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基于积分因子方法研究 Chaplygin 非完整系统的守恒律*

张毅†
(苏州科技大学 土木工程学院, 苏州 215011)

摘要 提出并研究了构建 Chaplygin 非完整系统守恒律的积分因子方法.基于正则形式的 Chaplygin 方程,定义了积分因子,给出了系统存在守恒量的必要条件,建立了 Chaplygin 非完整系统的守恒定理及其逆定理.研究表明:对应于必要条件的每一组非奇异函数解,系统存在一个守恒量;反之,对于一个已知守恒量,可找到相应的积分因子,且解是不唯一的.文末以匀质圆球在粗糙水平面上纯滚动为例,讨论了该方法的应用.

关键词 非完整系统, Chaplygin 方程, 积分因子, 守恒定理, 逆定理

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引言

守恒律在复杂系统动力学研究中具有基础重要性.其一,即使运动微分方程难以求解,某个守恒量的存在使我们有可能了解系统局部的物理性态或动力学行为;其二,利用守恒量可以实现运动微分方程的约化;其三,守恒量在复杂动力学系统的运动稳定性分析方面也发挥重要作用.因此,守恒律一直是分析力学研究的一个重要方面[1-4].

积分因子方法是Djukić于 1984 年提出的构造经典非保守动力学系统守恒律的一个方法[5].该方法通过类似于保守系统获得能量积分的方式,即运动微分方程乘以适当的积分因子来构建非保守动力学系统的守恒律.此后,积分因子方法被进一步推广应用于非完整约束系统[6-11], Birkhoff 系统[12-15]等.利用积分因子方法研究非完整约束系统的守恒律,一般都是从带乘子形式的运动微分方程出发来进行讨论的.实际上,非完整系统动力学方程更多是不带乘子的,如 Chaplygin 方程等.

这里将利用积分因子方法构建 Chaplygin 非完整系统的守恒律. Chaplygin 非完整系统是指用 Chaplygin 方程来描述运动的一类特殊而又重要的非完整系统[16,17].自然界与工程实际问题中遇到的很多非完整系统是 Chaplygin 系统,其运动微分方程的数目等于系统的自由度数目,并且构成独立于非完整约束方程的系统.基于正则形式的 Chaplygin 方程定义了积分因子,给出了守恒律存在的必要条件,建立了 Chaplygin 非完整系统的守恒定理及其逆定理.文章结构安排如下:第 1 节将列出非完整系统的 Chaplygin 方程并化为正则形式;第 2 节将给出 Chaplygin 正则方程的积分因子的定义;第 3 节建立 Chaplygin 系统的守恒定理;第 4 节给出计算积分因子的广义 Killing 方程;第 5 节建立守恒定理的逆定理;第 6 节以匀质圆球在粗糙水平上的纯滚动问题为例讨论积分因子方法对 Chaplygin 非完整系统的应用.

1 Chaplygin 方程及其正则形式

研究 Chaplygin 非完整系统,其位形由 n 个广义坐标 $q_s(s=1,2,\cdots,n)$ 确定,受有 g 个一阶线性、齐次、定常的非完整约束[16]

$$\dot q_{\varepsilon+\beta}=B_{\varepsilon+\beta,\sigma}\dot q_\sigma \quad (\beta=1,2,\cdots,g;\ \varepsilon=n-g;\ \sigma=1,2,\cdots,\varepsilon) \tag{1}$$

这里及以后两个相同指标表示求和.约束方程(1)中 $B_{\varepsilon+\beta,\sigma}$ 仅为 $q_1,q_2,\cdots,q_\varepsilon$ 的函数,且 $q_{\varepsilon+1},q_{\varepsilon+2},\cdots,$

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* 国家自然科学基金资助项目(11572212,11272227)
† 通讯作者 E-mail:zhy@mail.usts.edu.cn

q_n 是系统的循环坐标,并设系统是保守的,则系统的运动可由 Chaplygin 方程来描述,有[16]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}-\frac{\partial\tilde{L}}{\partial q_\sigma}+\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\dot{q}_\nu=0$$
$$(\sigma,\nu=1,2,\cdots,\varepsilon) \tag{2}$$

