第 16 卷第 5 期 2018 年 10 月
1672-6553/2018/16(5)/440-8
动力学与控制学报
JOURNAL OF DYNAMICS AND CONTROL
Vol.16 No.5
Oct. 2018

压电复合材料悬臂板 1:3内共振的非线性动力学分析*

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摘要　本文以飞行器机翼为工程背景,将机翼简化为悬臂板模型,在应用经典板理论和 Hamilton 原理建立横向和面内激励共同作用下压电复合材料悬臂板的无量纲非线性偏微分方程的基础上,利用 Galerkin 方法将系统离散为两自由度的非线性常微分方程.然后考虑主参数共振-1:3内共振,运用多尺度法将两自由度的系统控制方程进行摄动分析,推导出四维平均方程.基于四阶 Runge-Kutta 法,使用 MATLAB 软件研究了横向外激励幅值和压电参数项对压电复合材料悬臂板非线性动力学行为的影响.结果表明,系统存在周期和混沌运动,所得结论对实际工程具有指导意义.

关键词　悬臂板，　内共振，　摄动分析，　多尺度法，　非线性动力学

DOI:　10.6052/1672-6553-2018-028

引言

实际工作环境中,飞行器机翼是复杂的非线性系统,传统分析方法是忽略系统控制方程中的非线性项,利用线性系统近似描述工程问题,但在工程应用中发现,这样的研究结果与实际相差甚远,因此选择合适的方法对其进行研究是很重要的.对于大多数非线性系统的运动控制方程而言,由于非线性项的存在和复杂的边界条件,使得求其精确解几乎是不可能的,因此只能用数值方法或者是近似解析方法来研究.

摄动方法就是近似解析方法的一种,它是重要的求解非线性方程的数学工具.多尺度法是摄动法的一种,自 20 世纪 60 年代以来多尺度法经过 Nayfeh[1,2]等人的发展与完善,成为很有效的近似分析方法,已被广泛的应用于各种非线性问题的分析中.多尺度法的基本思想是根据变量的变化不同,来区分快慢时间尺度,将响应的展开式考虑成为多个时间变量,即多个时间尺度的函数,而不是单个时间变量的函数.多尺度法具有很多优点,可以方便地处理多种类型的非线性系统,它既能计算周期运动也能分析衰减振动,既能分析稳态响应,也能用于非稳态过程的研究中.

1998 年 Abe[3,4]分析了在四边简支矩形层合板受到谐波激励时的非线性动力学,用多尺度法分析了系统的稳态响应,得到了激励的振幅和线性固有频率在三阶 Galerkin 离散下的关系.2008 年 Hao[5]等人利用高阶板壳理论和多尺度法研究了面内激励和横向外激励联合作用下四边简支矩形功能梯度板的非线性动力学行为.2009 年 Nejad[6]等人研究了粘弹性悬臂板的非线性振动行为,利用多尺度和摄动方法对系统进行了分析,研究了系统的混沌运动,分叉等现象.2013 年陈建恩[7]等人应用多尺度法得到了横向载荷和面内载荷联合作用下四边简支点阵夹芯板的平均方程.通过数值方法研究了不同共振情况下点阵夹芯板的非线性振动响应.2017 年郭翔鹰[8]等人将机翼简化为压电纤维复合材料悬臂中厚壳模型,采用多尺度法得到极坐标形式的平均方程.通过数值方法得到了中厚壳结构的幅频响应曲线、分叉图等.

以上文献建立的动力学模型多为简支板模型,悬臂板较少,而压电复合材料层合悬臂板模型未

2017-09-29 收到第 1 稿,2017-11-13 收到修改稿.
*国家自然科学基金(11290152,11322214)资助项目
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见.本文以飞行器机翼为工程背景,在推导出压电复合材料悬臂板的无量纲非线性偏微分方程的基础上,将系统离散为两自由度的非线性常微分方程;运用多尺度法进行摄动分析,推导出系统四维直角坐标形式的平均方程;使用 MATLAB 软件分析了横向外激励幅值和压电参数项对复合材料悬臂板在主参数共振-1:3内共振情况下的非线性动力学响应.

1　基本方程

将飞行器机翼简化为复合材料层合悬臂板,建立力学模型如图 1 所示.悬臂板由三层材料组成,中间为基底层,材料选用石墨/环氧(HT3/QY8911)树脂,上下两层为压电薄膜(PVDF)层.悬臂板总的长、宽、厚分别为 a, b, h,建立直角坐标系在板的中面,OB 方向为 x 方向,OA 方向为 y 方向,板受横向激励 $F_0+F_1\cos(\Omega_1 t)$ 和面内激励 $p_0+p_1\cos(\Omega_2 t)$ 的作用.

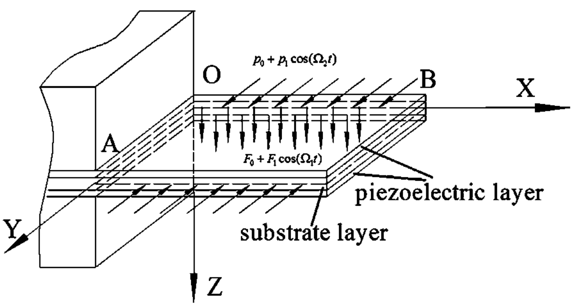


图 1　复合材料层合压电悬臂板的力学模型

Fig.1　Mathematical model of a cantilevered piezoelectric plate

根据经典板理论[9]及 Hamilton 原理,得内力表示的压电复合材料层合悬臂板的控制方程如下:

$$\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}=I_0\ddot{u}_0-I_1\frac{\partial \ddot{w}_0}{\partial x} \tag{1a}$$

$$\frac{\partial N_{yy}}{\partial y}+\frac{\partial N_{xy}}{\partial x}=I_0\ddot{v}_0-I_1\frac{\partial \ddot{w}_0}{\partial y} \tag{1b}$$

$$\frac{\partial N_{xx}}{\partial x}\frac{\partial w_0}{\partial x}+\frac{\partial N_{xy}}{\partial x}\frac{\partial w_0}{\partial y}+\frac{\partial N_{xy}}{\partial y}\frac{\partial w_0}{\partial x}+\frac{\partial N_{yy}}{\partial y}\frac{\partial w_0}{\partial y}+\frac{\partial^2 M_{xx}}{\partial x^2}+\frac{\partial^2 M_{yy}}{\partial y^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+F_1\cos(\Omega_1 t)-c_3\dot{w}_0$$
$$=I_0\ddot{w}_0+I_1\left(\frac{\partial \ddot{u}_0}{\partial x}+\frac{\partial \ddot{v}_0}{\partial y}\right)-I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial x^2}+\frac{\partial^2 \ddot{w}_0}{\partial y^2}\right) \tag{1c}$$

