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有多余坐标的可控完整力学系统的自由运动与初始运动*

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摘要 对于完整力学系统,若选取的参数不是完全独立的,则称为有多余坐标的完整系统.本文研究有多余坐标的可控力学系统的自由运动与初始运动.首先,需由 d′Alembert-Lagrange 原理并利用 Lagrange 乘子法建立有多余坐标完整系统的运动微分方程;其次,由约束系统自由运动的定义,令所有乘子为零,得到系统实现自由运动的条件.第三,如果给定运动的初始条件和控制参数,就可以研究系统的初始运动.文末,举例并说明方法和结果的应用.

关键词 自由运动, 初始运动, 控制参数, 可控系统

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引言

完整系统的第二类 Lagrange 方程是受约束质点且不包含约束力的运动微分方程[1, 28,29],故为研究有多余坐标完整约束系统,需用第一类 Lagrange 方程或有多余坐标带乘子形式的 Lagrange 方程[2,3].对于完整力学系统通常采用第二类Lagrange 方程来组建系统的运动微分方程,其中的坐标是彼此独立的.但是,有多余坐标的完整系统力学不仅在运动学描述上有重要意义,而且在诸多动力学问题中,如四连杆机构的运动学描述[4,5],工程中振动仪器[5],多体系统动力学[6,7,27]等有重要意义.在多体系统的动力学中更多地采用微分—代数方程,即有多余坐标完整系统的方程,以便更好地实施计算[8,9].故有时选多余坐标反而会带来方便.

众所周知,力学系统的运动依赖于作用力以及所加的约束.因此,既可以借助力来控制运动,也可以借助于约束来控制运动.前者称为动力学控制,后者称为运动学控制.力学中所研究的约束通常表示物体间的接触作用.这类约束的反力显然是接触作用力,它属于被动力的范畴,以区别于加在物体上的主动力.本文研究的约束也表示接触条件,但不同于通常的约束它的约束力不纯粹是被动力,因为这些反力不仅依赖于主动力,而且一般来说还依赖于出现于约束方程中的控制参数,主动的作用可以借助这些参数加在系统的运动上[10].文献[11,12]研究了这类系统的 Lagrange 方程,Hamilton 方程和 Appell 方程.文献[13,14]证明,对参数约束系统的 Joundain 原理等价于 d′Alembert-Lagrange 原理.文献[15-22]研究了这类可控系统的原理、方程以及积分方法.

本文研究有多余坐标的可控完整力学系统的两类特殊的运动,一类是自由运动,另一类是初始运动.所谓自由运动是指约束力为零的一类特殊运动.文献[23]研究了非完整系统的自由运动,文献[24]研究了有多余坐标完整系统的自由运动.文献[25,26]发展了文献[23]的思想.本文进一步研究有多余坐标的可控完整系统的自由运动.所谓的初始运动,最初是由 Whittaker 在其名著[27]中提出的,他研究了完整保守系统的初始运动问题,并把它当作刚体动力学的一个可解问题.一般来说,动力学系统的运动微分方程很难求解,特别是表示为已知函数的有限形式.然而,可以用幂级数的形式来求解系统的一组微分方程,当然这里不考虑某些

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奇点的附近.这些级数将提供系统初始运动特性的任何所需信息.在最初的研究中并没有考虑到幂级数展开项的收敛性问题,这关乎到系统解是否是唯一的.用运动学控制以及动力学控制的方法可以解决这个问题,也就是说通过合适控制参数的选取,可以控制幂级数展开项的收敛性,精确到第几项,从而求解出系统的解.这种方法求解出的运动不仅仅是运动初始时刻的运动轨迹,也是系统的全局轨迹.

1　系统的运动微分方程

设系统的位形由 r 个广义坐标 $q_d(d=1,2,\cdots,r)$ 来确定.由于某些要求,需选 g 个多余坐标 $q_{r+1},q_{r+2},\cdots q_{r+g}(g=n-r)$,并有 g 个双面理想约束.

$$f_\beta(q_s,t,u_\rho)=0$$
$$(\beta=1,2,\cdots,g;\ s=1,2,\cdots n;\ \rho=1,2,\cdots,\alpha)\tag{1}$$

其中,u_ρ是控制参数.系统的运动微分方程有形式

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot q_s}-\frac{\partial T}{\partial q_s}=Q_s+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}$$
$$(\beta=1,2,\cdots,g;\ s=1,2,\cdots,n)\tag{2}$$

这儿及后面相同指标表示求和.方程(2)右端带乘子的项 λ_β 代表约束力.由方程(2)和约束方程(1)可求系统的运动,并且可求约束力.

