

# 一类弦-梁耦合非线性振动系统的动力学数值 模拟研究<sup>\*</sup>

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**摘要** 本文研究了一类具有参数激励和外激励弦-梁耦合非线性系统.首先,运用多尺度法分析弦-梁耦合非线性系统的响应,求得系统平均方程.其次,基于求得的方程,以系统的阻尼系数作为分叉参数,并对系统平衡点的稳定性进行分析,得到平衡点的分叉曲线.为了验证理论预测的正确性数值模拟了不同分叉参数下的相空间轨线.利用四阶龙格库塔方法验证了弦-梁耦合非线性系统混沌运动的存在性,从数值模拟看出系统存在单倍周期运动、多倍周期运动和混沌运动.

**关键词** 耦合非线性系统, 稳定性, 多尺度方法, 周期运动

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## 引言

非线性系统自被提出以来一直都是非常具有吸引力和发展前景的研究领域.然而随着深入的研究,人们发现许多实际系统中存在激励振动而不能用传统自由振动和受迫振动模型进行模拟,为了解决这个问题在文献[1-4]分别建立了激励系统的非线性模型,并对模型进行相关研究.另外越来越多的研究表明共振能够反映系统线性模态之间的相互作用,有着非常重要的研究价值.文献[5]通过研究弹性悬浮电缆模型的共振,构建了该模型在1:1主共振和1:2亚谐共振情形下的非线性模态.文献[6]研究了一个极限环振子系统发生的1:3共振的Hopf分岔,并研究了非线性不同共振情形下对其动力学行为的影响.文献[7]研究了一个弦-梁模型的弦和梁在2:1内共振且梁在主参数共振-1/2亚谐共振和弦在1:1主共振情形下的非线性模态,并研究了外激励对共振附近动力学行为的影响.文献[8]研究了微谐振器模型在不同参数响应下的非线性模态.同时,研究者们已应用多种方法研究系统在共振点处的动力学行为,并取得了丰硕的成果:文献[9]应用多尺度方法研究了磁场力作用下

的双稳态曲梁压电发电离散系统的动力学行为.文献[10]运用多尺度法研究了两自由度和三自由度下系统的高阶近似.文献[11]讨论了轴向加速运动粘弹性梁问题的稳定性,该文献主要运用多尺度法和数值方法.文献[12]利用多尺度法研究了高阶多参数模型的响应和稳态.上述研究基本上是运用多尺度方法.可以看到多尺度方法不仅能计算稳态响应和计算非稳态响应,还能分析稳态响应的稳定性.因此经常被用来求解非线性系统的近似理论解.除了多尺度法还有许多其它有效的方法来求解非线性系统的理论解.比如文献[13]应用平均法研究Duffing-van der Pol系统的稳态响应,文献[14]应用摄动法分析了含有阻尼项的拟线性系统在噪声激励下的响应,文献[15]对单边碰撞悬臂梁系统,在基础激励实验中,变换多次激励频率运用实验法的定性研究.数值方法通过数值求解非线性微分方程得到非线性系统在特定的参数条件和初始条件下的运动规律.不仅具有检验理论结果的作用,还能得到直观的结果为理论研究提供启示.本篇文章继承了文献[7]的工作,但改变了系统的共振情况,增加应用多尺度方法微分方程定性理论研究了弦-梁耦合非线性振动系统的稳态与分叉,并进一

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步运用数值模拟分析了系统的运动情形。

文章结构如下:第一部分,介绍弦-梁耦合非线性振动系统模型.第二部分,对本文所研究梁和弦之间产生1:1主共振,梁在主参数主共振下原系统的平均方程.第三部分,通过阻尼系数分析参数变化对系统稳态响应的影响.第四部分,用数值模拟探讨外激励下系统运动形式.第五部分,对全文研究内容进行总结概括.

## 1 模型介绍

对弦-梁系统将梁和弦两端刚性连接,梁两端固定简谐激励  $p=p_0-F_2\cos(\Omega_2 t)$  和载荷联合作用,弦线两端的轴向初始张力为  $f(\lambda)=a_0\lambda^4+a_1\lambda^3+a_2\lambda^2+a_3\lambda+a_4$ ,同时整个弦-梁耦合系统受到简谐激励  $\ddot{y}_s=F_1\cos(\Omega_1 t)$  的作用.假设弦和梁在运动的过程中均只在  $a_1=\mu_1+\mu_2$  平面内运动,为了弦和梁之间不发生碰撞让他们保持一定的距离.

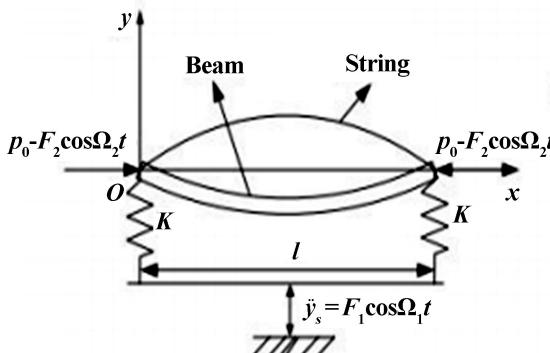


图1 模型图

Fig.1 Model of a string-beam coupled system

由以上假设,运用弹性力学方法建立弦-梁耦合系统的非线性动力学方程为:

