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变截面粘弹性旋转梁非线性参数振动研究

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摘要　研究了变截面粘弹性旋转梁的非线性参数振动.基于 Kelvin-Voigt 粘弹性本构关系,考虑几何非线性建立了变截面粘弹性旋转梁的非线性振动方程,用 Galerkin 法将其转化为常微分方程.运用多重尺度法得到其幅频响应.用数值方法讨论了转速和轮毂半径对梁固有频率和幅频响应的影响.研究表明:不稳定域随轮毂半径、转速的增大而增大,随锥度的增大而减小.

关键词　变截面梁,　参数振动,　多重尺度法,　非线性,　幅频响应

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引言

变截面梁在航空机械中有很多应用,对其振动特性和响应研究很多.Gupta 等[1]用有限元法得到了截面直径线性变化的圆截面梁固有频率和模态;Heidebrecht[2]基于变截面梁振动方程用傅里叶级数法得到其固有频率和模态;崔灿等[3]提出了快速计算变截面梁振动特性的半解析法;Ghafari[4]用多项式降阶法得到旋转复合材料梁的固有频率.对粘弹性梁,杨晓东等[5]用多重尺度法研究了粘弹性变速运动梁的稳定性;Martin 等[6]用修正变分迭代法对粘弹性梁进行分析,得到其振幅;Mahmood 等[7]研究了粘弹性梁的非线性自由振动,得到阻尼对振幅的影响;Abolghasemi 等[8]对不同倾角的旋转粘弹性梁进行研究,讨论了吸引子的稳定性;蒋宝坤等[9]对旋转粘弹性夹层梁的非线性自由振动特性进行了研究,得到其固有频率和响应.刘金建等[10]研究了轴向运动功能梯度粘弹性梁横向振动的稳定性,讨论了不同因素对稳定性的影响.目前对变截面粘弹性旋转梁的研究较少.Vinod 等[11]提出了一种适用于有锥度的旋转欧拉梁谱单元的计算公式,验证了其正确性,并表明该单元在波传播问题中有更好的收敛性.Zolkiewski[12]研究了承受横向变载荷并且固定在刚性盘上的变截面梁的振动问题.朱由锋等[13]用有限差分法对变截面旋转梁的弯曲振动进行了研究,得到其幅频特性和相频特性.本文将基于 Kelvin-Voigt 粘弹性本构,考虑几何非线性建立其振动方程,研究其振动特性和参数振动,讨论其参数振动的稳定性.

1　数学模型

1.1　变截面粘弹性旋转梁模型

图 1 为高度随长度线性变化的变截面粘弹性旋转梁模型,刚性转毂半径 R,梁左端高度 a,右端高度为 b,锥度 $\Delta=b/a$,长度 l,宽度 d,绕轮毂旋转角速度 Ω,材料弹性模量 E,粘性系数 η,密度 ρ.

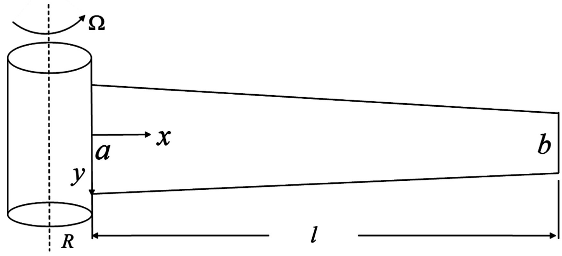

图 1　变截面粘弹性旋转梁模型

Fig.1　Rotating viscoelastic beam with variable cross-section

1.2　控制方程

仅考虑挥舞振动,旋转梁平衡方程[9]:

$$M_{,xx}-\rho A w_{,tt}+(N w_{,x})_{,x}=0 \tag{1}$$

式中,M 为弯矩,w 为 y 方向挠度,A 为横截面积,N 为轴力,$w_{,x}$ 和 $M_{,xx}$ 表示 w 和 M 对 x 的一阶、二阶偏

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导数，位于 x 处横截面积和惯性矩分别为：

$$A=(a-\frac{a-b}{l}x)d,\ I=\frac{d}{12}(a-\frac{a-b}{l}x)^3$$

对于粘弹性材料，用 Kelvin-Voigt 方程描述其本构关系：

$$\sigma=E\varepsilon+\eta\dot{\varepsilon} \tag{2}$$

几何非线性关系：

$$\varepsilon=\frac{1}{2}w_{,x}^2-yw_{,xx} \tag{3}$$

离截面 x 处离心力为：

$$N_1=\rho\Omega^2\int_x^l A(R+x)\mathrm{d}x \tag{4}$$

轴力为：

$$\begin{aligned}N&=N_1+\iint_A\sigma dydz\\&=\rho\Omega^2AR(l-x)+\frac{\rho\Omega^2A(l^2-x^2)}{2}+\\&\quad\frac{EA}{2}w_{,x}^2+\eta Aw_{,xt}w_{,x}\end{aligned} \tag{5}$$

