

# 石墨烯树脂复合材料板 1:3 内共振非线性动力学分析\*

张博 郭翔鹰<sup>†</sup> 姜盼

(北京工业大学 机电学院,北京 100124)

**摘要** 石墨烯是一种新型的碳纳米材料,将石墨烯添加到不同基体中可达到增强基体力学性能,并优化结构的效果.本文主要研究了在悬臂边界条件下石墨烯复合材料板的非线性动力学行为,通过 Halpin-Tsai 模型计算了复合材料板的等效杨氏模量,运用一阶剪切板理论和 Hamilton 原理得到复合材料板的非线性动力学方程.通过模态缩聚,得到复合材料板横向位移的运动控制方程.应用 Rayleigh-Ritz 法计算出复合材料板的固有频率,发现第二阶与第四阶固有频率间存在 1:3 的关系.利用多尺度法研究了复合材料板在 1:3 内共振情况下的非线性振动响应,通过数值模拟分析了外激励对板结构非线性振动响应的影响.结果表明:横向外激励对复合材料板的非线性动力学行为影响较大,对实际工程具有一定的指导意义.

**关键词** 石墨烯, Rayleigh-Ritz 法, 悬臂板, 非线性振动

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## 引言

石墨烯是一种新型二维纳米材料,具有很多优异的性能.将其作为增强材料添加到环氧树脂中,形成的复合材料具有较高的拉伸、弯曲强度,以及导电、导热性能,同时克服了环氧树脂易断裂的脆性问题.Rafiee 等人<sup>[1]</sup>通过实验研究了石墨烯增强梁的力学特性,发现石墨烯含量仅为 0.1wt% 时,基体材料的强度增强了 50%.Parashar 和 Mertiny<sup>[2]</sup>证实了随着石墨烯含量增加,基体材料屈曲性能随之增强,在添加体积分数 6% 时的石墨烯,能使复合材料的屈曲性能比原基体材料提高 26%.Song 等人<sup>[3]</sup>研究了石墨烯/环氧树脂组成的功能梯度板,发现加入质量分数为 1.2% 的石墨烯后,四种石墨烯分布不同的功能梯度板的基频值都高于纯环氧树脂板.Zhan 等人<sup>[4]</sup>表明石墨烯/压电叠层膜的非线性振动振幅可以通过调整参数来控制,且与非线性谐振频率阶数和振型相关.

Shen 等人<sup>[5]</sup>研究了在不同温度下,石墨烯分布形式不同的功能梯度板的动态响应,并对比了不同边界条件对结构动态特性的影响.用石墨烯增强的复合材料,可以明显地抑制其弯曲变形.Xu 等人<sup>[6]</sup>研究了功能梯度石墨烯增强纳米复合材料层

合板(FG-GRC),得出板中石墨烯的分布规律、质量分数、温度和几何参数对 FG-GRC 板的振动声特性有显著的影响. Shingare 和 Kundalwal<sup>[7]</sup>研究了具有挠曲电效应的石墨烯增强纳米复合材料(GRNC)板的静态和动态行为,表明挠曲电效应和板厚在薄板的静态和动力学过程中起着重要的作用.Feng 等人<sup>[8]</sup>在复合材料梁中分别添加质量分数为 0.5% 和 1.0% 的石墨烯,发现复合材料梁的挠度明显降低,分别减少了 13.7% 和 26.1%.Sobhy 等人<sup>[9]</sup>运用正弦剪切变形理论建立了单层石墨烯片非局部参数的偏微分本构方程,研究了石墨烯含量对结构的弯曲和振动特性的影响.Li 等人<sup>[10]</sup>研究了波纹状的石墨烯片层与压电(CGP)材料的叠层结构,通过等效正交异性板法分析波纹膜的共振频率,得出几何非线性和外加控制电压对结构动态特性有显著影响.

本文利用 Halpin-Tsai 模型计算了石墨烯/环氧树脂复合材料板的等效杨氏模量,通过一阶剪切理论与 Hamilton 原理建立了复合材料板的动力学方程,并应用 Galerkin 方法得到关于横向位移的两自由度常微分方程.采用 Rayleigh-Ritz 法计算了复合材料悬臂板的固有频率,发现第二阶与第四阶固有频率之间存在 1:3 比例关系.利用多尺度法研究了

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<sup>†</sup> 通讯作者 E-mail: eagle2008guo@yeah.net

1:3内共振中阻尼、外激励对频响曲线的影响,同时还研究了外激励对板结构非线性振动的响应。

## 1 复合材料动力学方程

石墨烯/环氧树脂复合材料悬臂板结构的力学模型如图1所示,板的长、宽、高分别为 $a$ 、 $b$ 、 $h$ ,在复合材料板的中面上建立直角坐标系.板上受到横向简谐激励 $f = F \cos \Omega t$ 作用。