假设系统非奇异,即设

$$\det\left(\frac{\partial^2 T}{\partial\dot q_s\partial\dot q_k}\right)\neq 0$$

则方程(2)展开为

$$A_{sk}\ddot q_k+[k,m;s]\dot q_k\dot q_m=\left(\frac{\partial B_k}{\partial q_s}-\frac{\partial B_s}{\partial q_k}\right)\dot q_k+Q_s-\frac{\partial B_s}{\partial t}+\frac{\partial T_0}{\partial q_s}-\frac{\partial A_{sk}}{\partial t}\dot q_k+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}$$
$$(s=1,2,\cdots,n)\tag{3}$$

因 $\det(A_{sk})\neq 0$,故可由式(3)求出所有广义加速度

$$\ddot q_l=A^{ls}\left\{-[k,m;s]\dot q_k\dot q_m+\left(\frac{\partial B_k}{\partial q_s}-\frac{\partial B_s}{\partial q_k}\right)\dot q_k+Q_s-\frac{\partial B_s}{\partial t}+\frac{\partial T_0}{\partial q_s}-\frac{\partial A_{sk}}{\partial t}\dot q_k+\lambda_\beta\frac{\partial f_\beta}{\partial q_s}\right\}$$
$$(l=1,2,\cdots,n)\tag{4}$$

其中,

$$A^{ls}A_{sk}=\delta_k^l\tag{5}$$

式(3)和式(4)中 A_{sk} 为动能中速度二次型系数,B_s 为一次型系数,T_0 为不含速度的项,而

$$[k,m;s]=\frac{1}{2}\left(\frac{\partial A_{ks}}{\partial q_m}+\frac{\partial A_{ms}}{\partial q_k}-\frac{\partial A_{km}}{\partial q_s}\right)\tag{6}$$

为系数矩阵(A_{sk})的第一类 Christoffel 记号.

为求得约束力,将方程(1)对 t 求两次导数,得

$$\frac{\partial^2 f_\beta}{\partial t^2}+2\frac{\partial^2 f_\beta}{\partial t\partial q_s}\dot q_s+2\frac{\partial^2 f_\beta}{\partial t\partial u_\rho}\dot u_\rho+\frac{\partial^2 f_\beta}{\partial q_s\partial q_k}\dot q_s\dot q_k+2\frac{\partial^2 f_\beta}{\partial q_s\partial u_\rho}\dot q_s\dot u_\rho+\frac{\partial^2 f_\beta}{\partial u_\rho\partial u_\alpha}\dot u_\rho\dot u_\alpha+\frac{\partial f_\beta}{\partial q_s}\ddot q_s+\frac{\partial f_\beta}{\partial u_\rho}\ddot u_\rho=0\tag{7}$$

将式(4)代入式(7),消去广义加速度,得到

$$\frac{\partial^2 f_\beta}{\partial t^2}+2\frac{\partial^2 f_\beta}{\partial t\partial q_s}\dot q_s+2\frac{\partial^2 f_\beta}{\partial t\partial u_\rho}\dot u_\rho+\frac{\partial^2 f_\beta}{\partial q_s\partial q_k}\dot q_s\dot q_k+\frac{\partial^2 f_\beta}{\partial q_s\partial u_\rho}\dot q_s\dot u_\rho+\frac{\partial^2 f_\beta}{\partial u_\rho\partial u_\alpha}\dot u_\rho\dot u_\alpha+\frac{\partial f_\beta}{\partial u_\rho}\ddot u_\rho+\frac{\partial f_\beta}{\partial q_l}A^{ls}\left\{-[k,m;s]\dot q_k\dot q_m+\left(\frac{\partial B_k}{\partial q_s}-\frac{\partial B_s}{\partial q_k}\right)\dot q_k+Q_s-\frac{\partial B_s}{\partial t}+\frac{\partial T_0}{\partial q_s}-\frac{\partial A_{sk}}{\partial t}\dot q_k+\lambda_\gamma\frac{\partial f_\gamma}{\partial q_s}\right\}=0$$
$$(\beta=1,2,\cdots,g)\tag{8}$$

当 $\det\left(\frac{\partial f_\beta}{\partial q_l}A^{ls}\frac{\partial f_\gamma}{\partial q_s}\right)\neq 0$ 时,可由式(8)求出所有 $\lambda_\gamma=\lambda_\gamma(t,q_s,\dot q_s,u_\rho,\dot u_\rho,\ddot u_\rho)(\gamma=1,2,\cdots,g)$,进而可求出约束力

$$\Lambda_s=\lambda_\gamma\frac{\partial f_\gamma}{\partial q_s}\quad(\gamma=1,2,\cdots,g;\ s=1,2,\cdots,n)\tag{9}$$