$$m_1 \frac{\partial^2 w_1}{\partial t^2} + c_1 \frac{\partial w_1}{\partial t} - \left\{ p_0 + \frac{EA}{2l} \int_0^l \left( \frac{\partial w_1}{\partial x} \right)^2 dx - F_2 \cos \Omega_2 t + \left[ T_0 + \frac{K_s}{2} \int_0^l \left( \frac{\partial w_2}{\partial x} \right)^2 dx \right] \right\} \frac{\partial^2 w_1}{\partial x^2} + EI \frac{\partial^4 w_1}{\partial t^4} = m_1 F_1 \cos \Omega_1 t \quad (1a)$$

$$m_1 \frac{\partial^2 w_2}{\partial t^2} - \left[ T_0 + \frac{K_s}{2} \int_0^l \left( \frac{\partial w_2}{\partial x} \right)^2 dx \right] \frac{\partial^2 w_2}{\partial x^2} + c_2 \frac{\partial w_2}{\partial t} = m_2 F_1 \cos \Omega_1 t \quad (1b)$$

其中  $m_1$  和  $m_2$  分别为梁和弦单位长度质量,  $w_1$  和

$w_2$  分别是梁和弦的横向位移,  $l$  是梁和弦的长度,  $P_0$  是梁两端所受的轴向压力,  $T_0$  是弦的初始张力,  $A$  是梁变形前的横截面积,  $c_1$  和  $c_2$  分别是梁和弦的线性外阻尼系数,  $K_s$  是弦的弹性系数,  $E$  和  $I$  分别是梁的杨氏模量和转动惯量.

梁的边界条件:

$$x=0, \frac{\partial^2 w_1}{\partial x^2}=0, EI \frac{\partial^3 w_1}{\partial x^3} = -Kw_1(0, t)$$

$$x=l, \frac{\partial^2 w_1}{\partial x^2}=0, EI \frac{\partial^3 w_1}{\partial x^3} = Kw_1(l, t)$$

弦的边界条件:

$$x=0, w_2(0, t)=w_1(0, t)$$

$$x=l, w_2(l, t)=w_1(l, t)$$

引入如下无量纲变量:

$$t^* = t \sqrt{\frac{T_0 l^2 + EI}{(m_1 + m_2) l^4}}$$

$$x^* = \frac{x}{l}, w_1^* = \frac{w_1}{l}, w_1^* = \frac{w_1}{l}$$

$$\Omega_1^* = \Omega_1 \sqrt{\frac{(m_1 + m_2) l^4}{T_0 l^2 + EI}}$$

$$\Omega_2^* = \Omega_2 \sqrt{\frac{(m_1 + m_2) l^4}{T_0 l^2 + EI}}$$

$$F_1^* = \frac{(m_1 + m_2) F_1 l^3}{T_0 l^2 + EI}$$

$$F_2^* = \frac{(m_1 + m_2) F_2 l^2}{m_1 (T_0 l^2 + EI)} \quad (2)$$

将上述变换代入方程(1),化简去掉(\*),可以得到弦-梁耦合系统无量纲形式运动方程:

$$\frac{\partial^2 w_1}{\partial t^2} + \mu_1 \frac{\partial w_1}{\partial t} - \beta_2 \frac{\partial^2 w_1}{\partial x^2} + F_2 \cos(\Omega_2 t) \frac{\partial^2 w_1}{\partial x^2} - \beta_3 \int_0^l \left( \frac{\partial w_1}{\partial x} \right)^2 dx \cdot \frac{\partial^2 w_1}{\partial x^2} - \beta_3 \int_0^l \left( \frac{\partial w_2}{\partial x} \right)^2 dx \cdot \frac{\partial^2 w_1}{\partial x^2} + \beta_1 \frac{\partial^4 w_1}{\partial x^4} = F_1 \cos(\Omega_1 t) \quad (3a)$$

$$\alpha_1 l_{12} y_1 - \alpha_2 (g_{22} l_{12} + 2g_{21} l_{22}) y_2^2 y_1 - \alpha_2 g_{11} l_{12} y_1^3 - \frac{\partial^2 w_2}{\partial t^2} + \mu_1 \frac{\partial w_2}{\partial t} - \alpha_2 \int_0^l \left( \frac{\partial w_2}{\partial x} \right)^2 dx \cdot \frac{\partial^2 w_2}{\partial x^2} - \alpha_1 \frac{\partial^2 w_1}{\partial x^2} = F_1 \cos(\Omega_1 t) \quad (3b)$$

其中:

$$\begin{aligned}\mu_1 &= \frac{c_1}{m_1} \sqrt{\frac{(m_1+m_2)l^4}{T_0l^2+EI}}, \quad \beta_1 = \frac{(m_1+m_2)EI}{m_1(T_0l^2+EI)} \\ \beta_2 &= \frac{(m_1+m_2)(p_0+T_0)l^2}{m_1(T_0l^2+EI)}, \quad \beta_3 = \frac{(m_1+m_2)EAl^2}{2m_1(T_0l^2+EI)} \\ \beta_4 &= \frac{(m_1+m_2)Kl^3}{2m_1(T_0l^2+EI)}, \quad \mu_2 = \frac{c_2}{m_2} \sqrt{\frac{(m_1+m_2)l^4}{T_0l^2+EI}} \\ \alpha_1 &= \frac{(m_1+m_2)T_0l^2}{m_2(T_0l^2+EI)}, \quad \alpha_2 = \frac{(m_1+m_2)K_s l^3}{2m_2(T_0l^2+EI)}\end{aligned}$$