其中,L 为系统的 Lagrange 函数,$\tilde{L}$ 为 L 中借助约束(1)消去不独立的广义速度 $\dot{q}_{\varepsilon+\beta}$ 而得到的表达式.令

$$p_\sigma=\frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\ (\sigma=1,2,\cdots,\varepsilon) \tag{3}$$

$$\widetilde{H}=\widetilde{H}(q_\sigma,p_\sigma)=p_\sigma\dot{q}_\sigma-\tilde{L} \tag{4}$$

则 Chaplygin 方程(2)可表为正则形式:

$$\dot{q}_\sigma=\frac{\partial\widetilde{H}}{\partial p_\sigma} \tag{5}$$

$$\dot{p}_\sigma=-\frac{\partial\widetilde{H}}{\partial q_\sigma}-\left(\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\right)\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\frac{\partial\widetilde{H}}{\partial p_\nu} \tag{6}$$

其中,$\left(\dfrac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\right)$ 表示 $\dfrac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}$ 中借助约束方程(1)以及方程(3)消去全部广义速度所得到的表达式.

2 Chaplygin 正则方程的积分因子

定义 如果存在函数 $G_\sigma=G_\sigma(t,q_\nu,p_\nu)(\sigma=1,2,\cdots,\varepsilon)$,使得不变式

$$\left[\dot{p}_\sigma+\frac{\partial\widetilde{H}}{\partial q_\sigma}+\left(\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\right)\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\frac{\partial\widetilde{H}}{\partial p_\nu}\right]G_\sigma \tag{7}$$

恒等地化为

$$\left[\dot{p}_\sigma+\frac{\partial\widetilde{H}}{\partial q_\sigma}+\left(\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\right)\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\frac{\partial\widetilde{H}}{\partial p_\nu}\right]G_\sigma$$
$$\equiv\frac{\mathrm{d}}{\mathrm{d}t}(p_\sigma G_\sigma-\widetilde{H}R-\Lambda)+\mu_\sigma\left[\dot{p}_\sigma+\frac{\partial\widetilde{H}}{\partial q_\sigma}+\right.$$
$$\left.\left(\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\right)\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\frac{\partial\widetilde{H}}{\partial p_\nu}\right] \tag{8}$$

其中,R,Λ 和 μ_σ 是 q_σ,p_σ 和 t 的任意函数.则称函数 G_σ 为 Chaplygin 正则方程(6)的积分因子.

3 Chaplygin 系统的守恒定理

下面研究由 Chaplygin 正则方程的积分因子来构建系统的守恒律.

设函数 G_σ 为方程(6)的积分因子,将方程(6)代入恒等式(8),有

$$\frac{\mathrm{d}}{\mathrm{d}t}(p_\sigma G_\sigma-\widetilde{H}R-\Lambda)=0 \tag{9}$$

于是有:

定理1 对于所论 Chaplygin 非完整系统,如果函数 G_σ 是方程(6)的积分因子,那么沿着由正则方程(5)(6)确定的 Chaplygin 系统的运动轨道,如下表达式

$$I=p_\sigma G_\sigma-\widetilde{H}R-\Lambda=\mathrm{const} \tag{10}$$

是系统的一个守恒量.

对于一个已知的 Chaplygin 非完整系统,如果函数 G_σ 是方程(6)的积分因子,则每一个函数组 G_σ,R,Λ 和 μ_σ 必须满足必要条件(8).展开式(8),有

$$-\left[\dot{p}_\sigma+\frac{\partial\widetilde{H}}{\partial q_\sigma}+\left(\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\right)\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\frac{\partial\widetilde{H}}{\partial p_\nu}\right]\cdot$$
$$G_\sigma+\dot{p}_\sigma G_\sigma+p_\sigma\dot{G}_\sigma-\dot{\widetilde{H}}R-\widetilde{H}\dot{R}-\dot{\Lambda}+\Phi=0 \tag{11}$$

其中,

$$\Phi=\mu_\sigma\left[\dot{p}_\sigma+\frac{\partial\widetilde{H}}{\partial q_\sigma}+\left(\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\right)\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\frac{\partial\widetilde{H}}{\partial p_\nu}\right] \tag{12}$$