式中:

$$\begin{Bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \end{Bmatrix}=\int_{-h/2}^{h/2}\sigma_{\alpha\beta}\begin{Bmatrix} 1 \\ z \end{Bmatrix}dz,(\alpha,\beta \text{ 分别代表 } x,y) \tag{2}$$

$$I_i=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}\rho_k(z)^i dz,(i=0,1,2) \tag{3}$$

压电复合材料层合悬臂板内力与应变关系为:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix}=\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}-\begin{Bmatrix} N_{xx}^{\ p} \\ N_{yy}^{\ p} \\ N_{xy}^{\ p} \end{Bmatrix},$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix}=\begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}-\begin{Bmatrix} M_{xx}^{\ p} \\ M_{yy}^{\ p} \\ M_{xy}^{\ p} \end{Bmatrix} \tag{4}$$

其中,

$$(A_{ij},B_{ij})=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}(Q_{ij})_k(1,z)dz,(i,j=1,2,6) \tag{5}$$

式(5)中$(Q_{ij})_k$为第 k 层材料的刚度系数.

将式(4)带入到式(1)中,得到广义位移表示的压电复合材料层合悬臂板的控制方程如下:

$$A_{11}\frac{\partial^2 u_0}{\partial x^2}+A_{66}\frac{\partial^2 u_0}{\partial y^2}+(A_{12}+A_{66})\frac{\partial^2 v_0}{\partial x\partial y}+A_{11}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}+$$
$$A_{66}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}+(A_{12}+A_{66})\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x\partial y}-B_{11}\frac{\partial^3 w_0}{\partial x^3}-$$
$$(B_{12}+2B_{66})\frac{\partial^3 w_0}{\partial x\partial y^2}=I_0\frac{\partial^2 u_0}{\partial t^2}-I_1\frac{\partial^3 w_0}{\partial x\partial t^2} \tag{6a}$$

$$A_{22}\frac{\partial^2 v_0}{\partial y^2}+A_{66}\frac{\partial^2 v_0}{\partial x^2}+(A_{21}+A_{66})\frac{\partial^2 u_0}{\partial x\partial y}+A_{22}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}+$$
$$A_{66}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x^2}+(A_{21}+A_{66})\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}-B_{22}\frac{\partial^3 w_0}{\partial y^3}-$$
$$(B_{21}+2B_{66})\frac{\partial^3 w_0}{\partial x^2\partial y}=I_0\frac{\partial^2 v_0}{\partial t^2}-I_1\frac{\partial^3 w_0}{\partial y\partial t^2} \tag{6b}$$

$$-D_{11}\frac{\partial^4 w_0}{\partial x^4}+(-D_{21}-4D_{66}-D_{12})\frac{\partial^4 w_0}{\partial x^2\partial y^2}-D_{22}\frac{\partial^4 w_0}{\partial y^4}-$$
$$(B_{12}-B_{21})\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x\partial y^2}+(B_{12}-B_{21})\frac{\partial w_0}{\partial y}\frac{\partial^3 w_0}{\partial y\partial x^2}+$$
$$(2B_{66}-B_{12}-B_{21})\frac{\partial^2 w_0}{\partial y^2}\frac{\partial^2 w_0}{\partial x^2}+\frac{3A_{11}}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\frac{\partial^2 w_0}{\partial x^2}+$$
$$\frac{3}{2}A_{22}\left(\frac{\partial w_0}{\partial y}\right)^2\frac{\partial^2 w_0}{\partial y^2}+\left(\frac{1}{2}A_{21}+A_{66}\right)\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w_0}{\partial y^2}+$$

$$(\frac{1}{2}A_{12}+A_{66})(\frac{\partial w_0}{\partial y})^2\frac{\partial^2 w_0}{\partial x^2}+(-2B_{66}+B_{12}+B_{21})(\frac{\partial^2 w_0}{\partial x\partial y})^2+$$
$$2(\frac{1}{2}A_{21}+2A_{66}+\frac{1}{2}A_{12})\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x\partial y}+A_{11}\frac{\partial u_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}+$$
$$A_{22}\frac{\partial v_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}+A_{21}\frac{\partial u_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}+A_{12}\frac{\partial v_0}{\partial y}\frac{\partial^2 w_0}{\partial x^2}+$$
$$2A_{66}\frac{\partial u_0}{\partial y}\frac{\partial^2 w_0}{\partial x\partial y}+2A_{66}\frac{\partial v_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}+(A_{21}+A_{66})\frac{\partial w_0}{\partial y}\frac{\partial^2 u_0}{\partial y\partial x}+$$
$$(A_{12}+A_{66})\frac{\partial w_0}{\partial x}\frac{\partial^2 v_0}{\partial y\partial x}+A_{11}\frac{\partial w_0}{\partial x}\frac{\partial^2 u_0}{\partial x^2}+A_{66}\frac{\partial w_0}{\partial x}\frac{\partial^2 u_0}{\partial y^2}+$$
$$A_{66}\frac{\partial w_0}{\partial y}\frac{\partial^2 v_0}{\partial x^2}+A_{22}\frac{\partial w_0}{\partial y}\frac{\partial^2 v_0}{\partial y^2}+(2B_{66}+B_{21})\frac{\partial^3 u_0}{\partial x\partial y^2}+$$
$$(2B_{66}+B_{12})\frac{\partial^3 v_0}{\partial y\partial x^2}+B_{11}\frac{\partial^3 u_0}{\partial x^3}+B_{22}\frac{\partial^3 v_0}{\partial y^3}-$$
$$N_{xx}{}^{p}\cos(\Omega_p t)\frac{\partial^2 w_0}{\partial x^2}-N_{yy}{}^{p}\cos(\Omega_p t)\frac{\partial^2 w_0}{\partial y^2}-$$
$$(p_0+p_1\cos(\Omega_2 t))\frac{\partial^2 w_0}{\partial y^2}-\frac{\partial N_{xx}{}^{p}}{\partial x}\frac{\partial w_0}{\partial x}-\frac{\partial N_{yy}{}^{p}}{\partial y}\frac{\partial w_0}{\partial y}-$$
$$\frac{\partial^2 M_{xx}{}^{p}}{\partial x^2}-\frac{\partial^2 M_{yy}{}^{p}}{\partial y^2}+F_0+F_1\cos(\Omega_1 t)=c_3\dot{w}_0+I_0\ddot{w}_0-$$
$$I_2\frac{\partial^2\ddot{w}}{\partial x^2}-I_2\frac{\partial^2\ddot{w}}{\partial y^2}+(I_1\frac{\partial\ddot{u}}{\partial x}+I_1\frac{\partial\ddot{v}}{\partial y}) \tag{6c}$$