由于弦和梁的方程是耦合的,假设弦的模态函数包含有梁的相对位移,对梁和弦各取一阶模态进行 Galerkin 方法截断,梁和弦的位移  $w_1$  和  $w_2$  有如下形式

$$\begin{aligned}w_1(x, t) &= Y_1(x) y_1(t) \\ w_2(x, t) &= Y_2(x) y_2(t) + Y_1(x) y_1(t)\end{aligned}\quad (4)$$

把式(4)代入无量纲形式弦-梁耦合系统运动方程,并且利用 Galerkin 方法进行截断,得到在参数激励和外激励联合作用下弦-梁耦合系统二自由度非线性常微分方程:

$$\begin{aligned}\ddot{y}_1 + (\beta_1 K^{*4} - \beta_2 l_{11}) y_1 - (\beta_3 g_{11} l_{11} + \beta_4 g_{11} l_{11}) y_1^3 + \\ F_2 l_{11} \cos \Omega_2 t \cdot y_1 - \beta_4 g_{22} l_{11} y_2^2 y_1 + \mu_1 \dot{y}_1 - \\ 2\beta_4 g_{12} l_{11} y_2 y_1^2 = f_{11} \cos \Omega_1 t\end{aligned}\quad (5a)$$

$$\begin{aligned}\ddot{y}_2 + \lambda_{21} \ddot{y}_1 + \mu_2 \lambda_{21} \dot{y}_1 - \alpha_1 l_{22} y_2 - \alpha_2 g_{22} l_{22} y_2^2 + \\ \mu_2 \dot{y}_2 + \alpha_2 (g_{11} l_{22} + 2g_{21} l_{12}) y_2 y_1^2 = f_{12} \cos \Omega_1 t\end{aligned}\quad (5b)$$

其中:

$$\lambda_{mn} = \int_0^l Y_m(x) Y_n(x) dx$$

$$l_{mn} = \int_0^l Y''_m(x) Y_n(x) dx$$

$$g_{mn} = \int_0^l Y'_m(x) Y'_n(x) dx$$

$$f_{1m} = \int_0^l F_1(x) Y_m(x) dx, \quad m, n = 1, 2$$

## 2 摆动分析

在本节中将运用多尺度方法分析方程(5). 为了便于利用多尺度法对弦-梁耦合系统进行揆动分析,引入如下的尺度变换:

$$\begin{aligned}\mu_1 \rightarrow \varepsilon \mu_1, \quad a_{11} \rightarrow \varepsilon a_{11}, \quad a_{13} \rightarrow \varepsilon a_{13}, \quad a_{14} \rightarrow \varepsilon a_{14} \\ b_{21} \rightarrow \varepsilon b_{21}, \quad b_{22} \rightarrow \varepsilon b_{22}, \quad b_{23} \rightarrow \varepsilon b_{23}, \quad a_{21} \rightarrow \varepsilon a_{21} \\ a_{24} \rightarrow \varepsilon a_{24}, \quad f_2 \rightarrow \varepsilon f_2, \quad a_{22} \rightarrow \varepsilon a_{22}, \quad a_{23} \rightarrow \varepsilon a_{23} \\ f_{11} \rightarrow \varepsilon f_{11}, \quad f_{12} \rightarrow \varepsilon f_{12}\end{aligned}\quad (6)$$

把变换代入上述得到含有小参数  $\varepsilon$  的运动方

程:

$$\begin{aligned}\ddot{y}_1 + \varepsilon \mu_1 \dot{y}_1 + (\omega_1^2 + \varepsilon f_2 \cos \Omega_2 t) y_1 - \varepsilon a_{11} y_1 y_2^2 - \\ \varepsilon a_{13} y_1^2 y_2 - \varepsilon a_{14} y_1^3 = \varepsilon f_{11} \cos \Omega_1 t\end{aligned}\quad (7a)$$

$$\begin{aligned}\ddot{y}_2 + \varepsilon b_{21} \ddot{y}_1 + \varepsilon \mu_2 \dot{y}_2 + \varepsilon b_{22} \dot{y}_1 + \omega_2^2 y_2 - \varepsilon b_{23} y_1 - \\ \varepsilon a_{21} y_2^3 - \varepsilon a_{22} y_1 y_2^2 - \varepsilon a_{23} y_1^2 y_2 - \\ \varepsilon a_{24} y_1^3 = \varepsilon f_{12} \cos \Omega_1 t\end{aligned}\quad (7b)$$

其中  $\dot{y}_i (i=1, 2)$  上面的点代表对时间  $t$  的微分,  $a_{11}, a_{13}, a_{14}, a_{21}, a_{22}, a_{23}, a_{24}, b_{21}, b_{22}, b_{23}$  是已知参数,  $\mu_1$  和  $\mu_2$  为阻尼系数,  $f_{11}$  和  $f_{12}$  为外激励振幅,  $\Omega_1$  为外激励频率,  $f_2$  为激励振幅,  $\Omega_2$  为激励频率,  $\omega_1$  和  $\omega_2$  分别是梁和弦的横向位移.