利用正则方程(5)和(6),得

$$\dot{\widetilde{H}}=\frac{\partial\widetilde{H}}{\partial q_\sigma}\dot{q}_\sigma+\frac{\partial\widetilde{H}}{\partial p_\sigma}\dot{p}_\sigma$$
$$=-\frac{\partial\widetilde{H}}{\partial p_\sigma}\left(\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\right)\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\frac{\partial\widetilde{H}}{\partial p_\nu} \tag{13}$$

将式(13)代入式(11),整理得

$$\left(\frac{\partial L}{\partial\dot{q}_{\varepsilon+\beta}}\right)\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\frac{\partial\widetilde{H}}{\partial p_\nu}\left(R\frac{\partial\widetilde{H}}{\partial p_\sigma}-G_\sigma\right)+$$
$$p_\sigma\dot{G}_\sigma-\frac{\partial\widetilde{H}}{\partial q_\sigma}G_\sigma-\widetilde{H}\dot{R}-\dot{\Lambda}+\Phi=0 \tag{14}$$

如果必要条件(14)的解 G_σ,R,Λ 和 μ_σ 使式(10)等号右边成为一个常数,称此函数组 G_σ,R,Λ 和 μ_σ 为奇异函数组.于是有:

定理2 对于所论 Chaplygin 非完整系统,相应于满足必要条件(14)的每一个非奇异函数组 G_σ,R,Λ 和 μ_σ,该系统存在一个形如式(10)的守恒量.

定理1和定理2是 Chaplygin 非完整系统基于

积分因子的守恒定理.

4　计算积分因子的广义 Killing 方程

利用定理 1 和定理 2 来寻找 Chaplygin 非完整系统的守恒律,关键在于找到积分因子 G_σ 以及函数 R 和 Λ.

将方程(14)展开,令 $\dot{p}_\sigma$ 项的系数和不含 $\dot{p}_\sigma$ 的项分别等于零,有

$$p_\nu \frac{\partial G_\nu}{\partial p_\sigma}-\widetilde{H}\frac{\partial R}{\partial p_\sigma}-\frac{\partial \Lambda}{\partial p_\sigma}+\mu_\sigma=0\ (\sigma=1,2,\cdots,\varepsilon) \tag{15}$$

$$\left(\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\right)\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\frac{\partial \widetilde{H}}{\partial p_\nu}\left(R\frac{\partial \widetilde{H}}{\partial p_\sigma}-G_\sigma\right)+ p_\sigma\left(\frac{\partial G_\sigma}{\partial t}+\frac{\partial G_\sigma}{\partial q_\nu}\frac{\partial \widetilde{H}}{\partial p_\nu}\right)-\frac{\partial \widetilde{H}}{\partial q_\sigma}G_\sigma- \widetilde{H}\left(\frac{\partial R}{\partial t}+\frac{\partial R}{\partial q_\nu}\frac{\partial \widetilde{H}}{\partial p_\nu}\right)-\frac{\partial \Lambda}{\partial t}-\frac{\partial \Lambda}{\partial q_\nu}\frac{\partial \widetilde{H}}{\partial p_\nu}+ \mu_\sigma\left[\frac{\partial \widetilde{H}}{\partial q_\sigma}+\left(\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\right)\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\frac{\partial \widetilde{H}}{\partial p_\nu}\right]=0 \tag{16}$$

一般情况下,式(15)和(16)是关于 $2(\varepsilon+1)$ 个未知函数 G_σ,R,Λ 和 μ_σ 的 $(\varepsilon+1)$ 个偏微分方程,可称为广义 Killing 方程.如果能够找到广义 Killing 方程(15)(16)的一个解,且它们构成一个非奇异函数组,则由上述守恒定理可得到系统的一个守恒量.由于方程数目小于未知量数目,故方程的解是不唯一的,因此,可以通过适当选择函数 G_σ,R 和 Λ 来得到不同的守恒量.

