathrm{d}t=(-1)^n\int_{t_1}^{t_2}f(t)A_{P^*}^{\alpha}g(t)\mathrm{d}t+\sum_{j=0}^{n-1}(-1)^jA_P^{\alpha+j-n}g(t)D^{n-1-j}f(t)\Big|_{t_1}^{t_2} \tag{15}$$

其中,$P=<t_1,t,t_2,p,q>$,$P^*=<t_1,t,t_2,p,q>$,$n-1<\alpha<n$.

当 $k_{\alpha}(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$,$k_{\alpha}(\tau,t)=(\tau-t)^{\alpha-1}/\Gamma(\alpha)$时,可得特例.此时,当 $P=P_1=<t_1,t,t_2,1,0>$时,有

$$\begin{aligned} A_{P_1}^{\alpha}f(t)&=D^nK_{P_1}^{n-\alpha}f(t)\\ &=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_{t_1}^{t}(t-\tau)^{n-\alpha-1}f(\tau)\mathrm{d}\tau\\ &={}_{t_1}^{\mathrm{RL}}D_t^{\alpha}f(t) \end{aligned}\tag{16}$$

$$\begin{aligned} B_{P_1}^{\alpha}f(t)&=K_{P_1}^{n-\alpha}D^nf(t)\\ &=\frac{1}{\Gamma(n-\alpha)}\int_{t_1}^{t}(t-\tau)^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^nf(\tau)\mathrm{d}\tau\\ &={}_{t_1}^{\mathrm{C}}D_t^{\alpha}f(t) \end{aligned}\tag{17}$$

当 $P=P_2=<t_1,t,t_2,0,1>$时,有

$$\begin{aligned} A_{P_2}^{\alpha}f(t)&=D^nK_{P_2}^{n-\alpha}f(t)\\ &=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_{t}^{t_2}(\tau-t)^{n-\alpha-1}f(\tau)\mathrm{d}\tau\\ &=(-1)^n\,{}_{t}^{\mathrm{RL}}D_{t_2}^{\alpha}f(t) \end{aligned}\tag{18}$$

$$\begin{aligned} B_{P_2}^{\alpha}f(t)&=K_{P_2}^{n-\alpha}D^nf(t)\\ &=\frac{1}{\Gamma(n-\alpha)}\int_{t}^{t_2}(\tau-t)^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^nf(\tau)\mathrm{d}\tau\\ &=(-1)^n\,{}_{t}^{\mathrm{C}}D_{t_2}^{\alpha}f(t) \end{aligned}\tag{19}$$

当 $P=P_3=<t_1,t,t_2,\frac{1}{2},\frac{1}{2}>$时,有

$$\begin{aligned} A_{P_3}^{\alpha}f(t)&=D^nK_{P_3}^{n-\alpha}f(t)\\ &=\frac{1}{2\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_{t_1}^{t}(t-\tau)^{n-\alpha-1}f(\tau)\mathrm{d}\tau+\\ &\quad\frac{1}{2\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_{t}^{t_2}(\tau-t)^{n-\alpha-1}f(\tau)\mathrm{d}\tau\\ &=\frac{1}{2}\left[{}_{t_1}^{\mathrm{RL}}D_t^{\alpha}f(t)+(-1)^n\,{}_{t}^{\mathrm{RL}}D_{t_2}^{\alpha}f(t)\right]\\ &={}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}f(t) \end{aligned}\tag{20}$$

$$\begin{aligned} B_{P_3}^{\alpha}f(t)&=K_{P_3}^{n-\alpha}D^nf(t)\\ &=\frac{1}{2\Gamma(n-\alpha)}\int_{t_1}^{t}(t-\tau)^{n-\alpha-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^nf(\tau)\mathrm{d}\tau+\\ &\quad\frac{1}{2\Gamma(n-\alpha)}\int_{t}^{t_2}(\tau-t)^{n-\alpha-1}\left(\frac{d}{\mathrm{d}t}\right)^nf(\tau)\mathrm{d}\tau\\ &=\frac{1}{2}\left[{}_{t_1}^{\mathrm{C}}D_t^{\alpha}f(t)+(-1)^n\,{}_{t}^{\mathrm{C}}D_{t_2}^{\alpha}f(t)\right]\\ &={}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}f(t) \end{aligned}\tag{21}$$

即,左(右)Riemann-Liouville 分数阶导数、左(右)Caputo 分数阶导数、Riesz-Riemann-Liouville 分数阶导数和 Riesz-Caputo 分数阶导数是分数阶微分算子 A_P^{α} 和 B_P^{α} 的特例.