首先,令  $y_1 = y_{10} + \varepsilon y_{11}, y_2 = y_{20} + \varepsilon y_{21}$  并且引入导算子得:

$$\begin{aligned}(D_0^2 + 2\varepsilon D_0 D_1)(y_{10} + \varepsilon y_{11}) + \varepsilon \mu_1 (D_0 + \varepsilon D_1) \times \\ (y_{10} + \varepsilon y_{11}) + (\Omega_1^2 + \varepsilon \sigma_1 + \varepsilon f_2 \cos 2\Omega_1 t) - \\ \varepsilon a_{13} (y_{10} + \varepsilon y_{11})^2 (y_{20} + \varepsilon y_{21}) - \\ \varepsilon a_{14} (y_{10} + \varepsilon y_{11})^3 = \varepsilon f_{11} \cos \Omega_1 t\end{aligned}\quad (8a)$$

$$\begin{aligned}(D_0^2 + 2\varepsilon D_0 D_1)(y_{20} + \varepsilon y_{21}) + \varepsilon b_{21} (D_0^2 + 2\varepsilon D_0 D_1) \times \\ (y_{10} + \varepsilon y_{11}) + \varepsilon \mu_2 (D_0 + \varepsilon D_1) (y_{20} + \varepsilon y_{21}) + \\ \varepsilon a_{22} (1/9\Omega_1^2 + \varepsilon \sigma_2) (y_{10} + \varepsilon y_{11}) (y_{20} + \varepsilon y_{21})^2 - \\ \varepsilon b_{22} (D_0 + \varepsilon D_1) (y_{10} + \varepsilon y_{11}) - \varepsilon a_{24} (y_{10} + \varepsilon y_{11})^3 + \\ \varepsilon a_{23} (y_{10} + \varepsilon y_{11})^2 (y_{20} + \varepsilon y_{21}) = \varepsilon f_{12} \cos \Omega_1 t\end{aligned}\quad (8b)$$

比较  $\varepsilon$  同次幂有:

$\varepsilon^0$  阶:

$$D_0^2 y_{10} + \omega_1^2 y_{10} = 0 \quad (9a)$$

$$D_0^2 y_{20} + \omega_2^2 y_{20} = 0 \quad (9b)$$

$\varepsilon^1$  阶:

$$\begin{aligned}D_0^2 y_{11} + \omega_1^2 y_{11} = -2D_0 D_1 y_{10} - f_2 y_{10} \cos \Omega_2 t + \\ a_{11} y_{10} y_{20}^2 - \mu_1 D_0 y_{10} + a_{13} y_{10}^2 y_{20} + \\ a_{14} y_{10}^3 + f_{11} \cos \Omega_1 t\end{aligned}\quad (10a)$$

$$\begin{aligned}D_0^2 y_{21} + \omega_2^2 y_{21} = -2D_0 D_1 y_{20} - b_{21} D_0^2 y_{10} - \\ \mu_2 D_0 y_{20} - b_{22} D_0 y_{10} + a_{23} y_{10}^2 y_{20} + a_{22} y_{10} y_{20}^2 + \\ b_{23} y_{10} + a_{21} y_{20}^3 + a_{24} y_{10}^3 + f_{12} \cos \Omega_1 t\end{aligned}\quad (10b)$$

由(9a), (9b)得:

$$y_{10} = A_1(T_1) e^{\omega_1 i T_0} + \bar{A}_1(T_1) e^{-\omega_1 i T_0} \quad (11a)$$

$$y_{20} = A_2(T_1) e^{\omega_2 i T_0} + \bar{A}_2(T_1) e^{-\omega_2 i T_0} \quad (11b)$$

代入(10a), (10b)整理得:

$$\begin{aligned}D_0^2 y_{11} + \omega_1^2 y_{11} = \frac{1}{2} f_{11} e^{i T_0 \Omega_1} + 2a_{13} A_1 A_2 \bar{A}_1 e^{i T_0 \omega_2} + \\ (-i\mu_1 \omega_1 A_1 + 3a_{14} A_1^2 \bar{A}_1 + 2a_{11} A_1 A_2 \bar{A}_2 -\end{aligned}$$

$$2i\omega_1 A_1' \times e^{iT_0\omega_1} + a_{13} A_1^2 A_2 e^{iT_0(\omega_2+2\omega_1)} + \\ a_{13} A_1^2 \bar{A}_2 e^{iT_0(2\omega_1-\omega_2)} - a_{11} A_1 \bar{A}_2^2 e^{iT_0(\omega_1-2\omega_2)} + \\ cc + NST \quad (12a)$$

$$D_0^2 y_{21} + \omega_2^2 y_{21} = \frac{1}{2} f_{12} e^{iT_0\Omega_1} - ib_{22} \omega_1 A_1 e^{i\omega_1 T_0} + \\ 3a_{24} A_1^2 \bar{A}_1 e^{i\omega_1 T_0} + a_{22} A_1 A_2^2 e^{iT_0(\omega_1+2\omega_2)} + \\ b_{23} A_1 e^{i\omega_1 T_0} + b_{21} \omega_1^2 A_1 e^{i\omega_1 T_0} + a_{24} A_1^3 e^{3i\omega_1 T_0} + \\ a_{22} \bar{A}_1 A_2^2 e^{iT_0(2\omega_2-\omega_1)} + 2a_{22} A_1 A_2 \bar{A}_2 e^{iT_0\omega_1} + \\ a_{23} A_2 \bar{A}_1^2 e^{iT_0(\omega_2-2\omega_1)} + cc + NST \quad (12b)$$