用Halpin-Tsai模型<sup>[11]</sup>计算复合材料板结构的等效杨氏模量,计算公式如下

$$E_c = E_m \left( \frac{3}{8} \frac{1 + \eta_L \xi \phi_g}{1 - \eta_L \phi_g} + \frac{5}{8} \frac{1 + 2\eta_w \phi_g}{1 - \eta_w \phi_g} \right) \quad (1)$$

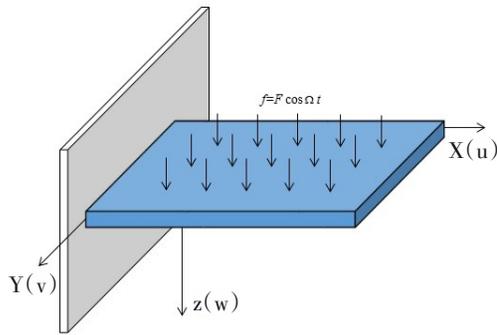


图1 石墨烯环氧树脂复合材料悬臂板

Fig.1 Graphene epoxy resin composite cantilever plate

其中,下标 $m$ 表示环氧树脂, $g$ 表示石墨烯, $c$ 表示石墨烯环氧树脂复合材料.公式(1)中参数的 $\phi_g$ 为石墨烯添加进复合材料中的体积分,  $\eta_L$ 、 $\eta_w$ 表示和石墨烯材料相关的参数,其计算公式为

$$\phi_g = \frac{W_g}{W_g + (\rho_g/\rho_m)(1 - W_g)} \quad (2)$$

其中, $\rho_g$ 、 $\rho_m$ 分别是石墨烯和环氧树脂的密度, $W_g$ 为添加石墨烯的质量分数。

系数 $\eta_L$ 、 $\eta_w$ 计算公式分别为

$$\eta_L = \frac{(E_g/E_m) - 1}{(E_g/E_m) + \xi} \quad (3)$$

$$\eta_w = \frac{(E_g/E_m) - 1}{(E_g/E_m) + 2}$$

其中, $\xi$ 由石墨烯几何尺寸决定,可由公式(4)计算得到

$$\xi = 2 \left( \frac{(W + L)/2}{t_g} \right) \quad (4)$$

其中, $L$ 、 $W$ 和 $t_g$ 分别表示石墨烯的长、宽、厚度。

复合材料板泊松比与密度通过混合率<sup>[12]</sup>求解

$$\mu_c = \mu_g \phi_g + \mu_m (1 - \phi_g) \quad (5)$$

$$\rho_c = \rho_g \phi_g + \rho_m (1 - \phi_g) \quad (6)$$

根据一阶剪切板的位移方程<sup>[13]</sup>,可得到如下位移场表达式:

$$u(x, y, z, t) = u_0(x, y, t) + z \phi_x(x, y, t) \quad (7a)$$

$$v(x, y, z, t) = v_0(x, y, t) + z \phi_y(x, y, t) \quad (7b)$$

$$w(x, y, z, t) = w_0(x, y, t) \quad (7c)$$

位移应变关系表示为:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{Bmatrix} \quad (8a)$$

$$\begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2} \left( \frac{\partial w_0}{\partial x} \right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2} \left( \frac{\partial w_0}{\partial y} \right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x} \frac{\partial w_0}{\partial y} \end{Bmatrix} \quad (8b)$$

$$\begin{Bmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix}, \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w_0}{\partial x} + \phi_y \\ \frac{\partial w_0}{\partial y} + \phi_x \end{Bmatrix} \quad (8c)$$

由于本文研究的石墨烯树脂复合材料为各向同性材料,其应力与应变关系可表示为

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \quad (9)$$

其中,  $Q_{11} = Q_{22} = \frac{E_c}{1 - \mu_c^2}$ ,  $Q_{12} = Q_{21} = \frac{\mu_c E_c}{1 - \mu_c^2}$ ,

$$Q_{44} = Q_{55} = Q_{66} = \frac{E_c}{2(1 + \mu_c)}$$

内力及内力矩可表示为

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{Bmatrix} \quad (10a)$$

$$\begin{pmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{pmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{pmatrix} \quad (10b)$$

$$\begin{pmatrix} Q_y \\ Q_x \end{pmatrix} = K \begin{bmatrix} A_{44} & 0 \\ 0 & A_{55} \end{bmatrix} \begin{pmatrix} \gamma_{yz} \\ \gamma_{xz} \end{pmatrix} \quad (10c)$$

