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基于余量谐波平衡的两质点动力学系统振动频率与响应分析*

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摘要　为全面了解和准确预测两质点动力学系统运动特性. 本文以具有固定边界的两质点动力学系统为例,构建了用于研究双自由度质点运动系统的余量谐波平衡解程序.解程序融合了谐波平衡与同伦方法优势,其高阶近似仅依赖于初始谐波近似,不需要根据前一阶近似进行调整.研究结果表明:本文给出的 2-阶近似频率比已有的方法结果更加精确,相对误差不同程度减小,相应的近似响应与数值解更加吻合.因此, 余量谐波平衡方法可广泛应用于其它质点动力学问题研究中.

关键词　双自由度振动系统,　余量谐波平衡,　高阶近似,　频率响应

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引言

非线性动力学与振动分析对机械、结构等动力学问题研究是非常重要的,它能够全面了解和准确预测系统运动特性.近年来, 双自由度非线性振动系统的振动频率与周期响应已被广泛研究[1-5].通常,寻求此类复杂方程的精确解析解是比较困难的.因此,多自由度振动系统的频率、响应分析主要集中在近似分析方法方面,诸如:摄动法[3,4]、MLP 法[5]、同伦摄动[6]、同伦分析法[7,8]等.

谐波平衡法是不受小参数约束应用最广泛的定量分析方法.诸如:钱长照[9]运用谐波平衡法研究了含有单向离合器、两滑轮及附件的轮-带驱动系统稳定稳态周期响应. 杨荣刚等[10]基于谐波平衡法研究了摆线钢球行星传动系统的基频稳态响应及动态特性. 谐波平衡法对于一阶近似解求解很方便, 但精度不高. 因此, 许多研究者将谐波平衡法进行了一些推广, 发展了一些诸如增量谐波平衡[11]、摄动-增量[12]、谐波-能量平衡[13]、线性谐波平衡[14]、牛顿谐波平衡[15,16]、余量谐波平衡[17]、多层余量谐波平衡[18]等方法. 同伦[19]是拓扑理论的一个基本概念,用于描述两个对象间的连续变化.因此,借鉴同伦思想,通过引入影射参数,可将原始非线性问题扩展为一簇易于求解的问题.余量谐波平衡[17,18]通过引入阶层参数, 融合同伦思想到谐波平衡方法中, 继而余量延拓,易获得高阶近似解.

本文针对两类双自由度质点动力学运动系统构建了其振动频率、稳态响应求解的余量谐波平衡解程序,得到系统的高阶余量谐波平衡近似解频率及响应表达式,并与已有结果进行了比较分析.

1　动力学系统模型

1.1　两质点动力学系统

考虑如图 1 所示, 运动在光滑平面上的两质点系统:

假定非线性恢复力函数为:

$k_1(x_2-x_1)+k_2(x_2-x_1)^3$

振动系统的动能与势能分别为:

$$T=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2,$$

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$$V=\frac{1}{2}k_1(x_1-x_2)^2+\frac{1}{4}k_2(x_1-x_2)^4 \tag{1}$$

其中，k_1 和 k_2 分别为线性、非线性弹簧刚度系数，x_1 和 x_2 分别为两等质量质点的位移函数.

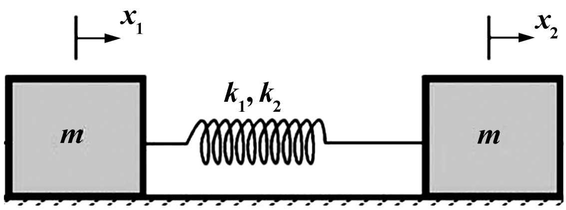

图 1　线性、非线性刚性连接的两质点动力学模型

Fig.1　Two-mass dynamic model connected by linear and nonlinear stiffnesses

继而运用欧拉-拉格朗日方程，可获得运动在光滑平面上的两质点运动方程为[2,4]：

$$\begin{aligned} &m\ddot{x}_1+k_1(x_1-x_2)+k_2(x_1-x_2)^3=0,\\ &\quad x_1(0)=x_{10},\dot{x}_1(0)=0\\ &m\ddot{x}_2+k_1(x_2-x_1)+k_2(x_2-x_1)^3=0,\\ &\quad x_2(0)=x_{20},\dot{x}_2(0)=0 \end{aligned} \tag{2}$$