值得指出,也可以直接由方程(14)来寻找非奇异函数组 G_σ,R 和 Λ.将方程(14)展开,并利用方程(5),(6)以及(12),得到

$$\left(\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\right)\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\frac{\partial \widetilde{H}}{\partial p_\nu}\left(R\frac{\partial \widetilde{H}}{\partial p_\sigma}-G_\sigma\right)+ p_\sigma\left(\frac{\partial G_\sigma}{\partial t}+\frac{\partial G_\sigma}{\partial q_\nu}\frac{\partial \widetilde{H}}{\partial p_\nu}\right)-\frac{\partial \widetilde{H}}{\partial q_\sigma}G_\sigma- \widetilde{H}\left(\frac{\partial R}{\partial t}+\frac{\partial R}{\partial q_\nu}\frac{\partial \widetilde{H}}{\partial p_\nu}\right)-\frac{\partial \Lambda}{\partial t}-\frac{\partial \Lambda}{\partial q_\nu}\frac{\partial \widetilde{H}}{\partial p_\nu}- \left[\frac{\partial \widetilde{H}}{\partial q_\sigma}+\left(\frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}\right)\left(\frac{\partial B_{\varepsilon+\beta,\nu}}{\partial q_\sigma}-\frac{\partial B_{\varepsilon+\beta,\sigma}}{\partial q_\nu}\right)\frac{\partial \widetilde{H}}{\partial p_\nu}\right]\cdot$$

$$\left(p_\nu\frac{\partial G_\nu}{\partial p_\sigma}-\widetilde{H}\frac{\partial R}{\partial p_\sigma}-\frac{\partial \Lambda}{\partial p_\sigma}\right)=0 \tag{17}$$

方程(17)是一个线性偏微分方程,只要找到方程的一个非奇异函数解,由定理 2 即可得到系统的一个守恒量.

5　守恒定理的逆定理

设 Chaplygin 非完整系统有一个已知守恒量

$$I=I(q_\sigma,p_\sigma,t)=\text{const} \tag{18}$$

将式(10)对 p_σ 求偏导数,有

$$-\frac{\partial I}{\partial p_\sigma}+G_\sigma+p_\nu\frac{\partial G_\nu}{\partial p_\sigma}-\frac{\partial \widetilde{H}}{\partial p_\sigma}R-\widetilde{H}\frac{\partial R}{\partial p_\sigma}-\frac{\partial \Lambda}{\partial p_\sigma}=0 \tag{19}$$

由式(19)和(15),得

$$G_\sigma=\frac{\partial I}{\partial p_\sigma}+\frac{\partial \widetilde{H}}{\partial p_\sigma}R+\mu_\sigma(\sigma=1,2,\cdots,\varepsilon) \tag{20}$$

由式(20)和(10),得

$$\Lambda=p_\sigma\frac{\partial I}{\partial p_\sigma}-I+R\left(p_\sigma\frac{\partial \widetilde{H}}{\partial p_\sigma}-\widetilde{H}\right)+p_\sigma\mu_\sigma \tag{21}$$

由此有:

定理 3　如果所论 Chaplygin 非完整系统有一个守恒量(18),则与此守恒量相应的积分因子 G_σ 以及函数 R 和 Λ 由式(20)和(21)确定.

定理 3 是 Chaplygin 非完整系统基于积分因子的守恒定理的逆定理.

6　方法的应用:以匀质圆球在粗糙水平面上的纯滚动问题为例

研究质量为 m,半径为 a 的匀质圆球在完全粗糙水平面上的纯滚动[16].取球心坐标 x,y 及三个 Euler 角 ψ,θ 和 φ 为广义坐标,圆球的 Lagrange 函数为

$$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{5}ma^2(\dot{\psi}^2+\dot{\theta}^2+\dot{\varphi}^2+2\dot{\psi}\dot{\varphi}\cos\theta) \tag{22}$$

圆球在粗糙水平面上作纯滚动,因此非完整约束为

$$\dot{x}+a(\dot{\varphi}\cos\psi\sin\theta-\dot{\theta}\sin\psi)=0$$
$$\dot{y}+a(\dot{\varphi}\sin\psi\sin\theta+\dot{\theta}\cos\psi)=0 \tag{23}$$