引入无量纲变换如下式：

$$u_0=\bar{u}_0 a,\ v_0=\bar{v}_0 b,\ w_0=\bar{w}_0 h,\ x=\bar{x}a,\ y=\bar{y}b,$$
$$t=\bar{t}\sqrt{\frac{ab\rho}{E}}\frac{1}{\pi^2},\ A_{ij}=\bar{A}_{ij}\frac{Eh^2}{\sqrt{ab}},\ B_{ij}=\bar{B}_{ij}\frac{Eh^3}{\sqrt{ab}},$$
$$D_{ij}=\bar{D}_{ij}\frac{Eh^4}{\sqrt{ab}},\ I_i=\bar{I}_i(ab)^{\frac{i+1}{2}}\rho,c_3=\bar{c}_3\frac{\sqrt{\rho E}\pi^2 h^4}{(ab)^2},$$
$$\Omega_i=\bar{\Omega}_i\pi^2\sqrt{\frac{E}{ab\rho}},\ F=\bar{F}\frac{Eh^7}{(ab)^{\frac{7}{2}}},\ p=\bar{p}\frac{El^3}{b^2} \tag{7}$$

将式(7)代入到式(6)中，得到系统的无量纲非线性偏微分运动控制方程(8)．为了表达简洁，将方程(8)及本文之后方程中的的 $\bar{u}_0,\bar{v}_0,\bar{w}_0,\bar{x},\bar{y},\bar{t}$ 等无量纲物理量上面的横线去掉.

$$a_{10}\frac{\partial^2 u_0}{\partial x^2}+a_{11}\frac{\partial^2 u_0}{\partial y^2}+a_{12}\frac{\partial^2 v_0}{\partial x\partial y}+a_{13}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}+$$
$$a_{14}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}+a_{15}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x\partial y}+a_{16}\frac{\partial^3 w_0}{\partial x\partial y^2}+a_{17}\frac{\partial^3 w_0}{\partial x^3}$$
$$=a_{18}\frac{\partial^2 u_0}{\partial t^2}+a_{19}\frac{\partial^3 w_0}{\partial x\partial t^2} \tag{8a}$$

$$b_{10}\frac{\partial^2 v_0}{\partial y^2}+b_{11}\frac{\partial^2 v_0}{\partial x^2}+b_{12}\frac{\partial^2 u_0}{\partial x\partial y}+b_{13}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x^2}+$$
$$b_{14}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}+b_{15}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}+b_{16}\frac{\partial^3 w_0}{\partial x^2\partial y}+b_{17}\frac{\partial^3 w_0}{\partial y^3}$$
$$=b_{18}\frac{\partial^2 v_0}{\partial t^2}+b_{19}\frac{\partial^3 w_0}{\partial y\partial t^2} \tag{8b}$$

$$c_{10}\frac{\partial^4 w_0}{\partial x^4}+c_{11}\frac{\partial^4 w_0}{\partial x^2\partial y^2}+c_{12}\frac{\partial^4 w_0}{\partial y^4}+c_{13}\frac{\partial w_0}{\partial x}\frac{\partial^3 w_0}{\partial x\partial y^2}+$$
$$c_{14}\frac{\partial w_0}{\partial y}\frac{\partial^3 w_0}{\partial x^2\partial y}+c_{15}\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2}+c_{16}(\frac{\partial w_0}{\partial x})^2\frac{\partial^2 w_0}{\partial x^2}+$$
$$c_{17}(\frac{\partial w_0}{\partial y})^2\frac{\partial^2 w_0}{\partial y^2}+c_{18}(\frac{\partial w_0}{\partial x})^2\frac{\partial^2 w_0}{\partial y^2}+c_{19}(\frac{\partial^2 w_0}{\partial x\partial y})^2+$$
$$c_{20}(\frac{\partial w_0}{\partial y})^2\frac{\partial^2 w_0}{\partial x^2}+c_{21}\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x\partial y}+c_{22}\frac{\partial u_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}+$$
$$c_{23}\frac{\partial v_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}+c_{24}\frac{\partial u_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}+c_{25}\frac{\partial v_0}{\partial y}\frac{\partial^2 w_0}{\partial x^2}+$$
$$c_{26}\frac{\partial u_0}{\partial y}\frac{\partial^2 w_0}{\partial x\partial y}+c_{27}\frac{\partial v_0}{\partial x}\frac{\partial^2 w_0}{\partial x\partial y}+c_{28}\frac{\partial^2 u_0}{\partial x\partial y}\frac{\partial w_0}{\partial y}+$$
$$c_{29}\frac{\partial^2 v_0}{\partial x\partial y}\frac{\partial w_0}{\partial x}+c_{30}\frac{\partial^2 u_0}{\partial x^2}\frac{\partial w_0}{\partial x}+c_{31}\frac{\partial^2 u_0}{\partial y^2}\frac{\partial w_0}{\partial x}+$$
$$c_{32}\frac{\partial^2 v_0}{\partial x^2}\frac{\partial w_0}{\partial y}+c_{33}\frac{\partial^2 v_0}{\partial y^2}\frac{\partial w_0}{\partial y}+c_{34}\frac{\partial^3 u_0}{\partial x\partial y^2}+c_{35}\frac{\partial^3 u_0}{\partial x^3}+$$
$$c_{36}\frac{\partial^3 v_0}{\partial x^2\partial y}+c_{37}\frac{\partial^3 v_0}{\partial y^3}+c_{38}\frac{\partial^2 w_0}{\partial x^2}+c_{39}\frac{\partial^2 w_0}{\partial y^2}+$$
$$c_{45}\cos(\bar{\Omega}_1\bar{t})=c_{40}\frac{\partial w_0}{\partial t}+c_{41}\frac{\partial^2 w_0}{\partial t^2}+c_{42}\frac{\partial^4 w_0}{\partial t^2\partial x^2}+$$
$$c_{43}\frac{\partial^4 w_0}{\partial t^2\partial y^2}+c_{44}(\frac{\partial^3 u_0}{\partial t^2\partial x}+\frac{\partial^3 v_0}{\partial t^2\partial y}) \tag{8c}$$

上式中系数的表达式见附录.