2　广义分数阶受迫 Birkhoff 方程

下面分别基于分数阶微分算子 A_P^{α} 和 B_P^{α},建立广义分数阶受迫 Birkhoff 方程.

2.1 带有算子 A_P^α 的广义分数阶受迫 Birkhoff 方程

考虑

$$J_A(a^\mu,F_{A\mu})=\int_{t_1}^{t_2}[R_v(t,a^\mu)\cdot A_P^\alpha a^v-B(t,a^\mu)+F_{A\mu}\cdot a^\mu]\mathrm{d}t\rightarrow \text{ext} \tag{22}$$

其中,满足的交换条件和边界条件分别为

$$\delta A_P^\alpha a^v=A_P^\alpha\delta a^v \tag{23}$$

$$\delta a^v\Big|_{t=t_1}=\delta a^v\Big|_{t=t_2}=0 \tag{24}$$

$B(t,a^\mu)$是 Birkhoff 函数,$R_v(t,a^\mu),v=1,2,\cdots,2n$ 是 Birkhoff 函数组,$F_{A\mu},\mu=1,2,\cdots,2n$ 为广义力,δ 表示等时变分,$n-1<\alpha<n$.

由式(22)得

$$\delta J_A=\int_{t_1}^{t_2}\left[\left(\frac{\partial R_v}{\partial a^\mu}\cdot A_P^\alpha a^v-\frac{\partial B}{\partial a^\mu}+F_{A\mu}\right)\delta a^\mu+R_v\cdot\delta A_P^\alpha a^v\right]\mathrm{d}t=0 \tag{25}$$

其中,由式(14)(23)(24)可得

$$\begin{aligned}\int_{t_1}^{t_2}R_v\cdot\delta A_P^\alpha a^v\mathrm{d}t&=\int_{t_1}^{t_2}R_v\cdot A_P^\alpha\delta a^v\mathrm{d}t\\&=-\int_{t_1}^{t_2}\delta a^v\cdot B_{P*}^\alpha R_v\mathrm{d}t+R_v\cdot A_P^{\alpha-1}\delta a^v\Big|_{t_1}^{t_2}\\&=-\int_{t_1}^{t_2}\delta a^v\cdot B_{P*}^\alpha R_v\mathrm{d}t\end{aligned} \tag{26}$$

值得注意的是,在式(14)中,已令 $n=1$,此时,$0<\alpha<1$.

将式(26)代入式(25)可得

$$\delta J_A=\int_{t_1}^{t_2}\left(\frac{\partial R_v}{\partial a^\mu}\cdot A_P^\alpha a^v-\frac{\partial B}{\partial a^\mu}+F_{A\mu}-B_{P*}^\alpha R_\mu\right)\cdot\delta a^\mu\mathrm{d}t=0 \tag{27}$$

因此,

$$\frac{\partial R_v}{\partial a^\mu}\cdot A_P^\alpha a^v-\frac{\partial B}{\partial a^\mu}+F_{A\mu}-B_{P*}^\alpha R_\mu=0,\quad \mu,v=1,2,\cdots,2n,0<\alpha<1 \tag{28}$$

方程(28)称为具有分数阶微分算子 A_P^α 的广义分数阶受迫 Birkhoff 方程.

注 1 文献[16]中的式(40)和(55)应该在条件 $n=1$ 和 $0<\alpha<1$ 下成立,但作者没有指出,这也导致文献[16]中的式(68)~(73)是多余的.

当 $k_\alpha(t,\tau)=(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, $k_\alpha(\tau,t)=(\tau-t)^{\alpha-1}/\Gamma(\alpha)$时,由参数集 P 的不同取值可以得到不同形式的分数阶受迫 Birkhoff 方程.

情形 1 当 $P=P_1=<t_1,t,t_2,1,0>$时,有

$$A_{P_1}^\alpha f(t)={}_{t_1}^{\mathrm{RL}}D_t^\alpha f(t),\ B_{P_1^*}^\alpha f(t)=-{}_t^{\mathrm{C}}D_{t_2}^\alpha f(t) \tag{29}$$

将式(29)代入方程(28),可得

$$\frac{\partial R_v}{\partial a^\mu}\cdot{}_{t_1}^{\mathrm{RL}}D_t^\alpha a^v-\frac{\partial B}{\partial a^\mu}+F_{A\mu}+{}_t^{\mathrm{C}}D_{t_2}^\alpha R_\mu=0,\quad \mu,v=1,2,\cdots,2n,0<\alpha<1 \tag{30}$$

方程(30)称为具有左 Riemann-Liouville 分数阶导数的分数阶受迫 Birkhoff 方程.