由于系统的非线性运动控制方程中即含有平方非线性项也有立方非线性项,引入小参数  $\varepsilon$ ,考虑梁和弦之间产生 1:1 主共振,梁在主参数 1:1 主共振情况.

$$\omega_1^2 = \omega^2 + \varepsilon \sigma_1, \omega_2^2 = \omega^2 + \varepsilon \sigma_2, \Omega_1 = \Omega_2 = \omega \quad (13)$$

这里的  $\omega_1$  和  $\omega_2$  是相应线性系统的第一阶和第二阶的固有频率,这里的  $\sigma_1$  和  $\sigma_2$  是调谐参数,为了简化分析,设  $\Omega_1 = \Omega_2 = 1$ .

由多尺度方法的应用<sup>[16]</sup>经过计算得到梁和弦耦合非线性系统直角坐标形式下的平均方程:

$$\dot{x}_1 = \frac{3a_{14}x_2}{4} (x_1^2 + x_2^2) + (x_1^2 x_4 + \sqrt{a^2 + b^2} + \\ 2x_1 x_2 x_3 + 3x_2^2 x_4) \frac{a_{13}}{4} + \frac{a_{11}x_4}{2} (x_1 x_3 + x_2 x_4) - \\ \frac{\mu_1 x_1}{2} - \sigma_1 x_2 \quad (14a)$$

$$\dot{x}_2 = -\frac{a_{13}}{4} (x_2^2 x_3 + 3x_1^2 x_3 + 2x_1 x_2 x_4) - \\ \frac{a_{11}x_3}{2} (x_1 x_3 + x_2 x_4) - \frac{3a_{14}x_1}{4} (x_1^2 + x_2^2) - \\ \frac{1}{2} \mu_1 x_2 + \sigma_1 x_1 - f_{11} \quad (14b)$$

$$\dot{x}_3 = \frac{3a_{21}x_4}{4} (x_3^2 + x_4^2) + \frac{a_{22}}{4} (x_2 x_3^2 + 3x_2 x_4^2 + \\ 2x_1 x_3 x_4) + \frac{a_{23}}{4} (x_1^2 x_4 + x_2^2 x_4 + 2x_1 x_2 x_3) + \\ \frac{3a_{24}x_2}{4} (x_1^2 + x_2^2) - \frac{b_{22}x_1}{2} + \frac{b_{21}x_2}{4} - \\ \frac{1}{2} \mu_2 x_3 - \sigma_2 x_4 \quad (14c)$$

$$\dot{x}_4 = -\frac{3a_{21}x_3}{4} (x_3^2 + x_4^2) - \frac{a_{23}}{4} (x_1^2 x_3 + x_2^2 x_3 +$$

$$2x_1 x_2 x_4) - \frac{a_{22}}{4} (x_1 x_3^2 + 3x_1 x_4^2 + 2x_2 x_3 x_4) - \\ \frac{3a_{24}x_1}{4} (x_1^2 + x_2^2) - \frac{b_{22}x_2}{2} - \frac{b_{21}x_1}{4} - \\ \frac{1}{2} \mu_2 x_4 + \sigma_2 x_3 - f_{12} \quad (14d)$$

### 3 稳定性和分叉分析

根据运动稳定性理论,一个非线性系统稳定性取决于它的线性化系统,如果要分析非线性运动方程的稳定性,首先需要求出方程的平衡点.由方程(14)可知  $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$  为方程的平凡解,对应的 Jacobian 矩阵为:

$$J = \begin{bmatrix} -\frac{1}{2} \mu_1 & -\sigma_1 & 0 & 0 \\ \sigma_1 & -\frac{1}{2} \mu_1 & 0 & 0 \\ -\frac{1}{2} b_{22} & \frac{1}{4} b_{21} & -\frac{1}{2} \mu_2 & -\sigma_2 \\ -\frac{1}{4} b_{21} & -\frac{1}{2} b_{22} & \sigma_2 & -\frac{1}{2} \mu_2 \end{bmatrix} \quad (15)$$

Jacobian 矩阵(15)对应的特征多项式为:

$$f(\lambda) = a_0 \lambda^4 + a_1 \lambda^3 + a_2 \lambda^2 + a_3 \lambda + a_4 \quad (16)$$

其中:

$$a_0 = 1 \quad (17a)$$

$$a_1 = \mu_1 + \mu_2 \quad (17b)$$

$$a_2 = \frac{1}{4} \mu_1^2 + \frac{1}{4} \mu_2^2 + \sigma_1 + \mu_1 \mu_2 \quad (17c)$$

$$a_3 = \frac{1}{4} \mu_1^2 \mu_2 + \frac{1}{4} \mu_1 \mu_2^2 + \mu_2 \sigma_1 \quad (17d)$$

$$a_4 = \frac{1}{16} \mu_1^2 \mu_2^2 + \frac{1}{4} \mu_2^2 \sigma_1 \quad (17e)$$

应用劳斯-霍尔维茨判据可知,如下条件均满足时:

$$a_1 > 0, a_1 a_2 - a_3 > 0 \quad (18)$$

$$a_1 a_2 a_3 - a_3^2 - a_1^2 a_4 > 0, a_4 > 0$$

系统的所有特征值的实部均为负数,平衡点  $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$  是稳定的.如果(17)中有一个系数出现负值,那么系统有特征值实部出现正值,此时系统处在不稳定状态,将会导致分叉现象.因此,接下来我们会讨论系统的稳定性和分叉行为.