其中,  $K$  为剪切修正因子, 由 Hamilton 原理

$$\delta \int_{t_0}^{t_1} (T - U + W) dt = 0 \quad (11)$$

从而得到广义位移表示的复合材料板的非线性运动方程为:

$$\begin{aligned} & A_{66}(u_{0,yy} + v_{0,xy} + w_{0,xy}w_{0,y} + w_{0,x}w_{0,yy}) \\ & + A_{11}(u_{0,xx} + w_{0,x}w_{0,xx}) + A_{12}(v_{0,xy} + w_{0,y}w_{0,xy}) \\ & = I_0 u_{0,t} + I_1 \phi_{x,xt} \end{aligned} \quad (12a)$$

$$\begin{aligned} & A_{66}(u_{0,xy} + v_{0,xx} + w_{0,xx}w_{0,y} + w_{0,x}w_{0,xy}) \\ & + A_{12}(u_{0,xy} + w_{0,x}w_{0,xy}) + A_{22}(v_{0,yy} + w_{0,y}w_{0,yy}) \\ & = I_0 v_{0,t} + I_1 \phi_{y,yy} \end{aligned} \quad (12b)$$

$$\begin{aligned} & 2A_{12}w_{0,x}w_{0,y}w_{0,xy} + 4A_{66}w_{0,x}w_{0,y}w_{0,xy} \\ & + A_{66}(w_{0,y})^2 w_{0,xx} + A_{66}(w_{0,x})^2 w_{0,yy} \\ & + 2A_{66}u_{0,y}w_{0,xy} + 2A_{66}v_{0,x}w_{0,xy} \\ & + A_{66}w_{0,y}v_{0,xx} + A_{66}w_{0,y}u_{0,xy} + A_{66}w_{0,x}v_{0,xy} \\ & + A_{66}w_{0,x}u_{0,yy} + A_{12}w_{0,y}u_{0,xy} + A_{22}w_{0,y}v_{0,yy} \\ & + \frac{3}{2}A_{22}(w_{0,y})^2 w_{0,yy} + \frac{1}{2}A_{12}(w_{0,x})^2 w_{0,yy} \\ & + \frac{1}{2}A_{12}(w_{0,y})^2 w_{0,xx} + \frac{3}{2}A_{11}(w_{0,x})^2 w_{0,xx} \\ & + A_{12}w_{0,yy}u_{0,x} + A_{22}v_{0,y}w_{0,yy} + A_{11}w_{0,x}u_{0,xx} \\ & + A_{12}w_{0,x}v_{0,xy} + A_{11}u_{0,x}w_{0,xx} + A_{12}w_{0,xx}v_{0,y} \\ & + KA_{55}(w_{0,xx} + \phi_{x,x}) + KA_{44}(w_{0,yy} + \phi_{y,y}) \\ & + f - rw_{0,t} = I_0 w_{0,t} \end{aligned} \quad (12c)$$

$$\begin{aligned} & D_{66}(\phi_{x,yy} + \phi_{y,xy}) + D_{11}\phi_{x,xx} + D_{12}\phi_{y,xy} \\ & - KA_{55}(w_{0,x} + \phi_x) = I_1 u_{0,t} + I_2 \phi_{x,t} \end{aligned} \quad (12d)$$

$$\begin{aligned} & D_{66}(\phi_{y,xx} + \phi_{x,xy}) + D_{22}\phi_{y,yy} + D_{12}\phi_{x,xy} \\ & - KA_{44}(w_{0,y} + \phi_y) = I_1 v_{0,t} + I_2 \phi_{y,t} \end{aligned} \quad (12e)$$

根据文献[14], 引入无量纲参数, 对方程(12)进行无量纲化.

$$\begin{aligned} & u_0 = \bar{u}a, v_0 = \bar{v}b, w_0 = \bar{w}h, x = \bar{x}a, y = \bar{y}b, \\ & t = \bar{t} \sqrt{\frac{ab\rho}{E}} \frac{1}{\pi^2}, A_{ij} = \bar{A}_{ij} \frac{Eh^2}{\sqrt{ab}}, D_{ij} = \bar{D}_{ij} \frac{Eh^4}{\sqrt{ab}}, \\ & I_i = \bar{I}_i(ab)^{\frac{i+1}{2}} \rho, f = \bar{f} \frac{Eh^7}{(ab)^{\frac{7}{2}}}, \phi_x = \bar{\phi}_x, \phi_y = \bar{\phi}_y \end{aligned}$$

将悬臂梁与自由梁的模态函数组合, 用来表达悬臂板的模态.