式(5)代入式(1)中得变截面粘弹性旋转梁非线性振动方程：

$$\begin{aligned}&\eta Iw_{,xxxxt}+EIw_{,xxxx}+2EI_{,x}w_{,xxx}+EI_{,xx}w_{,xx}+\\&\eta I_{,xx}w_{,xxt}+2\eta I_{,x}w_{,xxxt}+\rho Aw_{,tt}-(\frac{E}{2}A_{,x}w_{,x}^2+\\&EAw_{,xx}w_{,x}+\eta A_{,x}w_{,xt}w_{,x}+\eta A(w_{,xx}w_{,xt}+w_{,x}w_{,xxt})-\\&\rho\Omega^2A(R+x))w_{,x}-(\frac{E}{2}Aw_{,x}^2+\eta Aw_{,xt}w_{,x}+\\&\rho\Omega^2AR(l-x)+\frac{\rho\Omega^2A(l^2-x^2)}{2})w_{,xx}=0\end{aligned} \tag{6}$$

边界条件为：

$$\begin{aligned}&w(x,t)\big|_{x=0}=w(x,t)_{,x}\big|_{x=0}=0\\&w(x,t)_{,xx}\big|_{x=l}=w(x,t)_{,xxx}\big|_{x=l}=0\end{aligned} \tag{7}$$

2　振动特性

方程(6)对应的线性振动方程为：

$$\begin{aligned}&\eta Iw_{,xxxxt}+EIw_{,xxxx}+2EI_{,x}w_{,xxx}+EI_{,xx}w_{,xx}+\\&\eta I_{,xx}w_{,xxt}+2\eta I_{,x}w_{,xxxt}+\rho Aw_{,tt}+\\&\rho\Omega^2A(R+x)w_{,x}-(\rho\Omega^2AR(l-x)+\\&\frac{\rho\Omega^2A(l^2-x^2)}{2})w_{,xx}=0\end{aligned} \tag{8}$$

设：

$$w(x,t)=\sum_{i=1}^N Y_i(x)q_i(t) \tag{9}$$

其中，

$$\begin{aligned}Y_i(x)=&(\cos p_ix-\cosh p_ix)-\\&\frac{\cos p_il+\cosh p_il}{\sin p_il+\sinh p_il}(\sin p_ix-\sinh p_ix)\end{aligned} \tag{10}$$

为满足边界条件的试函数，$q_i(t)$ 为广义模态坐标，p_i 由方程(11)：

$$\cos pl\cosh pl+1=0 \tag{11}$$

确定，将方程(9)代入(8)，基于 Galerkin 方法得：

$$\boldsymbol{M}\ddot{\boldsymbol{q}}+\boldsymbol{C}\dot{\boldsymbol{q}}+\boldsymbol{K}\boldsymbol{q}=0 \tag{12}$$

式中 $\boldsymbol{M},\boldsymbol{C},\boldsymbol{K}$ 为广义质量，阻尼和刚度矩阵，其元素为：

$$\begin{aligned}&m_{ij}=\rho a_{ij}^{00}\\&c_{ij}=\eta f_{ij}^{02}+2\eta f_{ij}^{03}+\eta f_{ij}^{04}\\&k_{ij}=Ef_{ij}^{02}+2Ef_{ij}^{03}+Ef_{ij}^{04}-\rho\Omega^2R(la_{ij}^{02}-b_{ij}^{02}-a_{ij}^{01})-\\&\quad\rho\Omega^2(\frac{l^2}{2}a_{ij}^{02}-\frac{c_{ij}^{02}}{2}-b_{ij}^{01})\end{aligned} \tag{13}$$