情形 2 当 $P=P_2=<t_1,t,t_2,0,1>$时,有

$$A_{P_1}^\alpha f(t)=-{}_t^{\mathrm{RL}}D_{t_2}^\alpha f(t),\ B_{P_2^*}^\alpha f(t)={}_{t_1}^{\mathrm{C}}D_t^\alpha f(t) \tag{31}$$

将式(31)代入方程(28)可得

$$\frac{\partial R_v}{\partial a^\mu}\cdot{}_t^{\mathrm{RL}}D_{t_2}^\alpha a^v+\frac{\partial B}{\partial a^\mu}-F_{A\mu}+{}_{t_1}^{\mathrm{C}}D_t^\alpha R_\mu=0,\quad \mu,v=1,2,\cdots,2n,0<\alpha<1 \tag{32}$$

方程(32)称为具有右 Riemann-Liouville 分数阶导数的分数阶受迫 Birkhoff 方程.

情形 3 当 $P=P_3=<t_1,t,t_2,\frac{1}{2},\frac{1}{2}>$时,有

$$A_{P_3}^\alpha f(t)={}_{t_1}^{\mathrm{R}}D_{t_2}^\alpha f(t),\ B_{P_3^*}^\alpha f(t)={}_{t_1}^{\mathrm{RC}}D_{t_2}^\alpha f(t) \tag{33}$$

将式(33)代入方程(28)可得

$$\frac{\partial R_v}{\partial a^\mu}\cdot{}_{t_1}^{\mathrm{R}}D_{t_2}^\alpha a^v-\frac{\partial B}{\partial a^\mu}+F_{A\mu}-{}_{t_1}^{\mathrm{RC}}D_{t_2}^\alpha R_\mu=0,\quad \mu,v=1,2,\cdots,2n,0<\alpha<1 \tag{34}$$

方程(34)称为具有 Riesz-Riemann-Liouville 分数阶导数的分数阶受迫 Birkhoff 方程.

2.2 带有算子 B_P^α 的广义分数阶受迫 Birkhoff 方程

考虑

$$J_B(a^\mu,F_{B\mu})=\int_{t_1}^{t_2}[R_v(t,a^\mu)\cdot B_P^\alpha a^v-B(t,a^\mu)+F_{B\mu}\cdot a^\mu]\mathrm{d}t\rightarrow \exp \tag{35}$$

其中,满足的交换条件和边界条件分别为

$$\delta B_P^\alpha a^v=B_P^\alpha\delta a^v \tag{36}$$

$$\delta a^v\Big|_{t=t_1}=\delta a^v\Big|_{t=t_2}=0 \tag{37}$$

$B(t,a^\mu)$是 Birkhoff 函数,$R_v(t,a^\mu),v=1,2,\cdots,2n$ 是 Birkhoff 函数组,$F_{B\mu},\mu=1,2,\cdots,2n$ 为广义力,δ 表示等时变分,$n-1<\alpha<n$.

由式(35)得

$$\delta J_B=\int_{t_1}^{t_2}\left[\left(\frac{\partial R_v}{\partial a^\mu}\cdot B_P^\alpha a^v-\frac{\partial B}{\partial a^\mu}+F_{B\mu}\right)\delta a^\mu+R_v\cdot\delta B_P^\alpha a^v\right]\mathrm{d}t=0 \tag{38}$$

其中,由式(15)(36)(37)可得

$$\begin{aligned}\int_{t_1}^{t_2}R_v\cdot\delta B_P^\alpha a^v\mathrm{d}t&=\int_{t_1}^{t_2}R_v\cdot B_P^\alpha\delta a^v\mathrm{d}t\\&=-\int_{t_1}^{t_2}\delta a^v\cdot A_{P*}^\alpha R_v\mathrm{d}t+\end{aligned}$$

$$\delta a^v \cdot A_P^{\alpha-1} R_v \Big|_{t_1}^{t_2} = -\int_{t_1}^{t_2} \delta a^v \cdot A_{P*}^{\alpha} R_v \mathrm{d}t \qquad (39)$$

值得注意的是,在式(15)中,已令 $n=1$,此时,$0<\alpha<1$.

将式(39)代入式(38)可得

$$\delta J_B = \int_{t_1}^{t_2} \left(\frac{\partial R_v}{\partial a^\mu} \cdot B_P^\alpha a^v - \frac{\partial B}{\partial a^\mu} + F_{B\mu} - A_{P*}^\alpha R_\mu \right) \cdot \delta a^\mu \mathrm{d}t = 0 \qquad (40)$$

因此

$$\frac{\partial R_v}{\partial a^\mu} \cdot B_P^\alpha a^v - \frac{\partial B}{\partial a^\mu} + F_{B\mu} - A_{P*}^\alpha R_\mu = 0,$$
$$\mu, v = 1,2,\cdots,2n, 0<\alpha<1 \qquad (41)$$

方程(41)称为具有分数阶微分算子 B_P^α 的广义分数阶受迫 Birkhoff 方程.