$$u_0 = u_1 \sin \frac{\pi x}{2a} \cos \frac{\pi y}{b} + u_2 \sin \frac{3\pi x}{2a} \cos \frac{2\pi y}{b} \quad (13a)$$

$$v_0 = v_1 \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} + v_2 \cos \frac{3\pi x}{2a} \sin \frac{2\pi y}{b} \quad (13b)$$

$$w_0 = w_1(t) X_1(x) Y_1(y) + w_2(t) X_2(x) Y_2(y) \quad (13c)$$

$$\phi_x = \phi_1 \sin \frac{\pi x}{2a} \cos \frac{\pi y}{b} + \phi_2 \sin \frac{3\pi x}{2a} \cos \frac{2\pi y}{b} \quad (13d)$$

$$\phi_y = \phi_3 \cos \frac{\pi x}{2a} \sin \frac{\pi y}{b} + \phi_4 \cos \frac{3\pi x}{2a} \sin \frac{2\pi y}{b} \quad (13e)$$

其中,  $X_i(x), Y_i(y)$  的表达式如下:

$$X_i(x) = \sin \lambda_i x - \sinh \lambda_i x + \alpha_i (\cosh \lambda_i x - \cos \lambda_i x) \quad (14a)$$

$$Y_1(y) = 1, Y_2(y) = \sqrt{3} \left( 1 + \frac{2y}{b} \right) \quad (14b)$$

其中, 系数  $\lambda_i$  的计算如下

$$\cos \lambda_i a \cosh \lambda_i a + 1 = 0 \quad (15)$$

由 Galerkin 方法离散, 且忽略  $u_0, v_0, \phi_x, \phi_y$  的惯性项, 得到复合材料板两自由度的运动控制方程如下:

$$\begin{aligned} & \ddot{w}_1 + r_1 \dot{w}_1 + \omega_1^2 w_1 + c_{11} w_1^3 + c_{12} w_2^3 \\ & + c_{13} w_1^2 w_2 + c_{14} w_1 w_2^2 = f_1 \cos \Omega t \end{aligned} \quad (16a)$$

$$\begin{aligned} & \ddot{w}_2 + r_2 \dot{w}_2 + \omega_2^2 w_2 + c_{21} w_1^3 + c_{22} w_2^3 \\ & + c_{23} w_1^2 w_2 + c_{24} w_1 w_2^2 = f_2 \cos \Omega t \end{aligned} \quad (16b)$$

其中,  $r_1, f_1, \omega_1$  是第一阶模态的阻尼系数、横向激励、线性频率,  $r_2, f_2, \omega_2$  是第二阶模态的阻尼系数、横向激励、线性频率,  $c_{ij} (i=1, 2, j=1, 2, 3, 4)$  是方程中的系数, 通过代入材料参数计算得到.

## 2 Rayleigh-Ritz 法计算板结构低阶固有频率

Rayleigh-Ritz 法可用来计算结构的低阶固有频率, 以最小势能原理为理论基础, 通过选择试函数来逼近问题的精确解, 将试函数代入泛函中, 然后对泛函求驻值, 来确定试函数中的待定参数. 应用 Rayleigh-Ritz 法求解悬臂板结构的固有频率时, 将悬臂边界看作两个弹簧, 通过计算两个弹簧的势能矩阵, 模拟了悬臂的边界条件. 并叠加原来板的势能得到新的刚度矩阵, 从而计算得到含边界结构的频率和振型, 这种方法既降低了 Rayleigh-Ritz 法对试函数的选择要求, 同时也有更加精确的计算结果.

应用公式(9)计算复合材料板的应变能:

$$U = \frac{1}{2} \iint_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{xx} \varepsilon_{xx} + \sigma_{yy} \varepsilon_{yy} + \tau_{xy} \gamma_{xy}) dz dx dy$$

$$= \frac{1}{2} \iint_D \left\{ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \left( \frac{\partial^2 w}{\partial y^2} \right)^2 + 2\mu \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2(1-\mu) \left( \frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\} dx dy \quad (17)$$

同理求得动能为

$$T = \frac{1}{2} \iint_{-\frac{h}{2}}^{\frac{h}{2}} \rho \dot{w}^2 dz dx dy = \frac{1}{2} \iint \rho h \dot{w}^2 dx dy \quad (18)$$

设横向位移解的表达式为

$$w = Z(x, y) e^{i\omega t} = \sum_{i=1}^n \sum_{j=1}^m \alpha_{ij} U_i V_j \quad (19)$$