2　系统的自由运动

系统的自由运动是指约束力为零的运动,将 $\lambda_\gamma=0(\gamma=1,2,\cdots,g)$代入式(8),得到

$$\frac{\partial^2 f_\beta}{\partial t^2}+2\frac{\partial^2 f_\beta}{\partial t\partial q_s}\dot q_s+2\frac{\partial^2 f_\beta}{\partial t\partial u_\rho}\dot u_\rho+\frac{\partial^2 f_\beta}{\partial q_s\partial q_k}\dot q_s\dot q_k+\frac{\partial^2 f_\beta}{\partial q_s\partial u_\rho}\dot q_s\dot u_\rho+\frac{\partial^2 f_\beta}{\partial u_\rho\partial u_\alpha}\dot u_\rho\dot u_\alpha+\frac{\partial f_\beta}{\partial u_\rho}\ddot u_\rho+\frac{\partial f_\beta}{\partial q_l}A^{ls}\left\{-[k,m;s]\dot q_k\dot q_m+\left(\frac{\partial B_k}{\partial q_s}-\frac{\partial B_s}{\partial q_k}\right)\dot q_k+Q_s-\frac{\partial B_s}{\partial t}+\frac{\partial T_0}{\partial q_s}-\frac{\partial A_{sk}}{\partial t}\dot q_k\right\}=0\quad(\beta=1,2,\cdots,g)\tag{10}$$

这就是系统发生自由运动的条件.如果没有控制参数 u,则式(10)给出文献[24]的结果.

当系统发生自由运动时,方程(2)成为

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial\dot q_s}-\frac{\partial T}{\partial q_s}=Q_s\quad(s=1,2,\cdots,n)$$

3 系统的初始运动

将方程(9)带入方程(4),可求得所有广义加速度,记作

$$\ddot{q}_l = F_l(t, q_s, \dot{q}_s, u_\rho, \dot{u}_\rho, \ddot{u}_\rho) \quad (l=1,2,\cdots,n) \tag{11}$$

称方程(11)为与有多余坐标完整系统(1)(2)相应的完整系统的方程.如果运动的初始条件满足约束方程(1),则方程(11)的解就给出系统(1)(2)的运动.

初始运动的提法如下:

已知运动的初始条件

$$t=0,\ q_s = q_{s0},\ \dot{q}_s = \dot{q}_{s0} \tag{12}$$

及控制参数 $u_\rho(\rho=1,2,\cdots,\alpha)$,求初始加速度 $\ddot{q}_{l0}$,初始加加速度 $\dddot{q}_{l0}$,以及广义速度更高阶导数的初始值.

这里需要注意的是,式(12)中的 q_{s0} 不是彼此独立的,而是受到约束(1)的限制,$\dot{q}_{s0}$ 也要受到约束(1)对导数的限制.

为解上述问题,将式(12)代入式(11),得到初始加速度

$$\ddot{q}_{l0} = F_l(0, q_{s0}, \dot{q}_{s0}, u_{\rho 0}, \dot{u}_{\rho 0}, \ddot{u}_{\rho 0}) \quad (l=1,2,\cdots,n) \tag{13}$$

为求 $\dddot{q}_{l0}$,将方程(11)对 t 求导数,再代入式(12)及式(13),以及 $u_{\rho 0}, \dot{u}_{\rho 0}, \ddot{u}_{\rho 0}, \dddot{u}_{\rho 0}$.

4 算例

有多余坐标完整系统的动能,广义力和约束分别为

$$T = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2)$$

$$Q_1 = -q_1 - q_2,\ Q_2 = -2(\dot{q}_1 + \dot{q}_2),\ Q_3 = 0$$

$$f = q_3 - u(q_1 + q_2) = 0 \tag{14}$$

其中,u 为控制参数,而各量已无量纲化,试研究

1)在怎样的 u 下系统可实现自由运动;

2)在 $u=u_0+t$ 下的初始运动

首先,研究自由运动.将式(14)代入式(10),得到

$$-\ddot{u}(q_1+q_2) + u(q_1+q_2) + 2u(\dot{q}_1+\dot{q}_2) - 2(\dot{q}_1\dot{u} + \dot{q}_2\dot{u}) = 0$$

可找到控制参数

$$u = u_0 \exp t \tag{15}$$

其次,研究初始运动.运动微分方程(2)给出

$$\ddot{q}_1 = -q_1 - q_2 - \lambda u$$

$$\ddot{q}_2 = -u(\dot{q}_1 + \dot{q}_2) - \lambda u$$

$$\ddot{q}_3 = \lambda$$

解得约束乘子

$$\lambda = \frac{(\ddot{u}-u)(q_1+q_2) + 2(\dot{u}-u)(\dot{q}_1+\dot{q}_2)}{1+2u^2}$$