为分析阻尼参数对系统稳定性的影响,取阻

尼参数为分叉参数,通过计算假设参数值  $\mu_1=\mu_2=0, \sigma_1=1$ .此时系统方程有双零和一对纯虚的特征值,则方程(15)的特征多项式和特征值为:

$$f(\lambda)=\lambda^4+\lambda^2, \lambda_{1,2}=0, \lambda_{3,4}=\pm i \quad (19)$$

选取  $\mu_1$  和  $\mu_2$  作为摄动参数,引入摄动变换  $\mu_1=\varepsilon_1$  和  $\mu_2=\varepsilon_2$ ,则 Jacobian 矩阵(15)的特征多项式可以写成如下形式:

$$f(\lambda)=c_0\lambda^4+c_1\lambda^3+c_2\lambda^2+c_3\lambda+c_4 \quad (20)$$

其中:

$$c_1=\varepsilon_1+\varepsilon_2 \quad (21a)$$

$$c_2=\frac{1}{4}\left[(\varepsilon_1+\varepsilon_2)^2+2\varepsilon_1\varepsilon_2+4\right] \quad (21b)$$

$$c_3=\frac{\varepsilon_2}{4}\left[\varepsilon_1^2+\varepsilon_1\varepsilon_2+4\right] \quad (21c)$$

$$c_4=\frac{1}{16}\varepsilon_2^2(\varepsilon_1^2+4) \quad (21d)$$

由劳斯-霍尔维茨判据中平衡点是稳定条件可以将临界曲线定义为如下:

$$L_1: \varepsilon_1+\varepsilon_2=0 \quad (22a)$$

$$L_2: \frac{1}{4}(4\varepsilon_1+\varepsilon_1^3+4\varepsilon_1^2\varepsilon_2+7\varepsilon_1\varepsilon_2^2+\varepsilon_2^3)=0 \quad (22b)$$

$$L_3: \frac{1}{16}\varepsilon_1\varepsilon_2(16+8\varepsilon_1^2+\varepsilon_1^4+4\varepsilon_1\varepsilon_2+\varepsilon_1^3\varepsilon_2+20\varepsilon_2^2-3\varepsilon_1^2\varepsilon_2^2-14\varepsilon_1\varepsilon_2^3-2\varepsilon_2^4)=0 \quad (22c)$$

通过上式可以画平衡点的分叉曲线,如图 2. 临界曲线  $L_1, L_2, L_3$  将平面  $(\varepsilon_1, \varepsilon_2)$  分成稳定区域和不稳定区域,其中 I 为稳定区域, II 为不稳定区域.

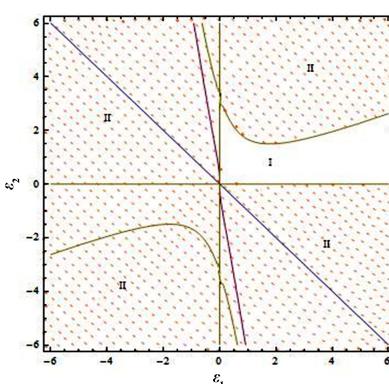


图 2 双零和一对纯虚特征值的稳定以及不稳定区域

Fig.2 Stable region and unstable region for the case of a double zero and a pair of purely imaginary eigenvalues

为了验证理论分析,在不同区域选取不同的参数值.首先,选取参数  $(\varepsilon_1, \varepsilon_2)=(0.5, 0.1)$  在稳定区域 I 中  $(z_1, z_2, z_3, z_4)=(0.004, 0.01, 0.003, 0.001)$ , 我们得到图 3. 在图 3 中,相空间轨线从初始点逐渐盘旋趋于平衡点达到稳定状态.选取  $(\varepsilon_1, \varepsilon_2)=(-0.5, 0.2)$  在不稳定区域 II 中和  $(z_1, z_2, z_3, z_4)=(0.0005, 0.001, -0.01, 0.001)$ , 得到图 4.相轨迹为不断向外扩展的螺线.