其中, 试函数  $U_i, V_j$  为

$$U_i = \cos \frac{i\pi x}{a}, V_j = \cos \frac{j\pi y}{b} \quad (20)$$

将表达式(19)代入到式(17)与式(18)中, 有

$$U_{\max} = \frac{1}{2} D \iint \left[ \left( \frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} \right)^2 - 2(1-\mu) \left( \frac{\partial^2 Z}{\partial x^2} \frac{\partial^2 Z}{\partial y^2} - \left( \frac{\partial^2 Z}{\partial x \partial y} \right)^2 \right) \right] dx dy \quad (21)$$

$$T_{\max} = \frac{\rho_c}{2} p^2 \iint h Z^2 dx dy \quad (22)$$

悬臂板边界势能计算如下:

$$V_{\max} = \int_0^b \left[ \frac{k_t}{2} W^2 \Big|_{x=0} + \frac{k_r}{2} \left( \frac{\partial W}{\partial x} \right)^2 \Big|_{x=0} \right] dy \quad (23)$$

上式中  $k_t$  是平动弹簧,  $k_r$  是转动弹簧.

$$\frac{\partial (U_{\max} + V_{\max} - T_{\max})}{\partial \alpha_{ij}} = 0, \left( \begin{matrix} i = 1, 2, \dots, n \\ j = 1, 2, \dots, m \end{matrix} \right) \quad (24)$$

将式(21)、(22)、(23)代入到式(24)中求解偏导数为零的值, 求解得到系数  $\alpha_{ij}$  和固有频率的值.

选取的材料参数见表1, 复合材料薄板几何参数  $a, b, h$  为 1.5m、0.5m、0.01m, 代入到上述方程(30)~(37)中, 计算添加石墨烯质量分数为 0.5% 时, 复合材料板的固有频率, 将计算结果同有限元计算的结果进行对比. 表2为 Rayleigh-Ritz 法计算结果和有限元仿真的对比结果.

由于计算得到的第一阶固有频率偏低, 所以在研究中, 我们比较了结构各阶频率同第二阶频率的比值, 如图2所示. 图3为 Rayleigh-Ritz 法计算得到

表1 选取材料参数

Material	Young modulus (GPa)	Poisson ratio	Density (kg/m <sup>3</sup> )
Epoxy	3	0.34	1200
Graphene	1010	0.186	1062.5

的结构前六阶频率的振型图, 与悬臂板实际振动的振型情况相符.

表2  $W_g$  为 0.5% 的复合材料板的固有频率

Table 2 Natural frequencies of composite plates with  $W_g = 0.5\%$

Method	Rayleigh-Ritz	Rayleigh-Ritz	Finite element
Mode	Epoxy	Composite plate	Composite plate
1	1.1952	1.500	1.491
2	7.4807	9.116	9.010
3	7.7435	9.359	9.308
4	21.0232	26.329	26.165
5	24.2261	28.701	28.356
6	41.3492	51.922	51.431

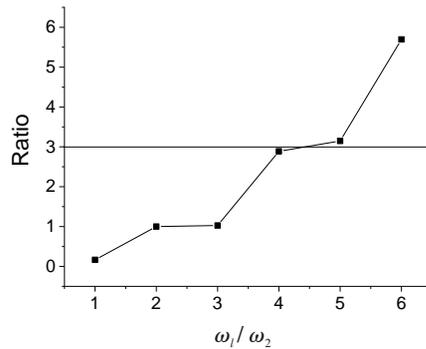


图2 各阶频率与第二阶的频率比

Fig.2 Frequency ratio of each order to the second order

### 3 摄动分析

通过以上计算得知复合材料板结构第二阶频率同第四阶频率存在 1:3 的关系, 而且内共振对结构破坏有很大的影响, 因此下面的分析中, 考虑复合材料板 1:3 的内共振的情况.

引入小参数  $\varepsilon$  变换尺度, 代入到式(16)中, 得到结构含小参量的非线性动力学方程.

$$\ddot{w}_1 + \varepsilon r_1 \dot{w}_1 + \omega_1^2 w_1 + \varepsilon c_{11} w_1^3 + \varepsilon c_{12} w_2^3 + \varepsilon c_{13} w_1^2 w_2 + \varepsilon c_{14} w_1 w_2^2 = \varepsilon f_1 \cos \Omega t \quad (25a)$$

$$\ddot{w}_2 + \varepsilon r_2 \dot{w}_2 + \omega_2^2 w_2 + \varepsilon c_{21} w_1^3 + \varepsilon c_{22} w_2^3 + \varepsilon c_{23} w_1^2 w_2 + \varepsilon c_{24} w_1 w_2^2 = \varepsilon f_2 \cos \Omega t \quad (25b)$$

设系统(25)的解为:

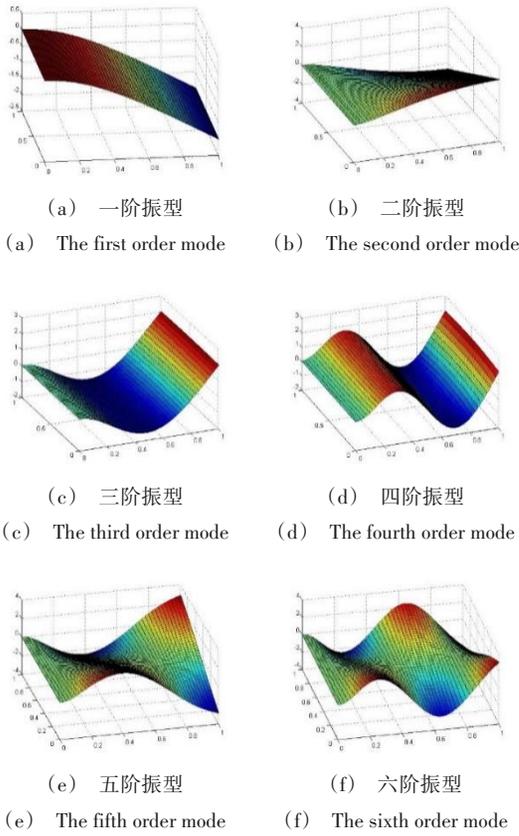


图3 Rayleigh-Ritz法计算得到的前六阶振型

Fig.3 The first six mode shapes calculated by Rayleigh-Ritz method

$$w_n(t, \varepsilon) = x_{n0}(T_0, T_1) + \varepsilon x_{n1}(T_0, T_1), (n = 1, 2) \quad (26)$$

考虑1:3的内共振,共振关系如下:

$$\omega_1^2 = \frac{1}{9}\Omega^2 + \varepsilon\sigma_1, \omega_2^2 = \Omega^2 + \varepsilon\sigma_2, \omega_2 \approx 3\omega_1 \quad (27)$$

其中,  $\omega_1$  和  $\omega_2$  表示结构第二阶和第四阶线性固有频率,  $\sigma_1$  和  $\sigma_2$  是调谐参数.

利用摄动分析,计算得到系统极坐标下的平均方程为:

$$\dot{a}_1 = -\frac{1}{2}r_1 a_1 - \frac{3}{8}c_{13} a_1^2 a_2 \sin(\varphi_2 - 3\varphi_1) \quad (28a)$$

$$a_1 \dot{\varphi}_1 = -3\sigma_1 a_1 + \frac{9}{8}c_{11} a_1^3 + \frac{3}{4}c_{14} a_1 a_2^2 + \frac{3}{8}c_{13} a_1^2 a_2 \cos(\varphi_2 - 3\varphi_1) \quad (28b)$$

$$\dot{a}_2 = -\frac{r_2}{2} a_2 - \frac{5}{8}c_{21} a_1^3 \sin(3\varphi_1 - \varphi_2) - \frac{1}{2}f_2 \sin\varphi_2 \quad (28c)$$

$$a_2 \dot{\varphi}_2 = -\sigma_2 a_2 - \frac{3}{8}c_{22} a_2^3 + \frac{1}{4}c_{23} a_1^2 a_2 \quad (28d)$$

$$+ \frac{1}{8}c_{21} a_1^3 \cos(3\varphi_1 - \varphi_2) - \frac{1}{2}f_2 \cos\varphi_2$$

考虑定常振动解时,系统在1:3内共振情况下的频响函数可得:

$$\left(3\sigma_1 a_1 - \frac{9}{8}c_{11} a_1^3 - \frac{3}{4}c_{14} a_1 a_2^2\right)^2 \quad (29a)$$

$$+ \left(\frac{1}{2}r_1 a_1\right)^2 = \left(\frac{3}{8}c_{13} a_1^2 a_2\right)^2$$

$$\left(a_2 \sigma_2 + \frac{3}{8}c_{22} a_2^3 - \frac{1}{4}c_{23} a_1^2 a_2 - \frac{1}{8}c_{21} a_1^3 \cos(3\varphi_1 - \varphi_2)\right)^2 + \left(\frac{1}{2}r_2 a_2 + \frac{5}{8}c_{21} a_1^3 \sin(3\varphi_1 - \varphi_2)\right)^2 = \left(\frac{1}{2}f_2\right)^2 \quad (29b)$$

#### 4 数值模拟

基于振动控制方程(16)和频响函数(29),分析了石墨烯/环氧树脂复合材料板的非线性振动特性.参数取值参照表1,添加的石墨烯质量分数为0.5%.在定常解的方程(29-b)中,考虑弱耦合效应,令  $a_1 = 1$ ,分析参数变化对幅频响应曲线的影响.通过图4可知,随着阻尼的增大,系统的振幅减小,硬弹簧特性减弱.与之相反的,随着横向力的增大,系统的振幅增加,硬弹簧特性变强,如图5所示.