当 $k_\alpha(t,\tau) = (t-\tau)^{\alpha-1}/\Gamma(\alpha)$, $k_\alpha(\tau,t) = (\tau-t)^{\alpha-1}/\Gamma(\alpha)$时,由参数集 P 的不同取值可以得到不同形式的分数阶受迫 Birkhoff 方程.

情形 4 当 $P=P_1=<t_1,t,t_2,1,0>$时,有

$$B_{P_1}^\alpha f(t) = {}_{t_1}^{\mathrm{C}}D_t^\alpha f(t), \ A_{P_1*}^\alpha f(t) = -{}_t^{\mathrm{RL}}D_{t_2}^\alpha f(t) \qquad (42)$$

将式(42)代入方程(41)中可得

$$\frac{\partial R_v}{\partial a^\mu} \cdot {}_{t_1}^{\mathrm{C}}D_t^\alpha a^v - \frac{\partial B}{\partial a^\mu} + F_{B\mu} + {}_t^{\mathrm{RL}}D_{t_2}^\alpha R_\mu = 0,$$
$$\mu, v = 1,2,\cdots,2n, 0<\alpha<1 \qquad (43)$$

方程(43)称为具有左 Caputo 分数阶导数的分数阶受迫 Birkhoff 方程.

情形 5 当 $P=P_2=<t_1,t,t_2,0,1>$时,有

$$B_{P_2}^\alpha f(t) = -{}_t^{\mathrm{C}}D_{t_2}^\alpha f(t), \ A_{P_2*}^\alpha f(t) = {}_{t_1}^{\mathrm{RL}}D_t^\alpha f(t) \qquad (44)$$

将式(44)代入方程(41)中可得

$$\frac{\partial R_v}{\partial a^\mu} \cdot {}_t^{\mathrm{C}}D_{t_2}^\alpha a^v + \frac{\partial B}{\partial a^\mu} - F_{B\mu} + {}_{t_1}^{\mathrm{RL}}D_t^\alpha R_\mu = 0,$$
$$\mu, v = 1,2,\cdots,2n, 0<\alpha<1 \qquad (45)$$

方程(45)称为具有右 Caputo 分数阶导数的分数阶受迫 Birkhoff 方程.

情形 6 当 $P=P_3=<t_1,t,t_2,\frac{1}{2},\frac{1}{2}>$时,有

$$B_{P_3}^\alpha f(t) = {}_{t_1}^{\mathrm{RC}}D_{t_2}^\alpha f(t), \ A_{P_3*}^\alpha f(t) = {}_{t_1}^{\mathrm{R}}D_{t_2}^\alpha f(t) \qquad (46)$$

将式(46)代入方程(41)中可得

$$\frac{\partial R_v}{\partial a^\mu} \cdot {}_{t_1}^{\mathrm{RC}}D_{t_2}^\alpha a^v - \frac{\partial B}{\partial a^\mu} + F_{B\mu} - {}_{t_1}^{\mathrm{R}}D_{t_2}^\alpha R_\mu = 0,$$
$$\mu, v = 1,2,\cdots,2n, 0<\alpha<1 \qquad (47)$$

方程(47)称为具有 Riesz-Caputo 分数阶导数的分数阶受迫 Birkhoff 方程.

注 2 方程(28)和方程(41)是本文的主要结果.如果广义力 $F_{A\mu}=F_{B\mu}=0, \mu=1,2,\cdots,2n$,则由方程(28)和方程(41)所得的结果与文献[16]里的结果是一致的.

注 3 当广义力满足 $F_{A\mu}=F_{B\mu}=0, \mu=1,2,\cdots,2n$ 时,由情形 1–情形 6 所得的结果与文献[10]里的结果是一致的.但是,本文使用的方法比文献[10]使用的方法简单.

3 特例

与 Hamilton 系统和 Lagrange 系统比起来, Birkhoff 系统更为广泛.因此,在一定条件下,可由本文的结果分别得到具有分数阶微分算子 A_P^α 和 B_P^α 的分数阶 Hamilton 方程和分数阶 Lagrange 方程.