利用 Galerkin 方法，采用悬臂梁和自由梁的模态函数的组合作为悬臂板的模态函数[10]，系统第一、二阶无量纲模态函数形式如下：

$$u_0=u_1(t)\sin(\frac{\pi x}{2})\cos(\pi y)$$
$$v_0=v_1(t)\sin(\frac{\pi x}{2})\cos(\pi y)$$
$$w_0=w_1(t)\alpha_1\beta_1+w_2(t)\alpha_1\beta_2 \tag{9}$$

式中 $u_1(t)$ 表示 x 方向的第一阶振幅，$v_1(t)$ 表示 y 方向的第一阶振幅，$w_1(t)$，$w_2(t)$ 分别表示横向即 z 方向的第一、二阶振幅，α_i，β_i 的表达式如下：

$$\alpha_1=\cosh k_1 x-\cos k_1 x-\varphi_1(\sinh k_1 x-\sin k_1 x)$$

$$\cos k_i \cosh k_i + 1 = 0,\ k_i^{\ 4} = \omega^2 \frac{\rho A}{EJ}(i=1)$$

$$\varphi_i = \frac{\sinh k_i - \sin k_i}{\cosh k_i + \cos k_i}(i=1)$$

$$\beta_1 = \cosh m_1 y + \cos m_1 y - \varphi_1(\sinh m_1 y + \sin m_1 y)$$

$$\beta_2 = \cosh m_2 y + \cos m_2 y - \varphi_2(\sinh m_2 y + \sin m_2 y)$$

$$\cos m_i \cosh m_i - 1 = 0,\ m_i^{\ 4} = \omega^2 \frac{\rho A}{EJ}(i=1,2)$$

$$\varphi_i = \frac{\cosh m_i - \cos m_i}{\sinh m_i - \sin m_i}(i=1,2) \tag{10}$$

忽略(8a)(8b)方程的所有惯性项和(8c)关于 u_0, v_0 的惯性项,利用 Galerkin 方法,将(9)和(10)代入方程(8),得压电复合材料层合悬臂板的两自由度的非线性常微分方程如下:

$$\ddot{w}_1 + \mu_1 \dot{w}_1 + (\omega_1^{\ 2} + q_1 \cos\Omega_2 t + q_p \cos\Omega_2 t) w_1 + q_2 w_2 + q_3 w_1^{\ 2} w_2 + q_4 w_2^{\ 2} w_1 + q_5 w_1^{\ 3} + q_6 w_2^{\ 3} = f_1 \cos\Omega_1 t \tag{11a}$$

$$\ddot{w}_2 + \mu_2 \dot{w}_2 + (\omega_2^{\ 2} + r_1 \cos\Omega_2 t + r_p \cos\Omega_2 t) w_2 + r_2 w_1 + r_3 w_2^{\ 2} w_1 + r_4 w_1^{\ 2} w_2 + r_5 w_2^{\ 3} + r_6 w_1^{\ 3} = f_2 \cos\Omega_1 t \tag{11b}$$

2　摄动分析

由于多尺度法是求解非线性系统很有效的近似分析方法,因此应用多尺度法对压电复合材料层合悬臂板两个自由度非线性常微分方程(11)进行摄动分析,引入如下的尺度变换:

$$\mu_1 \to \varepsilon\mu_1,\ \mu_2 \to \varepsilon\mu_2,\ q_1 \to \varepsilon q_1,\ q_p \to \varepsilon q_p,\ q_2 \to \varepsilon q_2,\ q_3 \to \varepsilon q_3,\ q_4 \to \varepsilon q_4,\ q_5 \to \varepsilon q_5,\ q_6 \to \varepsilon q_6,\ r_1 \to \varepsilon r_1,\ r_p \to \varepsilon r_p,\ r_2 \to \varepsilon r_2,\ r_3 \to \varepsilon r_3,\ r_4 \to \varepsilon r_4,\ r_5 \to \varepsilon r_5,\ r_6 \to \varepsilon r_6,\ f_1 \to \varepsilon f_1,\ f_2 \to \varepsilon f_2 \tag{12}$$

将变换公式(12)代入方程(11)得含小参数 ε 的方程:

$$\ddot{w}_1 + \varepsilon\mu_1 \dot{w}_1 + \omega_1^{\ 2} w_1 + \varepsilon q_1 \cos\Omega t w_1 + \varepsilon q_p \cos\Omega t w_1 + \varepsilon q_2 w_2 + \varepsilon q_3 w_1^{\ 2} w_2 + \varepsilon q_4 w_2^{\ 2} w_1 + \varepsilon q_5 w_1^{\ 3} + \varepsilon q_6 w_2^{\ 3} = \varepsilon f_1 \cos\Omega t \tag{13a}$$

$$\ddot{w}_2 + \varepsilon\mu_2 \dot{w}_2 + \omega_2^{\ 2} w_2 + \varepsilon r_1 \cos\Omega t w_2 + \varepsilon r_p \cos\Omega t w_2 + \varepsilon r_2 w_1 + \varepsilon r_3 w_2^{\ 2} w_1 + \varepsilon r_4 w_1^{\ 2} w_2 + \varepsilon r_5 w_2^{\ 3} + \varepsilon r_6 w_1^{\ 3} = \varepsilon f_2 \cos\Omega t \tag{13b}$$

为了获得压电复合材料层合悬臂板的四维平均方程,假设方程(13)有如下形式的解:

$$w_1(t,\varepsilon) = x_0(T_0, T_1) + \varepsilon x_1(T_0, T_1) \tag{14a}$$

$$w_2(t,\varepsilon) = y_0(T_0, T_1) + \varepsilon y_1(T_0, T_1) \tag{14b}$$

其中 $T_0 = t, T_1 = \varepsilon t$.