令 $q_1=\psi,q_2=\theta,q_3=\varphi,q_4=x,q_5=y$,则式(22)和(23)可表为:

$$L=\frac{1}{2}m(\dot{q}_4^2+\dot{q}_5^2)+$$

$$\frac{1}{5}ma^2(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2+2\dot{q}_1\dot{q}_3\cos q_2) \quad (24)$$

$$\dot{q}_4=-a(\dot{q}_3\cos q_1\sin q_2-\dot{q}_2\sin q_1)$$
$$\dot{q}_5=-a(\dot{q}_3\sin q_1\sin q_2+\dot{q}_2\cos q_1) \quad (25)$$

将约束(25)代入式(24),消去其中的 $\dot{q}_4$ 和 $\dot{q}_5$,得

$$\tilde{L}=\frac{1}{2}ma^2\left[\frac{7}{5}\dot{q}_2^2+\dot{q}_3^2\sin^2 q_2+\frac{2}{5}(\dot{q}_1^2+\dot{q}_3^2+2\dot{q}_1\dot{q}_3\cos q_2)\right] \quad (26)$$

式(3)给出

$$p_1=\frac{2}{5}ma^2(\dot{q}_1+\dot{q}_3\cos q_2)$$
$$p_2=\frac{7}{5}ma^2\dot{q}_2$$
$$p_3=\frac{2}{5}ma^2(\dot{q}_1\cos q_2+\dot{q}_3)+ma^2\dot{q}_3\sin^2 q_2 \quad (27)$$

由式(27)得

$$\dot{q}_1=\frac{5}{7ma^2\sin^2 q_2}\left(\frac{5}{2}p_1\sin^2 q_2+p_1-p_3\cos q_2\right)$$
$$\dot{q}_2=\frac{5}{7ma^2}p_2$$
$$\dot{q}_3=\frac{5}{7ma^2\sin^2 q_2}(p_3-p_1\cos q_2) \quad (28)$$

由式(4),并利用式(28),得

$$\tilde{H}=\frac{5}{14ma^2\sin^2 q_2}\left(\frac{5}{2}p_1^2\sin^2 q_2+p_2^2\sin^2 q_2+p_1^2+p_3^2-2p_1p_3\cos q_2\right) \quad (29)$$

方程(6)给出

$$\dot{p}_1=0$$
$$\dot{p}_2=\frac{5}{14ma^2\sin^2 q_2}\left[2\cot q_2(p_1^2+p_3^2-2p_1p_3\cos q_2)+5p_1^2\sin q_2\cos q_2-7p_1p_3\sin q_2\right]$$
$$\dot{p}_3=\frac{25p_1p_2\sin q_2}{14ma^2} \quad (30)$$

广义 Killing 方程(15),(16)分别给出:

$$p_1\frac{\partial G_1}{\partial p_1}+p_2\frac{\partial G_2}{\partial p_1}+p_3\frac{\partial G_3}{\partial p_1}-\tilde{H}\frac{\partial R}{\partial p_1}-\frac{\partial \Lambda}{\partial p_1}+\mu_1=0$$
$$p_1\frac{\partial G_1}{\partial p_2}+p_2\frac{\partial G_2}{\partial p_2}+p_3\frac{\partial G_3}{\partial p_2}-\tilde{H}\frac{\partial R}{\partial p_2}-\frac{\partial \Lambda}{\partial p_2}+\mu_2=0$$
$$p_1\frac{\partial G_1}{\partial p_3}+p_2\frac{\partial G_2}{\partial p_3}+p_3\frac{\partial G_3}{\partial p_3}-\tilde{H}\frac{\partial R}{\partial p_3}-\frac{\partial \Lambda}{\partial p_3}+\mu_3=0 \quad (31)$$