令广义力 $F_{A\mu}=F_{B\mu}=0, \mu=1,2,\cdots,2n$,则由方程(28)和方程(41)可得

$$\frac{\partial R_v}{\partial a^\mu} \cdot A_P^\alpha a^v - \frac{\partial B}{\partial a^\mu} - B_{P*}^\alpha R_\mu = 0,$$
$$\mu, v = 1,2,\cdots,2n, 0<\alpha<1 \qquad (48)$$

$$\frac{\partial R_v}{\partial a^\mu} \cdot B_P^\alpha a^v - \frac{\partial B}{\partial a^\mu} - A_{P*}^\alpha R_\mu = 0,$$
$$\mu, v = 1,2,\cdots,2n, 0<\alpha<1 \qquad (49)$$

在变换

$$a_\mu = \begin{cases} q_\mu, \mu = 1,2,\cdots,n \\ p_{\mu-n}, \mu = n+1, n+2,\cdots,2n \end{cases},$$

$$R_\mu = \begin{cases} p_\mu, \mu = 1,2,\cdots,n \\ 0, \mu = n+1, n+2,\cdots,2n \end{cases}, \ B = H \qquad (50)$$

下,由方程(48)可得

$$B_{P*}^\alpha p_i = -\frac{\partial H(t,q_j,p_j)}{\partial q_i}, \ A_P^\alpha q_i = \frac{\partial H(t,q_j,p_j)}{\partial p_i},$$
$$i,j = 1,2,\cdots,n, 0<\alpha<1 \qquad (51)$$

方程(51)称为具有分数阶微分算子 A_P^α 的分数阶 Hamilton 方程.

在变换

$$p_i = \frac{\partial L(t,q_j,A_P^\alpha q_j)}{\partial A_P^\alpha q_i}, \ H = p_i A_P^\alpha q_i - L(t,q_j,A_P^\alpha q_j) \qquad (52)$$

下,由方程(51)可得

$$\frac{\partial L(t,q_j,A_P^\alpha q_j)}{\partial q_i} - B_{P*}^\alpha \frac{\partial L(t,q_j,A_P^\alpha q_j)}{\partial A_P^\alpha q_i} = 0,$$
$$i,j = 1,2,\cdots,n, 0<\alpha<1 \qquad (53)$$

方程(53)称为具有分数阶微分算子 A_P^α 的分数阶

Lagrange 方程.

类似地,在变换式(50)下,由方程(49)可得

$$A_{P*}^{\alpha}p_i=-\frac{\partial H(t,q_j,p_j)}{\partial q_i},\ B_P^{\alpha}q_i=\frac{\partial H(t,q_j,p_j)}{\partial p_i},$$

$$i,j=1,2,\cdots,n,0<\alpha<1 \tag{54}$$

方程(54)称为具有分数阶微分算子 B_P^{α} 的分数阶 Hamilton 方程.

在变换

$$p_i=\frac{\partial L(t,q_j,B_P^{\alpha}q_j)}{\partial B_P^{\alpha}q_i},\ H=p_iB_P^{\alpha}q_i-L(t,q_j,B_P^{\alpha}q_j) \tag{55}$$

下,由方程(54)可得

$$\frac{\partial L(t,q_j,B_P^{\alpha}q_j)}{\partial q_i}-A_{P*}^{\alpha}\frac{\partial L(t,q_j,B_P^{\alpha}q_j)}{\partial B_P^{\alpha}q_i}=0,$$

$$i,j=1,2,\cdots,n,0<\alpha<1 \tag{56}$$

方程(56)称为具有分数阶微分算子 B_P^{α} 的分数阶 Lagrange 方程.

注 4 方程(56)与文献[12]里的结果一致.

4 应用

例 1 Lotka-生化振子模型[14] 的 Birkhoff 函数和 Birkhoff 函数组分别为

$$B=\alpha_2a^1-\alpha_1a^2-\beta_1\exp a^2+\beta_2\exp a^1,$$

$$R_1=-\frac{1}{2}a^2,\ R_2=\frac{1}{2}a^1 \tag{57}$$

试建立带有分数阶微分算子 A_P^{α} 的广义分数阶 Lotka-生化振子模型.

由方程(28)可得

$$\frac{1}{2}A_P^{\alpha}a^2-\alpha_2-\beta_2\exp a^1+F_{A1}+\frac{1}{2}B_{P*}^{\alpha}a^2=0$$

$$-\frac{1}{2}A_P^{\alpha}a^1+\alpha_1+\beta_1\exp a^2+F_{A2}-\frac{1}{2}B_{P*}^{\alpha}a^1=0 \tag{58}$$

方程(58)为具有分数阶微分算子 A_P^{α} 的广义分数阶 Lotka-生化振子模型.