考虑主参数共振和 1∶3内共振,则有如下形式的内共振关系:

$$\omega_1^{\ 2} = \frac{1}{9}\Omega_1^{\ 2} + \varepsilon\sigma_1,\ \omega_2^{\ 2} = \Omega_2^{\ 2} + \varepsilon\sigma_2,$$

$$\Omega_1 = \Omega_2 = \Omega = \omega,\ \omega_2 \approx 3\omega_1 \tag{15}$$

式中 ω_1 和 ω_2 为相应线性系统的第一阶和第二阶固有频率,σ_1 和 σ_2 为系统的调谐参数.

把式(14)和(15)代入式(13),并且比较方程两边小摄动参数 ε 同阶次的系数,得到下面的微分方程:

ε^0 阶 :

$$D_0^{\ 2} x_0 + \frac{1}{9}\omega^2 x_0 = 0 \tag{16a}$$

$$D_0^{\ 2} y_0 + \omega^2 y_0 = 0 \tag{16b}$$

ε^1 阶:

$$D_0^{\ 2} x_1 + \frac{1}{9}\omega^2 x_1 = -\mu_1 D_0 x_0 - \sigma_1 x_0 - 2D_0 D_1 x_0 - (q_1 + q_p) x_0 \cos\omega t - q_2 y_0 - q_3 x_0^{\ 2} y_0 - q_4 y_0^{\ 2} x_0 - q_5 x_0^{\ 3} - q_6 y_0^{\ 3} + f_1 \cos\omega t \tag{17a}$$

$$D_0^{\ 2} y_1 + \omega^2 y_1 = -\mu_2 D_0 y_0 - \sigma_2 y_0 - 2D_0 D_1 y_0 - (r_1 + r_p) y_0 \cos\omega t - r_2 x_0 - r_3 y_0^{\ 2} x_0 - r_4 x_0^{\ 2} y_0 - r_5 y_0^{\ 3} - r_6 x_0^{\ 3} + f_2 \cos\omega t \tag{17b}$$

将方程(13)的解用复数形式表示为:

$$x_0 = A(T_1) e^{i\omega_1 T_0} + \bar{A}(T_1) e^{-i\omega_1 T_0} = A(T_1) e^{\frac{1}{3} i\omega T_0} + \bar{A}(T_1) e^{-\frac{1}{3} i\omega T_0} \tag{18a}$$

$$y_0 = B(T_1) e^{i\omega_2 T_0} + \bar{B}(T_1) e^{-i\omega_2 T_0} = B(T_1) e^{i\omega T_0} + \bar{B}(T_1) e^{-i\omega T_0} \tag{18b}$$

式中 $\bar{A}, \bar{B}$ 分别是 A, B 的共轭.

将方程(18)带入方程(17),得下式:

$$D_0^{\ 2} x_1 + \frac{1}{9}\omega^2 x_1 = -\mu_1 A \frac{1}{3} i\omega e^{\frac{1}{3} i\omega T_0} - \sigma_1 A e^{\frac{1}{3} i\omega T_0} - D_1 A \frac{2}{3} i\omega e^{\frac{1}{3} i\omega T_0} - (q_1 + q_p) A e^{\frac{1}{3} i\omega T_0} \cos\omega t - q_2 B e^{i\omega T_0} - q_3(\bar{A}^2 B e^{\frac{1}{3} i\omega T_0} + 2A\bar{A}B e^{i\omega T_0}) - q_4(\bar{A}B^2 e^{\frac{5}{3} i\omega T_0} + 2AB\bar{B} e^{\frac{1}{3} i\omega T_0}) - q_5(A^3 e^{i\omega T_0} + 3A^2\bar{A} e^{\frac{1}{3} i\omega T_0}) - q_6(B^3 e^{3i\omega T_0} + 3B^2\bar{B} e^{i\omega T_0}) + \frac{1}{2} f_1 e^{i\omega T_0} + cc \tag{19a}$$

$$D_0^{\ 2} y_1 + \omega^2 y_1 = -\mu_2 B i\omega e^{i\omega T_0} - \sigma_2 B e^{i\omega T_0} - 2D_1 B i\omega e^{i\omega T_0} - (r_1 + r_p) B e^{i\omega T_0} \cos\omega t - r_2 A e^{\frac{1}{3} i\omega T_0} - r_3(AB^2 e^{\frac{7}{3} i\omega T_0} + \bar{A}B^2 e^{\frac{5}{3} i\omega T_0} + 2AB\bar{B} e^{\frac{1}{3} i\omega T_0}) - r_4(\bar{A}^2 B e^{\frac{1}{3} i\omega T_0} + 2A\bar{A}B e^{i\omega T_0}) - r_5(B^3 e^{3i\omega T_0} +$$

$$3B^2\bar{B}e^{i\omega T_0})-r_6(A^3e^{i\omega T_0}+3A^2\bar{A}e^{\frac{1}{3}i\omega T_0})+\frac{1}{2}f_2e^{i\omega T_0}+cc \tag{19b}$$

式中的符号 cc 和 NST 分别表示方程(19)右边函数的共轭项和非长期项,令方程(19)的长期项等于零,可以得到复数形式平均方程如下:

$$D_1A=-\frac{1}{2}\mu_1A+\frac{3i}{2\omega}\sigma_1A+\frac{3i}{2\omega}(q_1+q_p)A\cos\omega t+\frac{3i}{2\omega}q_3\bar{A}^2B+\frac{3i}{\omega}q_4AB\bar{B}+\frac{9i}{2\omega}q_5A^2\bar{A} \tag{20a}$$

$$D_1B=-\frac{1}{2}\mu_2B+\frac{i}{2\omega}\sigma_2B+\frac{i}{2\omega}(r_1+r_p)B\cos\omega t+\frac{i}{\omega}r_4A\bar{A}B+\frac{3i}{2\omega}r_5B^2\bar{B}+\frac{i}{2\omega}r_6A^3-\frac{i}{4\omega}f_2 \tag{20b}$$