以及

$$\frac{25}{14ma^2\sin^2 q_2}(p_1p_3-p_1^2\cos q_2)\left(\frac{5Rp_2}{7ma^2}-G_2\right)-\frac{25p_1p_2\sin q_2}{14ma^2}\left(\frac{5R(p_3-p_1\cos q_2)}{7ma^2\sin^2 q_2}-G_3\right)+\frac{5G_2}{14ma^2\sin^2 q_2}\cdot\left[2\cot q_2(p_1^2+p_3^2-2p_1p_3\cos q_2)-2p_1p_3\sin q_2\right]+$$
$$p_1\frac{\partial G_1}{\partial t}+p_2\frac{\partial G_2}{\partial t}+p_3\frac{\partial G_3}{\partial t}-\tilde{H}\frac{\partial R}{\partial t}-\frac{\partial \Lambda}{\partial t}+$$
$$\frac{5}{14ma^2\sin^2 q_2}\left(p_1\frac{\partial G_1}{\partial q_1}+p_2\frac{\partial G_2}{\partial q_1}+p_3\frac{\partial G_3}{\partial q_1}-\tilde{H}\frac{\partial R}{\partial q_1}-\frac{\partial \Lambda}{\partial q_1}\right)\cdot(5p_1\sin^2 q_2+2p_1-2p_3\cos q_2)+$$
$$\frac{5p_2}{7ma^2}\left(p_1\frac{\partial G_1}{\partial q_2}+p_2\frac{\partial G_2}{\partial q_2}+p_3\frac{\partial G_3}{\partial q_2}-\tilde{H}\frac{\partial R}{\partial q_2}-\frac{\partial \Lambda}{\partial q_2}\right)+$$
$$\frac{5}{7ma^2\sin^2 q_2}\left(p_1\frac{\partial G_1}{\partial q_3}+p_2\frac{\partial G_2}{\partial q_3}+p_3\frac{\partial G_3}{\partial q_3}-\tilde{H}\frac{\partial R}{\partial q_3}-\frac{\partial \Lambda}{\partial q_3}\right)(p_3-p_1\cos q_2)-\frac{5\mu_2}{14ma^2\sin^2 q_2}\cdot$$
$$\left[2\cot q_2(p_1^2+p_3^2-2p_1p_3\cos q_2)+5p_1^2\sin q_2\cos q_2-7p_1p_3\sin q_2\right]-\frac{25\mu_3}{14ma^2}p_1p_3\sin q_2=0 \quad (32)$$

联解方程(31)和(32),有

$$G_1=G_2=G_3=0,R=1,\Lambda=0,\mu_1=\mu_2=\mu_3=0 \quad (33)$$

$$G_1=1,G_2=G_3=0,R=0,\Lambda=0,\mu_1=\mu_2=\mu_3=0 \quad (34)$$

$$G_1=-\sin q_1\cot q_2,G_2=\cos q_1,G_3=\frac{\sin q_1}{\sin q_2},R=0,\Lambda=0,\mu_1=\mu_2=\mu_3=0 \quad (35)$$

$$G_1=5\cos q_2,G_2=G_3=0,R=0,\Lambda=0,\mu_1=\mu_2=\mu_3=0 \quad (36)$$

$$G_1=\cos q_1\cot q_2,G_2=\sin q_1,G_3=-\frac{\cos q_1}{\sin q_2},R=0,\Lambda=0,\mu_1=\mu_2=\mu_3=0 \quad (37)$$

根据守恒定理,与非奇异函数组(33)~(37)相应的守恒量分别为

$$I_1=-\frac{5}{14ma^2\sin^2 q_2}\left(\frac{5}{2}p_1^2\sin^2 q_2+p_2^2\sin^2 q_2+p_1^2+p_3^2-2p_1p_3\cos q_2\right)=\text{const} \quad (38)$$

$$I_2=p_1=\text{const} \quad (39)$$

$$I_3=-p_1\sin q_1\cot q_2+p_2\cos q_1+p_3\frac{\sin q_1}{\sin q_2}=\text{const} \quad (40)$$

$$I_4 = 5p_1\cos q_2 + 2p_3 = \text{const} \tag{41}$$

$$I_5 = p_1\cos q_1 \cot q_2 + p_2 \sin q_1 - p_3\frac{\cos q_1}{\sin q_2} = \text{const} \tag{42}$$

下面研究逆问题.设系统有如下守恒量

$$I = p_1\cos q_1 \cot q_2 + p_2 \sin q_1 - p_3\frac{\cos q_1}{\sin q_2} = \text{const} \tag{43}$$