当 $P=P_1=<t_1,t,t_2,1,0>$时,由式(29)和方程(58)可得

$$\frac{1}{2}{}_{t_1}^{RL}D_t^{\alpha}a^2-\alpha_2-\beta_2\exp a^1+F_{A1}-\frac{1}{2}{}_t^{C}D_{t_2}^{\alpha}a^2=0,$$

$$-\frac{1}{2}{}_{t_1}^{RL}D_t^{\alpha}a^1+\alpha_1+\beta_1\exp a^2+F_{A2}+\frac{1}{2}{}_t^{C}D_{t_2}^{\alpha}a^1=0 \tag{59}$$

方程(59)称为具有左 Riemann-Liouville 分数阶导数的广义分数阶 Lotka-生化振子模型.

当 $P=P_2=<t_1,t,t_2,0,1>$时,由式(31)和方程(58)可得

$$-\frac{1}{2}{}_t^{RL}D_{t_2}^{\alpha}a^2-\alpha_2-\beta_2\exp a^1+F_{A1}+\frac{1}{2}{}_{t_1}^{C}D_t^{\alpha}a^2=0,$$

$$\frac{1}{2}{}_t^{RL}D_{t_2}^{\alpha}a^1+\alpha_1+\beta_1\exp a^2+F_{A2}-\frac{1}{2}{}_{t_1}^{C}D_t^{\alpha}a^1=0 \tag{60}$$

方程(60)称为具有右 Riemann-Liouville 分数阶导数的广义分数阶 Lotka-生化振子模型.

当 $P=P_3=<t_1,t,t_2,\frac{1}{2},\frac{1}{2}>$时,由式(33)和方程(58)可得

$$-\frac{1}{2}{}_{t_1}^{R}D_{t_2}^{\alpha}a^2-\alpha_2-\beta_2\exp a^1+F_{A1}+\frac{1}{2}{}_{t_1}^{RC}D_{t_2}^{\alpha}a^2=0,$$

$$-\frac{1}{2}{}_{t_1}^{R}D_{t_2}^{\alpha}a^1+\alpha_1+\beta_1\exp a^2+F_{A2}-\frac{1}{2}{}_{t_1}^{RC}D_{t_2}^{\alpha}a^1=0 \tag{61}$$

方程(61)称为具有 Riesz-Riemann-Liouville 分数阶导数的广义分数阶 Lotka-生化振子模型.

注 5 当 $F_{A1}=F_{A2}=0$ 时,由方程(61)得到的结果与文献[10]里的结果是一致的.

例 2 Hojman-Urrutia 模型[27] 的 Birkhoff 函数和 Birkhoff 函数组分别为

$$B=\frac{1}{2}[(a^3)^2+2a^2a^3-(a^4)^2],$$

$$R_1=a^2+a^3,\ R_2=0,\ R_3=a^4,\ R_4=0 \tag{62}$$

试建立带有分数阶微分算子 B_P^{α} 的广义分数阶 Hojman-Urrutia 模型.

由方程(41)可得

$$F_{B1}-A_{P*}^{\alpha}a^2-A_{P*}^{\alpha}a^3=0,\ B_P^{\alpha}a^1-a^3+F_{B2}=0,$$

$$B_P^{\alpha}a^1-(a^2+a^3)+F_{B3}-A_{P*}^{\alpha}a^4=0,$$

$$B_P^{\alpha}a^3+a^4+F_{B4}=0 \tag{63}$$

方程(63)为具有分数阶微分算子 B_P^{α} 的广义分数阶 Hojman-Urrutia 模型.

当 $P=P_1=<t_1,t,t_2,1,0>$时,由式(42)和方程(63)可得

$$F_{B1}+{}_t^{RL}D_{t_2}^{\alpha}a^2+{}_t^{RL}D_{t_2}^{\alpha}a^3=0,\ {}_{t_1}^{C}D_t^{\alpha}a^1-a^3+F_{B2}=0,$$

$${}_{t_1}^{C}D_t^{\alpha}a^1-(a^2+a^3)+F_{B3}+{}_t^{RL}D_{t_2}^{\alpha}a^4=0,$$

$${}_{t_1}^{C}D_t^{\alpha}a^3+a^4+F_{B4}=0 \tag{64}$$

方程(64)称为具有左 Caputo 分数阶导数的广义分数阶 Hojman-Urrutia 模型.