令 $A=x_1+ix_2$, $B=x_3+ix_4$,代入(20)得直角坐标形式的四维平均方程:

$$\dot{x}_1=-\frac{1}{2}\mu_1x_1-\frac{3}{2\omega}\sigma_1x_2-\frac{3(q_1+q_p)}{2\omega}x_2\cos\omega t-\frac{3q_3}{2\omega}({x_1}^2x_4-2x_1x_2x_3-{x_2}^2x_4)-\frac{3q_4}{\omega}(x_2{x_3}^2+x_2{x_4}^2)-\frac{9q_5}{2\omega}({x_1}^2x_2+{x_2}^3) \tag{21a}$$

$$\dot{x}_2=-\frac{1}{2}\mu_1x_2+\frac{3}{2\omega}\sigma_1x_1+\frac{3(q_1+q_p)}{2\omega}x_1\cos\omega t+\frac{3q_3}{2\omega}({x_1}^2x_3+2x_1x_2x_4-{x_2}^2x_3)+\frac{3q_4}{\omega}(x_1{x_3}^2+x_1{x_4}^2)+\frac{9q_5}{2\omega}({x_2}^2x_1+{x_1}^3) \tag{21b}$$

$$\dot{x}_3=-\frac{1}{2}\mu_2x_3-\frac{1}{2\omega}\sigma_2x_4-\frac{r_1+r_p}{2\omega}x_4\cos\omega t-\frac{r_4}{\omega}({x_1}^2x_4+{x_2}^2x_4)-\frac{3r_5}{2\omega}({x_3}^2x_4+{x_4}^3)-\frac{r_6}{2\omega}(-{x_2}^3+3{x_1}^2x_2) \tag{21c}$$

$$\dot{x}_4=-\frac{1}{2}\mu_2x_4+\frac{1}{2\omega}\sigma_2x_3+\frac{r_1+r_p}{2\omega}x_3\cos\omega t+\frac{r_4}{\omega}({x_1}^2x_3+{x_2}^2x_3)+\frac{3r_5}{2\omega}({x_4}^2x_3+{x_3}^3)+\frac{r_6}{2\omega}({x_1}^3-3{x_2}^2x_1)-\frac{f_2}{4\omega} \tag{21d}$$

3 数值模拟

本节在得到压电复合材料层合悬臂板四维直角坐标平均方程(21)的基础上,选取无量纲横向激励幅值 f_2 和压电参数项 q_p 作为控制参数,基于四阶 Runge-Kutta 法,利用 MATLAB 软件对主参数共振-1:3内共振情况下的压电复合材料层合悬臂板进行数值模拟分析,得到系统的非线性动力学响应.

3.1 横向外激励幅值对系统非线性振动特性影响

由于横向外激励幅值是影响系统非线性振动响应最重要的因素之一,所以首先研究 f_2 对系统的影响.选取系统初始条件和参数值为:

$$x_{10}=-0.23,\ x_{20}=-0.48,\ x_{30}=0.02,\ x_{40}=-0.01,\ \mu_1=0.02,\ \mu_2=0.04,\ \sigma_1=0.25,\ \sigma_2=0.37,\ q_1=4.5,\ q_p=3.7,\ q_3=6.3,\ q_4=0.7,\ q_5=6.4,\ r_1=1.2,\ r_p=6.4,\ r_4=0.45,\ r_5=0.31,\ r_6=4.9 \tag{22}$$

将式(22)的值代入到方程(21),采用 MATLAB 软件,通过数值模拟,得到系统随横向外激励幅值 f_2 变化的分叉图如图 2 所示.由图可以看出,系统存在周期和混沌运动,并且周期与混沌交替出现,且混沌幅值远高于周期的幅值.当无量纲的横向激励幅值从 84 变化到 97 时,系统依次呈现单倍周期→混沌→单倍周期→混沌→单倍周期的变化规律.

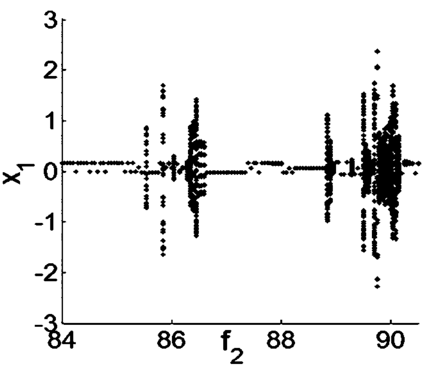
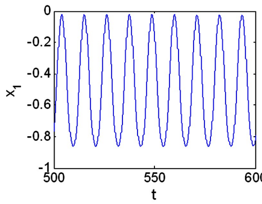


图 2 系统随横向激励幅值变化的分叉图

Fig.2 Bifurcation diagrams of system amplitude f_2 with the change of transverse excitation

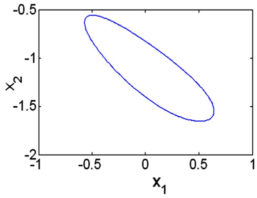
为了验证分叉图的正确性,取 $f_2=85$,绘制系统前两阶波形图、相图、庞加莱映射如图 3 所示.由图可以看出,系统运动呈现为单倍周期运动.

由分叉图可以看出,当系统发生混沌时,系统幅值会发生大幅跳跃,运动形式由单倍周期转换为混沌形式.当取 $f_2=90$ 时,绘制前两阶的波形图、相图、庞加莱映射如图 4 所示,由图 4 看出系统的运动确实为混沌运动.