其中 $a_{ij}^{00}=\int_0^l AY_i(x)Y_j(x)\mathrm{d}x$，

$$a_{ij}^{01}=\int_0^l AY_i'(x)Y_j(x)\mathrm{d}x,$$

$$a_{ij}^{02}=\int_0^l AY_i''(x)Y_j(x)\mathrm{d}x,$$

$$f_{ij}^{02}=\int_0^l I''Y_i''(x)Y_j(x)\mathrm{d}x,$$

$$f_{ij}^{03}=\int_0^l I'Y_i'''(x)Y_j(x)\mathrm{d}x,$$

$$f_{ij}^{04}=\int_0^l IY_i''''(x)Y_j(x)\mathrm{d}x,$$

$$b_{ij}^{01}=\int_0^l AxY_i'(x)Y_j(x)\mathrm{d}x,$$

$$b_{ij}^{02}=\int_0^l AxY_i''(x)Y_j(x)\mathrm{d}x,$$

$$c_{ij}^{02}=\int_0^l Ax^2Y_i''(x)Y_j(x)\mathrm{d}x,$$

$$d_{ij}^{02}=\int_0^l AY_i''(x)[Y_i'(x)]^2Y_j(x)\mathrm{d}x,$$

$$d_{ij}^{03}=\int_0^l A'[Y_i'(x)]^3Y_j(x)\mathrm{d}x$$

3　参数振动

在方程(9)中，取 $N=1$，代入(6)有：

$$m_{11}\ddot{q}_1+c_{11}\dot{q}_1+k_{11}q_1+d_{11}q_1^{\ 3}+g_{11}q_1^{\ 2}\dot{q}_1=0 \tag{14}$$

设：

$$\Omega=\Omega_0+\varepsilon cos(\omega t) \tag{15}$$

式(15)代入式(14)有：

$$\ddot{q}_1+\varepsilon c_1\dot{q}_1+(k_1+\varepsilon k_2cos(\omega t))q_1+$$

$$d_1 q_1^{\ 3}+g_1 q_1^{\ 2}\dot{q}_1=0 \quad (16)$$

式中，

$$c_1=\frac{c_{11}}{\rho a_{11}^{00}},\ k_1=\frac{k_{11}}{\rho a_{11}^{00}},$$

$$k_2=\frac{\Omega_0 R(-2la_{11}^{02}+2b_{11}^{02}+2a_{11}^{01})+\Omega_0(-l^2 a_{11}^{02}+c_{11}^{02}+2b_{11}^{01})}{a_{11}^{00}},$$

$$d_1=\frac{-(\frac{E}{2}d_{11}^{03}+\frac{3E}{2}d_{11}^{02})}{\rho a_{11}^{00}},\ g_1=\frac{-(\eta d_{11}^{03}+3\eta d_{11}^{02})}{\rho a_{11}^{00}}$$

用多尺度法求解方程(16).

假设：

$$q_1=q_0+\varepsilon q_2$$

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}=D_0+\varepsilon D_1$$

$$\frac{\partial^2}{\partial t^2}=\frac{\partial^2}{\partial T_0^{\ 2}}+\varepsilon\frac{\partial^2}{\partial T_1^{\ 2}}+2\varepsilon\frac{\partial^2}{\partial T_0 T_1}=D_0^{\ 2}+\varepsilon^2 D_1^{\ 2}+2\varepsilon D_0 D_1 \quad (17)$$

代入(16)中得：

ε^0 项

$$D_0^{\ 2}q_0+k_1 q_0=0 \quad (18)$$

ε^1 项：

$$D_0^{\ 2}q_2+k_1 q_2=-c_1 D_0 q_0-2D_0 D_1 q_0-k_2 q_0 cos(\omega t)-d_1 q_0^{\ 3}-g_1 q_0^{\ 3}D_0 q_0 \quad (19)$$