当 $P=P_2=<t_1,t,t_2,0,1>$时,由式(44)和方程(63)可得

$$F_{B1}-{}_{t_1}^{RL}D_t^{\alpha}a^2-{}_{t_1}^{RL}D_t^{\alpha}a^3=0,\ -{}_t^{C}D_{t_2}^{\alpha}a^1-a^3+F_{B2}=0,$$

$$-{}_{t}^{\mathrm{C}}D_{t_2}^{\alpha}a^1-(a^2+a^3)+F_{B3}-{}_{t_1}^{\mathrm{RL}}D_{t}^{\alpha}a^4=0,$$
$$-{}_{t}^{\mathrm{C}}D_{t_2}^{\alpha}a^3+a^4+F_{B4}=0 \tag{65}$$

方程(65)称为具有右 Caputo 分数阶导数的广义分数阶 Hojman-Urrutia 模型.

当 $P=P_3=\langle t_1,t,t_2,\frac{1}{2},\frac{1}{2}\rangle$时,由式(46)和方程(63)可得

$$F_{B1}-{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^2-{}_{t_1}^{\mathrm{RL}}D_{t_2}^{\alpha}a^3=0,\ {}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^1-a^3+F_{B2}=0,$$
$${}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^1-(a^2+a^3)+F_{B3}-{}_{t_1}^{\mathrm{R}}D_{t_2}^{\alpha}a^4=0,$$
$${}_{t_1}^{\mathrm{RC}}D_{t_2}^{\alpha}a^3+a^4+F_{B4}=0 \tag{66}$$

方程(66)称为具有 Riesz-Caputo 分数阶导数的广义分数阶 Hojman-Urrutia 模型.

注 6　当 $F_{B1}=F_{B2}=F_{B3}=F_{B4}=0$ 时,由方程(66)得到的结果与文献[10]里的结果是一致的.

5　结论

本文建立了具有分数阶微分算子 A_P^α 和 B_P^α 的广义分数阶受迫 Birkhoff 方程.作为特例,得到了具有左(右)Riemann-Liouville 分数阶导数、左(右)Caputo 分数阶导数、Riesz-Riemann-Liouville 分数阶导数、Riesz-Caputo 分数阶导数的分数阶受迫 Birkhoff 方程,以及具有分数阶微分算子 A_P^α 和 B_P^α 的分数阶 Hamilton 方程和分数阶 Lagrange 方程.

具有分数阶微分算子 A_P^α 和 B_P^α 的广义分数阶受迫 Birkhoff 方程、分数阶 Hamilton 方程及具有分数阶微分算子 A_P^α 的分数阶 Lagrange 方程是新结果.具有分数阶微分算子 B_P^α 的分数阶 Lagrange 方程与之前所得结果是一致的.

作为应用,文中对 Lotka-生化振子模型和 Hojman-Urrutia模型进行研究,分别建立了具有分数阶微分算子 A_P^α 的广义分数阶 Lotka-生化振子模型和具有分数阶微分算子 B_P^α 的广义分数阶 Hojman-Urrutia 模型.当这两个模型退化时,所得结果与之前已有的结论是一致的.

分数阶微分算子 A_P^α 和 B_P^α 具有统一的性质,因为它们统一了左(右)Riemann-Liouville 分数阶导数、左(右)Caputo 分数阶导数、Riesz-Riemann-Liouville分数阶导数和 Riesz-Caputo 分数阶导数.同时,它们也具有拓展的性质,因为它们可由核 k_α 和参数集 P 的不同取值而得到更多的结果.

参　考　文　献

1　Riewe F. Nonconservative Lagrangian and Hamiltonian mechanics. *Physical Review E*, 1996,53(2):1890~1899

2　Zhou S, Fu J L, Liu Y S. Lagranges equations of nonholonomic systems with fractional derivatives. *Chinese Physics B*, 2010,19(12):120301

3　Baleanu D, Muslih S I. Lagrangian formulation of classical fields with Riemann-Liouville fractional derivatives. *Physica Scripta*, 2005,72(2-3):119~121

4　Agrawal O P. Generalized Euler-Lagrange equations and transversality conditions for FVPs in terms of the Caputo derivative. *Journal of Vibration Control*, 2007,13(9-10):1217~1237

5　Almeida R, Torres D F M. Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives. *Communications in Nonlinear Science and Numerical Simulation*, 2011,16:1490~1500

6　周燕,张毅. 分数阶 Birkhoff 系统基于 Caputo 导数的 Noether 对称性与守恒量. 动力学与控制学报, 2015,13(6):410~417 (Zhou Y, Zhang Y. Noether symmetry and conserved quantity for fractional Birkhoffian systems in terms of Caputo derivatives. *Journal of Dynamics and Control*, 2015,13(6):410~417 (in Chinese))

7　Agrawal O P. Fractional variational calculus in terms of Riesz fractional derivatives. *Journal of Physics A: Mathematical and Theoretical*, 2007,40(24): 6287~6303