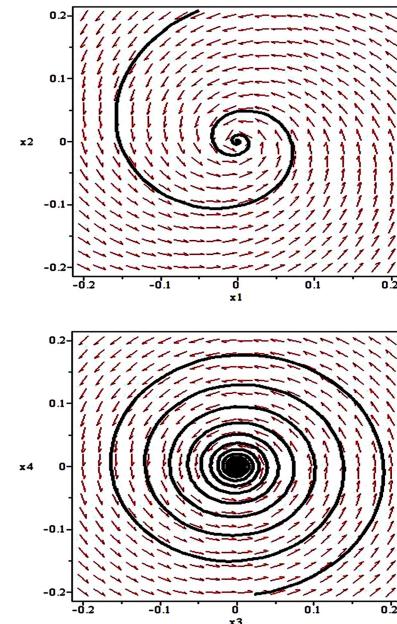


图 3 稳定区域内的轨迹

Fig.3 Trajectory of the stable region

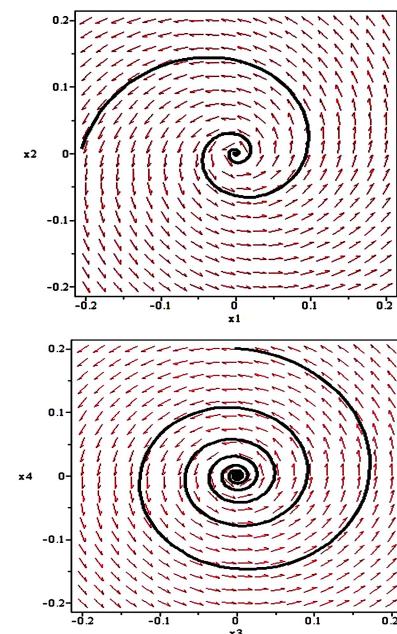


图 4 不稳定区域内的轨迹

Fig.4 Trajectory of the unstable region

## 4 数值模拟

为了对平均方程进一步研究利用四阶龙格-库塔算法<sup>[17]</sup>发现了弦-梁耦合系统混沌运动的存在. 设方程(14)中的基本参数为  $a_{11} = 20.1, a_{13} = 40, a_{14} = 56.2, a_{21} = 101.425, a_{21} = 10, a_{23} = -13.825, a_{24} = -3.8, b_{21} = 15, b_{22} = 27, \mu_1 = 0.88, \mu_2 = 0.0016, f_{12} = 42$  对外激励  $f_{11}$  赋予不同的数值, 由 matlab 作图得到图 5~8 的不同的运动情形. 其中, (a), (b) 分别为  $x_1, x_2$  和  $x_3, x_4$  相图, (c), (d) 分别为  $x_1$  和  $x_3$  的时间历程图, (e) 为  $x_1, x_2, x_3$  的三维相图, (f) 为庞加莱截面图.

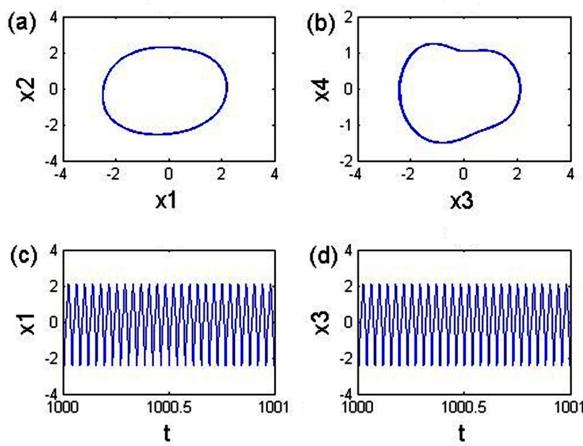


图 5 单倍周期解( $f_{11} = 50.8999$ )

Fig.5 Period-1 solution when  $f_{11} = 50.8999$

当外激励  $f_{11} = 50.8999$  时, 如图 5(f) 所示系统庞加莱截面只有一个孤立的点, 可判断此时弦-梁耦合系统产生单倍周期运动. 当外激励改变为  $f_{11} = 180$  时, 如图 6 此时弦-梁耦合系统产生两倍周期运动. 当外激励  $f_{11} = 58.0582$  时, 如图 7 系统产生四倍周期运动. 当改变外激励为  $f_{11} = 5.44$  时, 图 8(f) 给出的庞加莱截面表明弦-梁耦合系统产生混沌运动. 同样的, 从 5(e)、6(e)、7(e) 和 8(e) 可看出系统的运动情形, 因为轨道的跳跃发生在三维相图中.

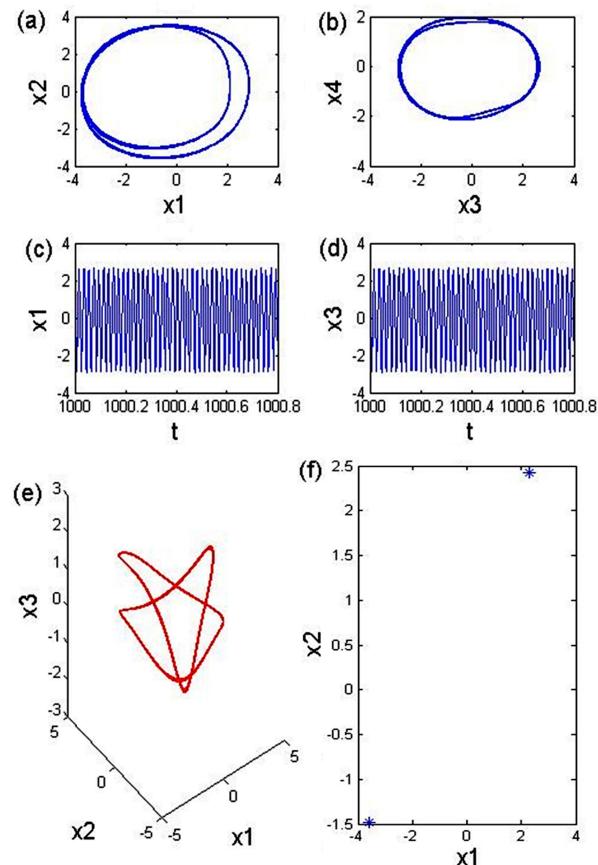


图 6 二倍周期解( $f_{11} = 180$ )