设 q_0 解的形式为：

$$q_0=A(T_1)e^{i\omega_0 T_0}+\overline{A(T_1)}e^{-i\omega_0 T_0} \quad (20)$$

并令：

$$\omega=2\omega_0+\varepsilon\sigma \quad (21)$$

将方程(20)、(21)代入到(19)得：

$$\begin{aligned}D_0^{\ 2}q_2+k_1 q_2=&-c_1(i\omega_0 A(T_1)e^{i\omega_0 T_0}-i\omega_0\bar{A}(T_1)e^{-i\omega_0 T_0})-\\&2(i\omega_0\dot{A}(T_1)e^{i\omega_0 T_0}-i\omega_0\dot{\bar{A}}(T_1)e^{-i\omega_0 T_0})-\\&k_2(A(T_1)e^{i\omega_0 T_0}+\\&\bar{A}(T_1)e^{-i\omega_0 T_0})\cos(2\omega_0 T_0+\sigma T_1)-\\&d_1([A(T_1)]^3 e^{3i\omega_0 T_0}+3[A(T_1)]^2\cdot\\&\bar{A}(T_1)e^{i\omega_0 T_0}+[\bar{A}(T_1)]^3 e^{-i\omega_0 T_0}+\\&3A(T_1)[\bar{A}(T_1)]^2 e^{-i\omega_0 T_0})-\\&g_1(i\omega_0[A(T_1)]^3 e^{3i\omega_0 T_0}+\\&i i\omega_0[A(T_1)]^2\bar{A}(T_1)e^{i\omega_0 T_0}-\\&i\omega_0 A(T_1)[\bar{A}(T_1)]^2 e^{-i\omega_0 T_0}-\\&i\omega_0[\bar{A}(T_1)]^3 e^{-3i\omega_0 T_0})\end{aligned} \quad (22)$$

消除(22)式久期项得：

$$-i\omega_0(c_1 A(T_1)+2\dot{A}(T_1))-\frac{k_2\bar{A}(T_1)}{2}e^{i\sigma T_1}-3d_1[A(T_1)]^2\bar{A}(T_1)-ig_1\omega_0[A(T_1)]^2\bar{A}(T_1)=0 \quad (23)$$

假设 $A(T_1)=\frac{1}{2}\alpha e^{i\beta}$，代入(23)得：

$$-i\omega_0(\frac{c_1\alpha}{2}+\dot{\alpha})+\omega_0\alpha\dot{\beta}-\frac{k_2}{4}\alpha(\cos(\sigma T_1-2\beta)+i\sin(\sigma T_1-2\beta))-\frac{3}{8}d_1\alpha^3-\frac{1}{8}ig_1\omega_0\alpha^3=0 \quad (24)$$

得到：

$$\begin{cases}\dot{\alpha}=-\frac{c_1\alpha}{2}-\frac{1}{8}g_1\alpha^3-\frac{k_2}{4\omega_0}\alpha\sin(\sigma T_1-2\beta)\\ \alpha\dot{\beta}=\frac{k_2}{4\omega_0}\alpha(\cos(\sigma T_1-2\beta)+\frac{3}{8\omega_0}d_1\alpha^3\end{cases} \quad (25)$$

令 $\gamma=\sigma T_1-2\beta$，

$$\begin{cases}\dot{\alpha}=-\frac{c_1\alpha}{2}-\frac{1}{8}g_1\alpha^3-\frac{k_2}{4\omega_0}\alpha\sin\gamma\\ \alpha\dot{\gamma}=\alpha\sigma-\frac{3}{4\omega_0}d_1\alpha^3-\frac{k_2}{2\omega_0}\alpha\cos\gamma\end{cases} \quad (26)$$

由 $\dot{\alpha}=0$ 和 $\dot{\gamma}=0$ 得：

$$(-\frac{c_1\alpha}{2}-\frac{1}{8}g_1\alpha^3)^2+(\frac{\alpha\sigma}{2}-\frac{3}{8\omega_0}d_1\alpha^3)^2=(\frac{k_2}{4\omega_0}\alpha)^2 \quad (27)$$

4 数值计算及讨论

本节通过数值方法讨论了梁的振动特性和幅频响应，计算中相关参数如表1.

表1 材料参数

Table 1 Material parameters

E(Pa)	η	ρ(kg/m^3)
2.1×10^{11}	0.02	7900

4.1 方法有效性验证

为验证本方法，计算其频率并与有限元结果对比，有限元采用一维梁单元进行模拟，数值计算中取 $l=2$m，$a=0.1$m，$b=0.05$m，$d=0.05$m，当模态阶数取12时，频率趋于稳定.表2给出了轮毂半径为0.1m，模态阶数为12时有限元法(FEM)及本文方法得到的不同转速下的前四阶频率.

表 2　不同转速下的频率

Table 2　Frequencies with various rotating speeds

Ω(r/s)	1st Frequency(Hz)		2nd Frequency(Hz)		3rd Frequency(Hz)		4th Frequency(Hz)	
	Present	FEM	Present	FEM	Present	FEM	Present	FEM
0	21.6	22.6	108.7	108.5	279.9	279.8	535.7	535.4
10	24.5	25.4	111.5	111.2	282.6	282.4	538.4	538.0
20	31.6	32.1	119.5	118.9	290.7	290.3	546.6	546.0
30	40.6	40.7	131.7	130.9	303.6	302.8	560.0	558.9

可见有限元与本文方法有非常好的一致性,说明本文方法有效.