8　Luo S K, Zhang X T, He J M. A general method of fractional dynamics, i. e., fractional Jacobi last multiplier method, and its applications. *Acta Mechanica*, 2017,228(1):157~174

9　Almeida R. Fractional variational problems with the Riesz-Caputo derivative. *Applied Mathematics Letters*, 2012,25(2):142~148

10　Luo S K, Xu Y L. Fractional Birkhoffian mechanics. *Acta Mechanica*, 2015,226(3):829~844

11　Song C J, Zhang Y. Noether symmetry and conserved quantity for fractional Birkhoffian mechanics and its applications. *Fractional Calculus & Applied Analysis*, 2018,21(2):509~526

12　Agrawal O P. Generalized variational problems and Euler-Lagrange equations. *Computers and Mathematics with Applications*, 2010,59(5):1852~1864

13　Galiullan A S. Analytical dynamics. Moscow: Nauka, 1989 (in Russian)

14 梅凤翔,史荣昌,张永发等. Birkhoff系统动力学. 北京:北京理工大学出版社, 1996 (Mei F X, Shi R C, Zhang Y F, et al. Dynamics of Birkhoff systems. Beijing: Beijing Institute of Technology, 1996 (in Chinese))

15 Santilli R M. Foundations of theoretical mechanics II. New York: Springer-Verlag, 1983

16 Zhang H B, Chen H B. Generalized variational problems and Birkhoff equations. *Nonlinear Dynamics*, 2016,83(1-2):347~354

17 梅凤翔. 广义Birkhoff系统动力学. 北京:科学出版社, 2013 (Mei F X. Dynamics of generalized Birkhoffian system. Beijing:Science Press, 2013 (in Chinese))

18 Kong X L, Wu H B, Mei F X. Variational integrators for forced Birkhoffian systems. *Applied Mathematics and Computation*, 2013,225(12):326~332

19 Jiang W A, Li L, Li Z J, et al. Lie symmetrical perturbation and a new type of non-Noether adiabatic invariants for disturbed generalized Birkhoffian systems. *Nonlinear Dynamics*, 2012,67(2):1075~1081

20 Zhang K J, Fang J H, Li Y. Symmetry and conserved quantities of discrete generalized Birkhoffian system. *Chinese Physics B*, 2010,19(12):124601

21 Wu H B, Mei F X. Type of integral and reduction for a generalized Birkhoffian system. *Chinese Physics B*, 2011, 20(10):104501

22 崔金超,梅凤翔. 广义Birkhoff方程的平衡稳定性. 动力学与控制学报, 2010,8(4):297~299 (Cui J C, Mei F X. Stability of equilibrium for generalized Birkhoff equation. *Journal of Dynamics and Control*, 2010,8(4):297~299 (in Chinese))

23 梅凤翔,吴惠彬. 广义Birkhoff方程的梯度表示. 动力学与控制学报, 2012,10(4):289~292 (Mei F X, Wu H B. A gradient representation for generalized Birkhoff system. *Journal of Dynamics and Control*, 2012,10(4):289~292 (in Chinese))

24 曹秋鹏,陈向炜. 二阶自治广义Birkhoff系统的奇点分析. 动力学与控制学报, 2016,14(5):391~394 (Cao Q P, Chen X W. Singular points analysis for second order autonomous generalized Birkhoff systems. *Journal of Dynamics and Control*, 2016,14(5):391~394 (in Chinese))

25 Jia Q L, Wu H B, Mei F X. Noether symmetries and conserved quantities for fractional forced Birkhoffian systems. *Journal of Mathematical Analysis & Applications*, 2016, 442(2):782~795

26 Song C J, Zhang Y. Conserved quantities and adiabatic invariants for fractional generalized Birkhoffian systems. *International Journal of Non-Linear Mechanics*, 2017,90:32~38

27 Hojman S, Urrutia L F. On the inverse problem of the calculus of variations. *Journal of Mathematical Physics*, 1981,22(9):1896~1903

GENERALIZED FRACTIONAL FORCED BIRKHOFF EQUATIONS *

Song Chuanjing†

(*School of Mathematics and Physics, Suzhou University of Science and Technology, Suzhou* 215009, *China*)

Abstract The fractional calculus of variations for a forced Birkhoffian system with two fractional differential operators was studied, and the generalized fractional forced Birkhoff equations were obtained. Then, the fractional Hamilton equations and fractional Lagrange equations with two fractional differential operators were presented, respectively. Finally, the generalized fractional Lotka biochemical oscillator model and the generalized fractional Hojman-Urrutia model were discussed, respectively.

Key words fractional differential operator, calculus of variation, fractional forced Birkhoff equation

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† Corresponding author E-mail: songchuanjingsun@126.com