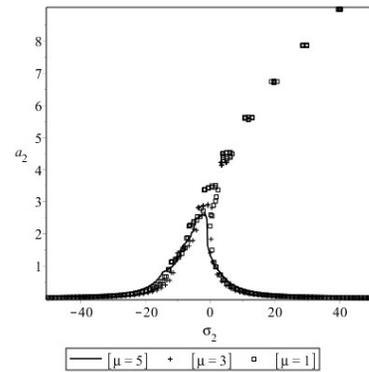


图4 二阶模态的频响曲线随阻尼的变化

Fig.4 Frequency-response curves of the second-order mode with different damping

使用Runge-Kutta法研究了横向激励对系统非线性振动特性的影响.通过图7发现,横向激励从0到120的过程中,结构出现了先周期运动、再混沌和周期交替出现、最后进入周期运动的过程.

为了更清晰地描述结构的动态特性,本文同时给出了系统在不同外激励作用下的周期和混沌运动的具体形式,其中,(a)、(b)为波形图,(c)、(d)为

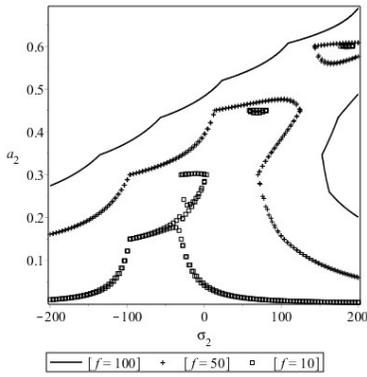
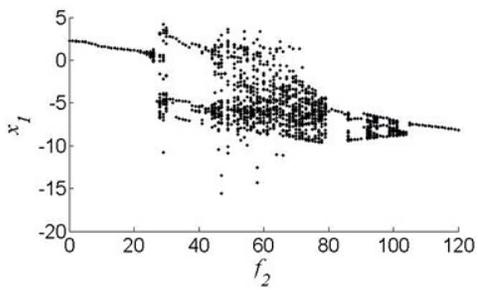
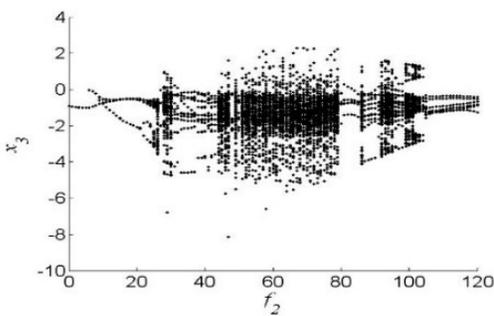


图 5 二阶模态的频响曲线随横向力的变化  
Fig.5 Frequency-response curves of the second-order mode under different cross force

二维的相图, (e) 为三维的相图. 基于图 6 系统解的分叉情况, 如图 7 为结构最初在  $f_2 = 20$  时发生周期运动的形式, 图 8 为  $f_2 = 60$  时, 结构发生的混沌运动, 图 9 为  $f_2 = 85$  时, 系统解处在周期窗口时的运动形式.



(a) 一阶分叉图  
(a) Bifurcation diagram of the first order



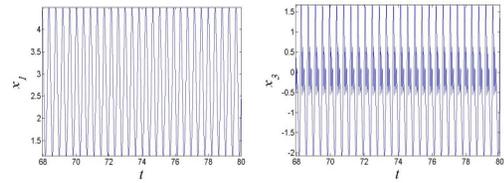
(b) 二阶分叉图  
(b) Bifurcation diagram of the second order

图 6 随横向激励变化分叉图

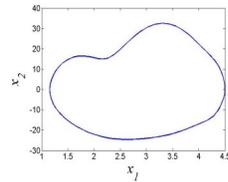
Fig.6 Bifurcation diagram with transverse excitation

### 5 结论

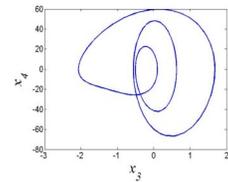
本文主要研究了石墨烯/环氧树脂复合材料板在横向激励作用下的非线性振动响应. 利用



(a) 波形图 (a) Waveform  
(b) 波形图 (b) Waveform

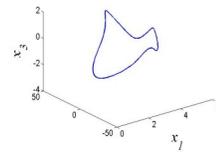


(c) 二维相图



(d) 二维相图

(c) Two-dimensional phase portrait (d) Two-dimensional phase portrait

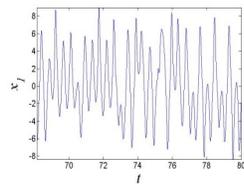


(e) 三维相图

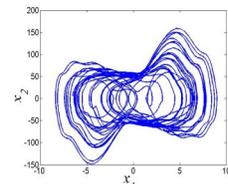
(e) Three-dimensional phase portrait

图 7  $f_2=20$  的周期运动

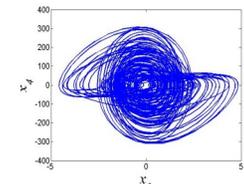
Fig.7 Periodic motion of the plate when  $f_2=20$