1.2　具有固定边界的两质点动力学系统

考虑如图 2 所示，运动在光滑平面上且与线性和非线性弹簧连接的两质点系统：

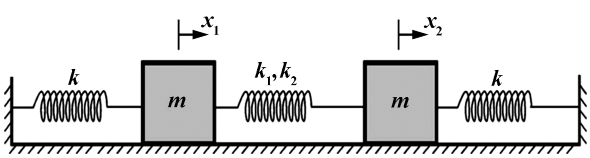

图 2　具有固定边界的两质点动力学模型

Fig.2　Two-mass dynamic system connected with the fixed boundaries

类似于模型系统(2)，此振动系统的动能和势能分别为：

$$\begin{aligned} &T=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2,\\ &V=\frac{1}{2}k(x_1^2+x_2^2)+\frac{1}{2}k_1(x_1-x_2)^2+\frac{1}{4}k_2(x_1-x_2)^4 \end{aligned} \tag{3}$$

其中，k 为线性弹簧刚度系数.

同样，运用欧拉-拉格朗日方程，可获得具有固定边界的两质点动力学系统为[2,4]：

$$\begin{aligned} &m\ddot{x}_1+kx_1+k_1(x_1-x_2)+k_2(x_1-x_2)^3=0,\\ &\quad x_1(0)=x_{10},\dot{x}_1(0)=0\\ &m\ddot{x}_2+kx_2+k_1(x_2-x_1)+k_2(x_2-x_1)^3=0,\\ &\quad x_2(0)=x_{20},\dot{x}_2(0)=0 \end{aligned} \tag{4}$$

2　基于余量谐波平衡的近似求解

假定 ω 是模型系统(4)的未知振动频率，引入变量 $\tau=\omega t$，得：

$$\begin{aligned} &m\omega^2x''_1+kx_1+k_1(x_1-x_2)+k_2(x_1-x_2)^3=0,\\ &\quad x_1(0)=x_{10},x'_1(0)=0 \end{aligned} \tag{5}$$

$$\begin{aligned} &m\omega^2x''_2+kx_2+k_1(x_2-x_1)+k_2(x_2-x_1)^3=0,\\ &\quad x_2(0)=x_{20},x'_2(0)=0 \end{aligned} \tag{6}$$

根据文献[3]，引入如下变量：

$$u=x_1,v=x_2-x_1 \tag{7}$$

方程(5)，(6)变为：

$$m\omega^2u''+ku-k_1v-k_2v^3=0 \tag{8}$$

$$m\omega^2(u''+v'')+k(u+v)+k_1v+k_2v^3=0 \tag{9}$$

且：

$$u(0)=x_{10},u'(0)=0,v(0)=x_{20}-x_{10},v'(0)=0 \tag{10}$$

将上述三式整理可得：

$$m\omega^2v''+(k+2k_1)v+2k_2v^3=0 \tag{11}$$

$$v(0)=x_{20}-x_{10}\triangleq A,v'(0)=0 \tag{12}$$

基于方程(11)的对称性，其周期解具有如下基本解级数形式：

$$\{\cos[(2k+1)\tau\,|\,k=0,1,2,\cdots\} \tag{13}$$

为方便计算，引入阶层参数 p，并将系统稳态下解响应及振动频率设为：

$$\begin{aligned} &v(\tau)=v_0(\tau)+pv_1(\tau)+p^2v_2(\tau)+\cdots\\ &\omega^2=\omega_0^2+p\omega_1+p^2\omega_2+\cdots \end{aligned} \tag{14}$$

其中 $\omega_i(i=0,1,2,\cdots)$ 为未知频率.

2.1　初始谐波近似

根据方程(11)与初始条件(12)，初始谐波解设为如下形式：

$$v_0(\tau)=A\cos(\tau),\tau=\omega_0t \tag{15}$$

将(15)代入(11)式，得初始余量项：

$$\begin{aligned} R_0(\tau)&=m\omega_0^2v''_0+(k+2k_1)v_0+2k_2v_0^3\\ &=(-m\omega_0^2A+kA+2k_1A+\frac{3}{2}k_2A^3)\cos(\tau)+\\ &\quad \frac{1}{2}k_2A^3\cos(3\tau) \end{aligned} \tag{16}$$

根据伽辽金法，消除久期项，易获得初始谐波近似频率及周期响应为：

$$\begin{aligned} &\omega_0=\frac{1}{\sqrt{2m}}\sqrt{2k+4k_1+3k_2A^2},\\ &v_0(\tau)=A\cos(\omega_0t) \end{aligned} \tag{17}$$

上述近似公式与振幅-频率公式[4]结果一致.