Fig.6 Period-2 solution when  $f_{11} = 180$

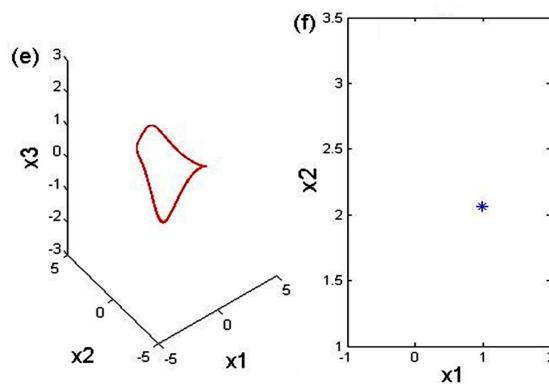
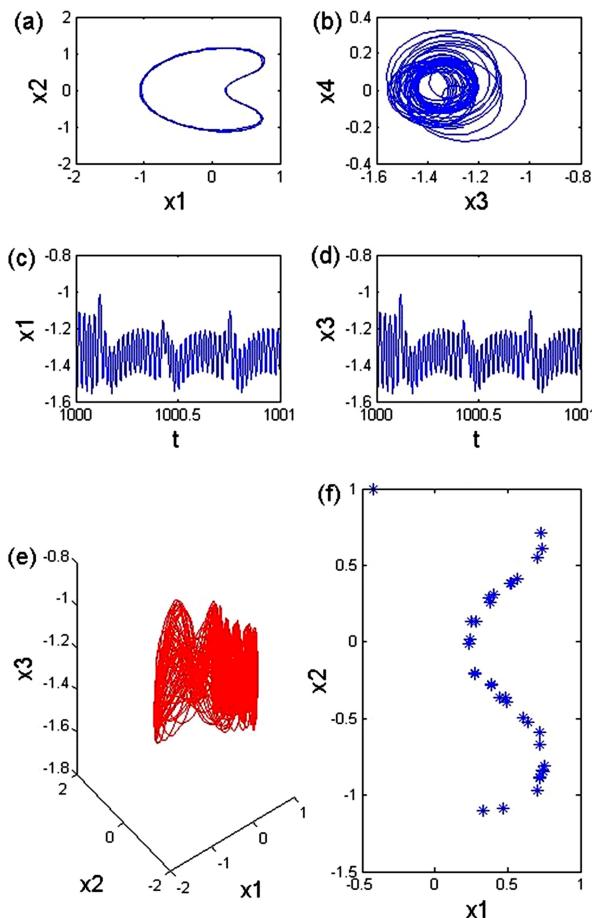


图 7 四倍周期解( $f_{11} = 58.0582$ )

Fig.7 Period-4 solution when  $f_{11} = 58.0582$

图 8 混沌运动( $f_{11} = 5.44$ )Fig.8 Chaotic motion when  $f_{11} = 5.44$ 

## 5 结论

文献[7]中用 Galerkin 方法,得到了弦-梁耦合系统的运动控制方程,进而简化为具有两自由度的常微分方程.本文运用解析和数值方法研究了一种基于参数和外部激励的弦-梁耦合系统的稳定性与分叉和混沌动力学.利用多尺度法,求得具有参数和外部激励系统的平均方程.本研究的重点是在梁和弦的模式之间存在 1:1 内部共振的情况下,这是弦的主参数共振和梁的主共振.在平均方程的基础上,对系统的稳定性进行分析,发现参数取某些值时有系统一个双零和一对纯虚特征值.分叉分析结果表明,该弦-梁耦合系统具有稳定的平衡点.为了进一步说明理论预测,利用四阶龙格-库塔算法进行数值模拟.绘制平面相图时间历程图、三维相图和庞加莱图.数值结果表明,该系统的混沌响应有不同的形状.研究还发现:外激励参数  $f_{11}$  对弦-梁耦合系统的混沌运动有重要影响.

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## NUMERICAL SIMULATION RESEARCH ON DYNAMICS OF A STRING-BEAM COUPLED NONLINEAR VIBRATION SYSTEM<sup>\*</sup>

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**Abstract** This paper studies nonlinear dynamic behavior of a string-beam coupled system subjected to parametric and external excitations. Firstly, the method of multiple scales is used to analyze the nonlinear responses of the string-beam system coupled system. Secondly, based on the average equation and taking the damping coefficient of the system as the bifurcation parameter, the stability of the equilibrium point of the system is analyzed and the bifurcation curve of the equilibrium point is obtained. In order to verify the correctness of theoretical prediction, the trajectories in phase space under different bifurcation parameters are simulated. Finally, the fourth-order Runge-Kutta method is utilized to verify the existence of the chaotic motions in the string-beam coupled system. From the results of numerical simulation, it is clearly found that the system exists period-1 motion, multi-periodic motion and chaotic motion.

**Key words** coupled nonlinear system, stability, multiple-scale method, periodic motion

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