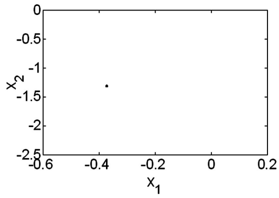
(a) 平面(t,x_1)的波形图

(a) Waveform on the plane (t,x_1)



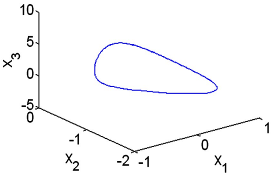
(b) 平面(x_1,x_2)的相图

(b) Phase portrait on the plane (x_1,x_2)



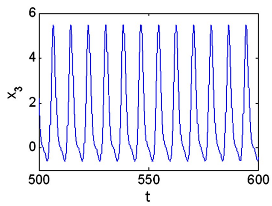
(c) 平面(x_1,x_2)的庞加莱映射

(c) Poincare map on the plane (x_1,x_2)



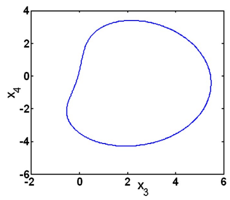
(d) 空间(x_1,x_2,x_3)的三维相图

(d) Three-dimensional phase portrait in the space (x_1,x_2,x_3)



(e) 平面(t,x_3)的波形图

(e) Waveform on the plane (t,x_3)



(f) 平面(x_3,x_4)的相图

(f) Phase portrait on the plane (x_3,x_4)

图 3　单倍周期运动

Fig.3　Periodic motion of the plate when $f_2=85$

3.2　压电参数项对系统非线性振动特性影响

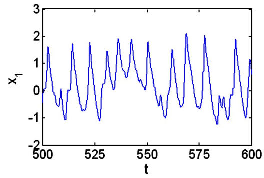
为了分析压电材料在系统振动过程中对系统非线性响应的影响,选取另一组初始条件和参数值如下:

$$x_{10}=1.4,\ x_{20}=2.7,\ x_{30}=0.05,\ x_{40}=-0.09,$$
$$\mu_1=0.02,\ \mu_2=0.04,\ \sigma_1=7.5,\ \sigma_2=9.3, q_1=11.2,$$
$$q_3=6.3, q_4=31.6,\ q_5=1.4, r_1=1.2,\ r_p=9.7,$$
$$r_4=6.1,\ r_5=5.6,\ r_6=2.7,\ f_2=85 \tag{23}$$

取压电参数项 q_p 从 0 变化到 15,得到系统的分叉图如图 5 所示.由图可知,随着压电参数项的增大,系统由三倍周期运动逐渐变为分叉运动,最后呈现为混沌运动,因此压电参数项对系统的非线性振动响应有很重要的调节作用.

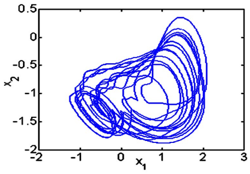
4　结论

本文研究了在横向激励和面内激励联合作用下,压电复合材料层合悬臂板的非线性动力学响应.应用经典板理论和 Hamilton 原理建立系统无量纲偏微分动力学方程,采用 Galerkin 方法对系统进行二阶离散,得到两自由度的无量纲常微分方程.



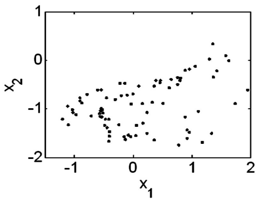
(a) 平面(t,x_1)波形图

(a) Waveform on the plane (t,x_1)



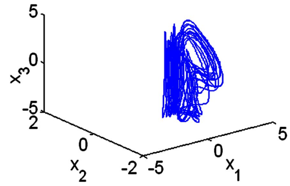
(b) 平面(x_1,x_2)的相图

(b) Phase portrait on the plane (x_1,x_2)



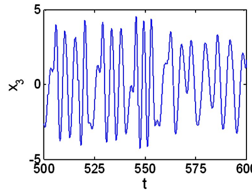
(c) 平面(x_1,x_2)的庞加莱映射

(c) Poincare map on the plane (x_1,x_2)



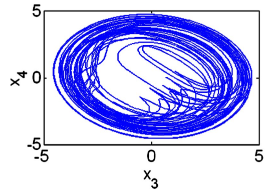
(d) 空间(x_1,x_2,x_3)的三维相图

(d) Three-dimensional phase portrait in the space (x_1,x_2,x_3)



(e) 平面(t,x_3)的波形图

(e) Waveform on the plane (t,x_3)



(f) 平面(x_3,x_4)的相图

(f) Phase portrait on the plane (x_3,x_4)

图 4　混沌运动

Fig.4　Chaotic motion of the plate when $f_2=90$

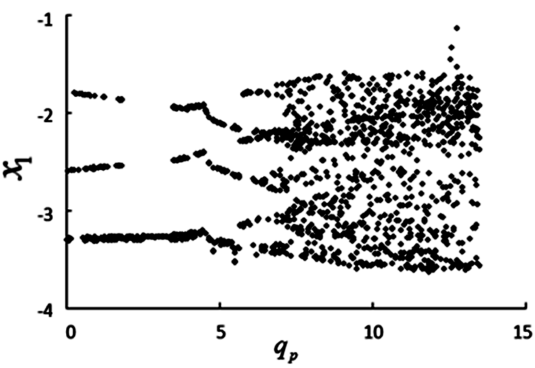


图 5　系统随压电参数项变化的分叉图

Fig.5　Bifurcation diagrams of system coefficient q_p with the change of piezoelectric

考虑主参数共振-1:3内共振,运用多尺度法进行摄动分析,推导出四维平均方程.选取一组合适的参数,数值模拟得到横向外激励幅值对复合材料悬臂板非线性动力学响应的影响.结果表明当无量纲的横向激励幅值从 84 变化到 97 时,系统存在周期和混沌运动,并且周期与混沌交替出现,且混沌幅值远高于周期的幅值.

选取另一组参数值,绘制出了系统幅值随压电参数项变化的分叉图,由图可知压电参数项对系统

的非线性振动响应有很重要的调节作用.因此,在实际工程中,可以通过改变系统外激励幅值或压电参数项,来调节系统振动的幅值,保持系统振动的稳定性.

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NONLINEAR DYNAMIC ANALYSIS OF A COMPOSITE CANTILEVER PIEZOELECTRIC PLATE WITH ONE-TO-THREE INTERNAL RESONANCE*

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Abstract The wing is simplified as the cantilever plate model based on the engineering background of the aircraft wing. The nonlinear dimensionless partial differential governing equations of the composite cantilever piezoelectric plate under the combination of transverse and in-plane excitations are established by using the Hamilton's principle. Galerkin approach is used to discretize the partial differential equations to the ordinary differential equations with two-degree-of-freedom under the transverse and parametric excitations. Meanwhile, the method of multiple scales is employed to obtain the four-dimensional averaged equation considering the primary parametric resonance-1∶3 internal resonance. Numerical method is utilized to find the nonlinear dynamics responses of the composite piezoelectric cantilever plate. The numerical results indicate the existence of the periodic and chaotic responses. The influences of the transverse excitation and piezoelectric parameter term on the bifurcations and chaotic behaviors of the composite piezoelectric cantilever plate are investigated. The conclusions are instructive to the practical engineering.