代入方程,得

$$\begin{aligned}
\ddot{q}_1 &= -q_1 - q_2 - \frac{u}{1+2u^2}\{(\ddot{u}-u)(q_1+q_2) + 2(\dot{u}-u)(\dot{q}_1+\dot{q}_2)\} \\
\ddot{q}_2 &= -2\dot{q}_1 - 2\dot{q}_2 - \frac{u}{1+2u^2}\{(\ddot{u}-u)(q_1+q_2) + 2(\dot{u}-u)(\dot{q}_1+\dot{q}_2)\} \\
\ddot{q}_3 &= \frac{1}{1+2u^2}\{(\ddot{u}-u)(q_1+q_2) + 2(\dot{u}-u)(\dot{q}_1+\dot{q}_2)\}
\end{aligned} \tag{16}$$

将

$$t=0, q_1=q_{10}, q_2=q_{20}, q_3=q_{30}, \dot{q}_1=\dot{q}_{10}, \dot{q}_2=\dot{q}_{20}, \dot{q}_3=\dot{q}_{30}$$

以及

$$u = u_0 + t$$

代入方程(16)得到初始加速度

$$\begin{aligned}
\ddot{q}_{10} &= -q_{10} - q_{20} - \frac{u_0}{1+2u_0^2}\{-u_0(q_{10}+q_{20}) + 2(1-u_0)(\dot{q}_{10}+\dot{q}_{20})\} \\
\ddot{q}_{20} &= -2\dot{q}_{10} - 2\dot{q}_{20} - \frac{u_0}{1+2u_0^2}\{-u_0(q_{10}+q_{20}) + 2(1-u_0)(\dot{q}_{10}+\dot{q}_{20})\} \\
\ddot{q}_{30} &= \frac{u_0}{1+2u_0^2}\{-u_0(q_{10}+q_{20}) + 2(1-u_0)(\dot{q}_{10}+\dot{q}_{20})\}
\end{aligned} \tag{17}$$

将方程(16)对 t 求导数并代入初值,可求得初始加加速度等,于是,初始运动可写成形式

$$\begin{aligned}
q_1 &= q_{10} + \dot{q}_{10}t + \frac{1}{2!}\ddot{q}_{10}t^2 + \frac{1}{3!}\dddot{q}_{10}t^3 + \cdots \\
q_2 &= q_{20} + \dot{q}_{20}t + \frac{1}{2!}\ddot{q}_{20}t^2 + \frac{1}{3!}\dddot{q}_{20}t^3 + \cdots \\
q_3 &= q_{30} + \dot{q}_{30}t + \frac{1}{2!}\ddot{q}_{30}t^2 + \frac{1}{3!}\dddot{q}_{30}t^3 + \cdots
\end{aligned} \tag{18}$$

其中,$\ddot{q}_{10}, \ddot{q}_{20}, \ddot{q}_{30}$ 由式(17)确定,而 q_{30} 和 $\dot{q}_{30}$ 由约束方程确定为

$$q_{30} = u_0(q_{10} + q_{20})$$

$$\dot{q}_{30}=q_{10}+q_{20}+u_0(\dot{q}_{10}+\dot{q}_{20}) \tag{19}$$

5 结论

本文研究了有多余坐标可控完整力学系统的两类特殊运动,即自由运动和初始运动.对自由运动,得到系统实现自由运动的条件(10),它是比较复杂的;在给定 T,Q_s 以及约束下来寻求控制参数 u,一般来说也是相对困难的.对初始运动问题,在给定运动的初始条件以及控制参数 u,来求初始运动相对容易.

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FREE MOTION AND INITIAL MOTION OF CONTROLLABLE HOLONOMIC SYSTEM WITH REDUNDANT COORDINATES *

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Abstract When the parameters of a system are not completely independent, it is called a holonomic system with redundant coordinates. This article explored the free motion and initial motion of a controllable holonomic system with redundant coordinates. Firstly, the differential equations of motion were established according to the d'Alembert-Lagrange principle, and the multipliers of constraints were obtained. Secondly, according to the definition of free motion of a constrained system, all the multipliers were set to zero, and the conditions for free motion of the system were obtained. Thirdly, the initial motion of the system was studied, when the initial condition of motion and control parameters were given. Finally, an example was given to illustrate the application of this method.

Key words free motion, initial motion, control parameter, controllable system

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