方程(20)和(21)分别给出:

$$G_1 = \cos q_1 \cot q_2 + \frac{5R}{14ma^2\sin^2 q_2}(5p_1\sin^2 q_2 + 2p_1 - 2p_3\cos q_2) + \mu_1$$

$$G_2 = \sin q_1 + \frac{5Rp_2}{7ma^2} + \mu_2$$

$$G_3 = -\frac{\cos q_1}{\sin q_2} + \frac{5R}{7ma^2\sin^2 q_2}(p_3 - p_1\cos q_2) + \mu_3 \tag{44}$$

以及

$$\Lambda = \frac{5R}{14ma^2\sin^2 q_2}\left(\frac{5}{2}p_1^2\sin^2 q_2 + p_2^2\sin^2 q_2 + p_1^2 + p_3^2 - 2p_1p_3\cos q_2\right) + p_1\mu_1 + p_2\mu_2 + p_3\mu_3 \tag{45}$$

方程(44)和(45)共有 4 个方程,但是含有 8 个未知量 $G_1, G_2, G_3, R, \Lambda, \mu_1, \mu_2, \mu_3$,因此方程的解不是唯一的,可以适当选择其中 4 个量,进而解出其余 4 个未知量.例如,取

$$R = 0,\ \mu_1 = \mu_2 = \mu_3 = 0 \tag{46}$$

则有

$$G_1 = \cos q_1 \cot q_2,\ G_2 = \sin q_1, G_3 = -\frac{\cos q_1}{\sin q_2}$$

$$\Lambda = 0 \tag{47}$$

如取

$$R = 0, \mu_1 = -\cos q_1 \cot q_2, \mu_2 = 0, \mu_3 = \frac{\cos q_1}{\sin q_2} \tag{48}$$

则有

$$G_1 = 0, G_2 = \sin q_1, G_3 = 0$$

$$\Lambda = -p_1\cos q_1\cot q_2 + p_3\frac{\cos q_1}{\sin q_2} \tag{49}$$

需要指出的是:与基于带乘子的运动微分方程不同,本问题中得到的守恒量以及相应的非奇异函数组都仅与 q_σ 和 p_σ 相关,而不依赖于 $q_{\varepsilon+\beta}$.

7　结论

寻找约束力学系统的守恒律是分析力学研究的一个重要方面.文章基于积分因子方法研究 Chaplygin 非完整系统的守恒律.主要贡献在于:一是定义了正则形式的 Chaplygin 方程的积分因子,建立了系统存在守恒量的必要条件;二是建立了 Chaplygin 非完整系统的守恒定理及其逆定理;三是以匀质圆球在完全粗糙水平面上的纯滚动为例,讨论了方法的应用.不同于以往的工作,文章基于经典的不带乘子的 Chaplygin 方程给出系统的守恒定理和逆定理,更能体现 Chaplygin 非完整系统的特点.由于非完整系统动力学方程更多是不带乘子的,因此文章的方法和结果可望进一步推广和应用.

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A STUDY OF CONSERVATION LAWS FOR CHAPLYGIN NONHOLONOMIC SYSTEMS BY MEANS OF INTEGRATING FACTORS METHOD*

Zhang Yi†

(*College of Civil Engineering*, *Suzhou University of Science and Technology*, *Suzhou* 215011, *China*)

Abstract The method of integrating factors for the construction of conservation laws of a Chaplygin nonholonomic system is presented and studied. Based on the Chaplygin equations with canonical form, the integrating factors of the equations is defined, the necessary condition for the existence of conserved quantities of the system is given, and the conservation theorem and its inverse for the Chaplygin nonholonomic system are established. The studies show that for each group of nonsingular functions that correspond to the necessary condition, the system has a conserved quantity, while for a known conserved quantity, the corresponding integrating factor can be found, and the solution is not unique. Finally, we take an uniform sphere rolling on a perfectly rough horizontal plane as an example to discuss the application of the method.

Key words nonholonomic system, Chaplygin equation, integrating factor, conservation theorem, inverse theorem

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† Corresponding author E-mail: zhy@mail.usts.edu.cn