4.2　轮毂半径和转速对频率的影响

图 2 给出了转速、锥度和轮毂半径对一阶频率影响.

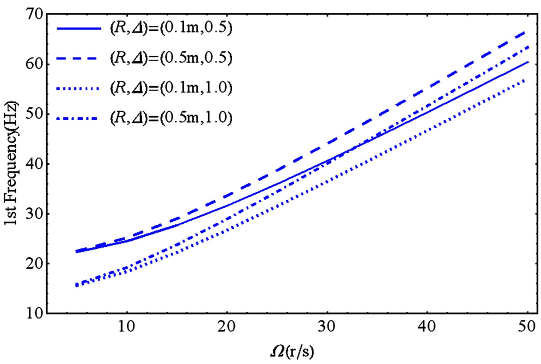

图 2　一阶固有频率随转速、锥度和轮毂半径变化

Fig.2　Development of first natural frequency with rotating speed, taper and radius of hub

可见,转速增大,频率增大,轮毂半径增大,频率增大,锥度增大,频率减小.

4.3　幅频响应

图 3 给出了轮毂半径为 0.1m,锥度为 0.25,转速为 1r/s 时的幅频响应曲线.图中实线和虚线分别代表系统的稳态和不稳态响应.可见系统从左到右存在两个分岔点,系统开始时只存在稳定的平凡零解,当到达第一分岔点时,系统发生音叉分岔,平凡解稳定性消失,出现一稳定的非平凡解,到达下一分岔点时,不稳定的平凡解变为稳定,同时分岔出一个不稳定的非平凡解.

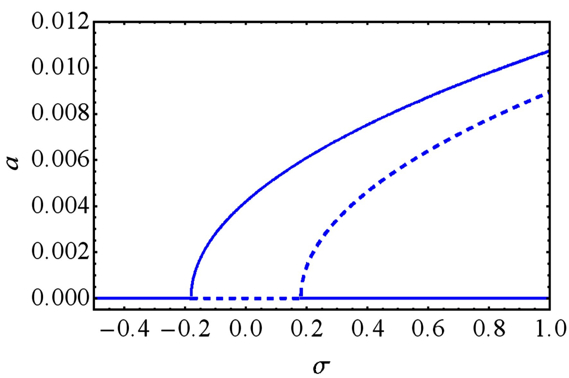

图 3　幅频响应曲线

Fig.3　Amplitude-frequency response curves

4.4　参数对稳定性的影响

图 4 给出了转速为 1.0r/s,锥度为 0.25,轮毂半径为 0.1m 和 0.7m 时的幅频响应曲线.可见轮毂半径增大,系统不稳定域增大.

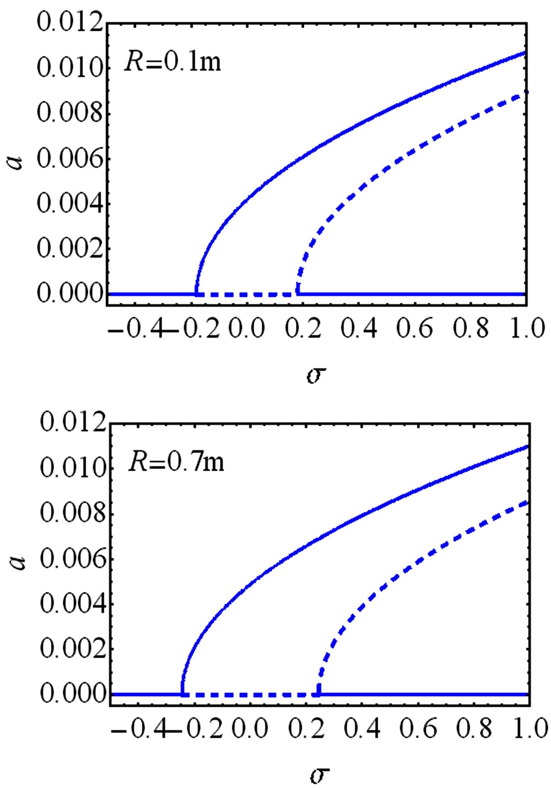

图 4　轮毂半径对稳定性影响

Fig.4　Effect of hub radius on stability

图 5 给出了轮毂半径为 0.1m,锥度为 0.25,转速为 0.6r/s 和 1.2r/s 时的幅频响应.可见随着转速增加,系统不稳定域增大,且使第一分岔点提前,第二分岔点延后.