Key words cantilever plate, internal resonance, perturbation analysis, multiple scales method, nonlinear dynamics

Received 29 September 2017, revised 13 November 2017.

* The project supported by the National Natural Science Foundation of China(11290152,11322214).

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附录:

式(8)中的系数如下:

$a_{10}=1$, $a_{11}=\frac{a^2\bar{A}_{66}}{b^2\bar{A}_{11}}$, $a_{12}=\frac{(\bar{A}_{12}+\bar{A}_{66})}{\bar{A}_{11}}$,

$a_{13}=\frac{h^2}{a^2}$, $a_{14}=\frac{h^2\bar{A}_{66}}{b^2\bar{A}_{11}}$, $a_{15}=\frac{h^2}{b^2\bar{A}_{11}}(\bar{A}_{12}+\bar{A}_{66})$,

$a_{16}=-\frac{h^2}{b^2\bar{A}_{11}}(\bar{B}_{12}+2\bar{B}_{66})$, $a_{17}=-\frac{h^2\bar{B}_{11}}{a^2\bar{A}_{11}}$,

$a_{18}=\frac{a^2\pi^4}{h^2}\frac{\bar{I}_0}{\bar{A}_{11}}$, $a_{19}=-\frac{\sqrt{ab}\,\pi^4}{h}\frac{\bar{I}_1}{\bar{A}_{11}}$,

$b_{10}=1$, $b_{11}=\frac{b^2\bar{A}_{66}}{a^2\bar{A}_{22}}$, $b_{12}=\frac{(\bar{A}_{21}+\bar{A}_{66})}{\bar{A}_{22}}$,

$b_{13}=\frac{h^2}{b^2}$, $b_{14}=\frac{h^2\bar{A}_{66}}{a^2\bar{A}_{22}}$, $b_{15}=\frac{h^2}{a^2\bar{A}_{22}}(\bar{A}_{21}+\bar{A}_{66})$,

$b_{16}=-\frac{h^2}{a^2\bar{A}_{22}}(\bar{B}_{21}+2\bar{B}_{66})$, $b_{17}=-\frac{h^2\bar{B}_{22}}{b^2\bar{A}_{22}}$,

$b_{18}=\frac{b^2\pi^4}{h^2}\frac{\bar{I}_0}{\bar{A}_{22}}$, $b_{19}=-\frac{\sqrt{ab}\,\pi^4}{h}\frac{\bar{I}_1}{\bar{A}_{22}}$,

$c_{10}=-\frac{h^3\bar{D}_{11}}{a^4}$, $c_{11}=-\frac{h^3}{a^2b^2}(\bar{D}_{21}+4\bar{D}_{66}+\bar{D}_{12})$,

$c_{12}=-\frac{h^3\bar{D}_{22}}{a^4}$, $c_{13}=-\frac{h^3}{a^2b^2}(\bar{B}_{12}-\bar{B}_{21})$,

$c_{14}=\frac{h^3}{a^2b^2}(\bar{B}_{12}-\bar{B}_{21})$, $c_{15}=\frac{h^3}{a^2b^2}(2\bar{B}_{66}-\bar{B}_{12}-\bar{B}_{21})$,

$c_{16}=\frac{3h^3}{2a^4}\bar{A}_{11}$, $c_{17}=\frac{3h^3}{2b^4}\bar{A}_{22}$,

$c_{18}=\frac{h^3}{a^2b^2}(\frac{1}{2}\bar{A}_{21}+\bar{A}_{66})$, $c_{19}=\frac{h^3}{a^2b^2}(\frac{1}{2}\bar{A}_{12}+\bar{A}_{66})$,

$c_{20}=\frac{h^3}{a^2b^2}(\bar{B}_{12}+\bar{B}_{21}-2\bar{B}_{66})$,

$c_{21}=\frac{h^3}{a^2b^2}(\bar{A}_{21}+4\bar{A}_{66}+\bar{A}_{12})$,

$c_{22}=\frac{h}{a^2}\bar{A}_{11}$, $c_{23}=\bar{A}_{22}$, $c_{24}=\frac{h}{b^2}\bar{A}_{21}$,

$c_{25}=\frac{h}{a^2}\bar{A}_{12}$, $c_{26}=\frac{2h}{b^2}\bar{A}_{66}$, $c_{27}=\frac{2h}{a^2}\bar{A}_{66}$,

$c_{28}=\frac{h}{b^2}(\bar{A}_{21}+\bar{A}_{66})$, $c_{29}=\frac{h}{a^2}(\bar{A}_{12}+\bar{A}_{66})$,

$c_{30}=\frac{h}{a^2}\bar{A}_{11}$, $c_{31}=\frac{h}{b^2}\bar{A}_{66}$, $c_{32}=\frac{h}{a^2}\bar{A}_{66}$,

$c_{33}=\frac{h}{b^2}\bar{A}_{22}$, $c_{34}=\frac{h}{b^2}(2\bar{B}_{66}+\bar{B}_{21})$,

$c_{35}=\frac{h}{a^2}(2\bar{B}_{66}+\bar{B}_{12})$, $c_{36}=\frac{h}{a^2}\bar{B}_{11}$, $c_{37}=\frac{h}{b^2}\bar{B}_{22}$,

$c_{38}=-N_{xx}{}^p\cos(\Omega_p t)\frac{1}{Eh}\sqrt{\frac{b}{a^3}}$,

$c_{39}=-[N_{yy}{}^p\cos(\Omega_p t)+p_0+p_1\cos(\Omega_2 t)]\frac{1}{Eh}\sqrt{\frac{a}{b^3}}$,

$c_{40}=\frac{\pi^4h^3}{a^2b^2}\bar{c}_3$, $c_{41}=\frac{\pi^4}{h}\bar{I}_0$, $c_{42}=-\frac{\pi^4 b}{ha}\bar{I}_2$,

$c_{43}=-\frac{\pi^4 a}{hb}\bar{I}_2$, $c_{44}=\frac{\pi^4(ab)^{\frac{1}{2}}}{h^2}\bar{I}_1$, $c_{45}=F_1 a\frac{\sqrt{ab}}{Eh^2}$