(a) 波形图 (a) Waveform  
(b) 波形图 (b) Waveform

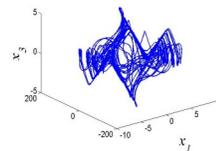


(c) 二维相图  
(c) Two-dimensional phase portrait



(d) 二维相图  
(d) Two-dimensional phase portrait

(c) Two-dimensional phase portrait (d) Two-dimensional phase portrait

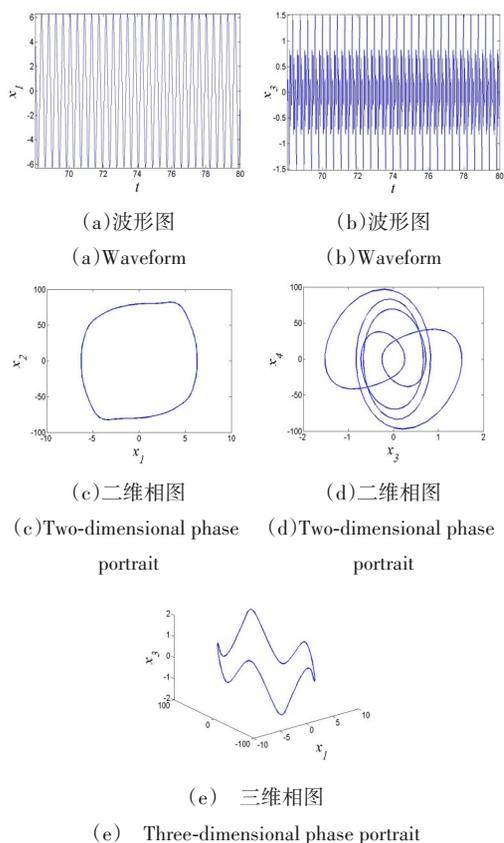


(e) 三维相图

(e) Three-dimensional phase portrait

图 8  $f_2=60$  的混沌运动

Fig.8 Chaotic motion of the plate when  $f_2=60$

图9  $f_2=85$  的周期运动Fig.9 Periodic motion of the plate when  $f_2=85$ 

Halpin-Tsai 模型及混合率计算复合材料板结构的物理参数. 通过优化的 Rayleigh-Ritz 法, 计算得到悬臂板结构的低阶频率和振型, 并与有限元的数值结果进行对比, 发现第二阶与第四阶的固有频率之间存在 1:3 的关系. 因此, 采用多尺度的方法研究板结构在 1:3 内共振下的频响方程. 考虑了阻尼和外激励对频响曲线的影响. 表明: 结构呈现硬弹簧特性. 随着阻尼的增大, 系统的振幅减小, 硬弹簧特性减弱. 与之相反的, 随着横向力的增大, 系统的振幅增加, 硬弹簧特性变强. 数值模拟研究了横向激励对板结构的非线性振动响应, 发现在共振情况下, 外激励幅值对结构振动响应的影响较大, 系统的稳定和不稳定运动状态交替出现.

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## NONLINEAR DYNAMIC ANALYSIS OF GRAPHENE RESIN COMPOSITES PLATE WITH 1:3 INTERNAL RESONANCE \*

Zhang Bo Guo Xiangying<sup>†</sup> Jiang Pan

(College of Mechanical Engineering Beijing University of Technology, Beijing, 100124, China)

**Abstract** Graphene is a new type of carbon nano-material, which can be utilized to enhance composites for excellent mechanical properties. In this paper, the nonlinear dynamic behavior of a graphene-reinforced composite plate under cantilever boundary condition was studied. The equivalent Young's modulus of the composite plate was calculated by Halpin-Tsai model. The nonlinear dynamic equations of the composite plate were obtained by using the first-order laminated plate theory and Hamilton's principle. The governing equations of transverse displacements of the composite plate were obtained by modal condensation. The Rayleigh-Ritz method was used to calculate the natural frequencies of the composite plate. It was found that there exists a relationship of 1:3 between the second and fourth order natural frequencies of the composite plate. The nonlinear vibration responses of the composite plate under 1:3 resonance was studied by using the multi-scale method. The influence of external excitation on the nonlinear vibration responses was analyzed by numerical simulation. The results showed that the transverse excitation has a great influence on the nonlinear dynamic behavior of the composite plate.

**Key words** graphene, Rayleigh-Ritz method, cantilever plates, nonlinear vibration

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<sup>†</sup> Corresponding author E-mail: eagle2008guo@yeah.net