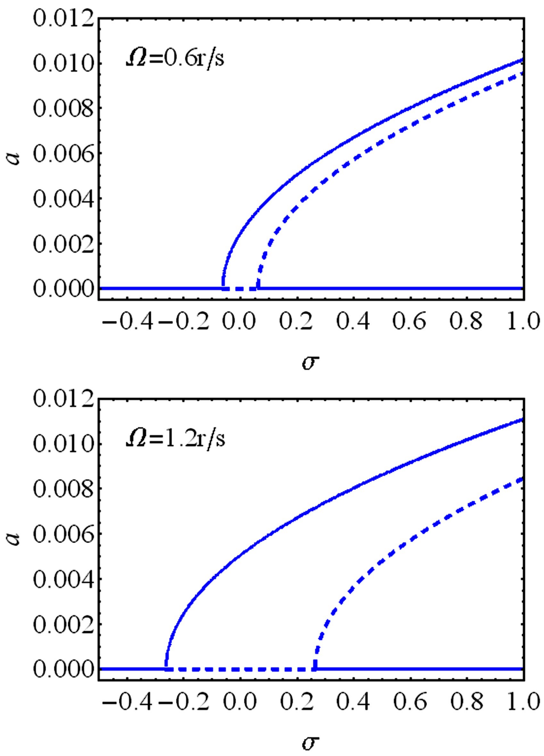

图 5 转速对稳定性影响

Fig.5 Effect of rotating speed on stability

图 6 给出了轮毂半径为 0.1m,转速为 1r/s,锥度为 0.25 和 0.15 时的幅频响应.可见随着锥度的增大,系统不稳定域减小.

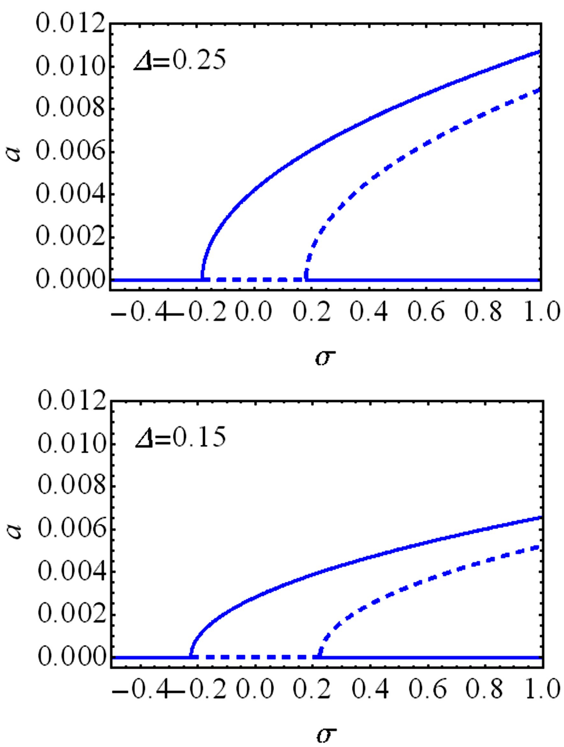

图 6 锥度对稳定性影响

Fig.6 Effect of taper on stability

5 结论

本文研究了变截面粘弹性旋转梁的振动特性和参数振动,得到了幅频响应.讨论了轮毂半径和转速对固有频率和幅频响应的影响,结果表明参数振动不稳定区域随轮毂半径、转速的增大而增大,随着锥度的增大而减小.

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STUDIES ON NONLINEAR PARAMETRIC VIBRATION OF A ROTATING VISCOELASTIC BEAM WITH VARIABLE CROSS-SECTIONS

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Abstract Nonlinear parametric vibration of a rotating viscoelastic beam with variable cross-section is studied. Nonlinear governing equation of the rotating viscoelastic beam with variable cross-section involving geometric nonlinearity is established based on Kelvin-Voigt constitutive relation. The governing equation is then transformed into an ordinary differential equation by using Galerkin method. Amplitude-frequency response is obtained by adopting the method of multiple scale. Finally, the influences of rotating velocityand radius of the hub on nature frequency and amplitude-frequency response are numerically discussed. It is shown that the unstable region increases with the increasing rotating velocity and radius of hub, but decreases with the increasing taper of the beam.

Key words variable cross-section beam, parametric vibration, multiple scale method, nonlinearity, amplitude-frequency response

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