2.2 1-阶余量谐波近似

将初始近似(17)代入余量项(16)时,$\cos(3\tau)$系数非零.因此,我们将(14)代入(11)合并阶层参数p的一次系数,得:

$$\Gamma_1(\tau)=m\omega_0^2v''_1+m\omega_1v''_0+(k+2k_2)v_1+6k_3v_0^2v_1 \tag{18}$$

式(18)为关于未知ω_1和$v_1(\tau)$的线性方程,根据周期解级数形式(13)及初始条件,假定:

$$v_1(\tau)=a_{11}[\cos(\tau)-\cos(3\tau)] \tag{19}$$

将(19)代入(18)式,并消除初始余量项,得:

$$\begin{aligned}R_1(\tau)&=\Gamma_1(\tau)+R_0(\tau)\\&=[a_{11}(k-3k_2A^2-m\omega_0^2+2k_1)-m\omega_1A]\cdot\\&\quad\cos(\tau)+[a_{11}(-k-2k_1-\frac{3}{2}k_2A^2+9m\omega_0^2)+\\&\quad\frac{1}{2}k_2A^3]\cos(3\tau)-\frac{3}{2}k_2a_{11}A^2\cos(5\tau)\end{aligned} \tag{20}$$

将初始余量项引入(20)式,提高解的精确性.

根据伽辽金法,消除久期项,解$\cos(\tau)$和$\cos(3\tau)$的系数方程得1-阶余量谐波近似为:

$$\begin{aligned}&\omega_{(1)}=\sqrt{\omega_0^2+\omega_1},\\&v_{(1)}(\tau)=v_{(0)}(\tau)+v_1(\tau)\\&\qquad=(A+a_{11})\cos(\tau)-a_{11}\cos(3\tau),\\&\tau=\omega_{(1)}t\end{aligned} \tag{21}$$

2.3 2-阶余量谐波近似

将1-阶余量谐波近似(21)代入余量项(20)时,$\cos(5\tau)$系数非零. 因此,我们将(14)代入(11)合并阶层参数p的2次系数,得:

$$\begin{aligned}\Gamma_2(\tau)=&m[\omega_0^2v''_2+\omega_1v''_1+\omega_2v''_0]+\\&(k+2k_1)v_2+6k_2[v_0^2v_2+v_0v_1^2]\end{aligned} \tag{22}$$

式子(22)为关于未知ω_2和$v_2(\tau)$的线性方程,根据周期解级数形式(3)及初始条件,假定:

$$\begin{aligned}v_2(\tau)=&a_{21}[\cos(\tau)-\cos(3\tau)]+\\&a_{22}[\cos(\tau)-\cos(5\tau)]\end{aligned} \tag{23}$$

将(23)代入(22)式,并消除1-阶余量项,得:

$$\begin{aligned}R_2(\tau)&=\Gamma_2(\tau)+R_1(\tau)\\&=[a_{21}(k+2k_1+3k_2A^2-m\omega_0^2)+a_{22}(k+2k_1+\\&\quad\frac{9}{2}k_2A^2-m\omega_0^2)+\frac{9}{2}k_2Aa_{11}^2-m\omega_1a_{11}-\\&\quad m\omega_2A]\cos(\tau)+[a_{21}(-k-2k_1+9m\omega_0^2-\\&\quad\frac{3}{2}k_2A^2)+9m\omega_1a_{11}-\frac{9}{2}k_2Aa_{11}^2]\cos(3\tau)+\\&\quad[a_{22}(-k-2k_1+25m\omega_0^2-3k_2A^2)-\\&\quad\frac{3}{2}k_2A(Aa_{21}+a_{11}^2+Aa_{11})]\cos(5\tau)+\\&\quad\frac{3}{2}k_2A(a_{11}^2-Aa_{22})\cos(7\tau)\end{aligned} \tag{24}$$

根据伽辽金法,消除久期项,我们解$\cos(\tau)$,$\cos(3\tau)$和$\cos(5\tau)$的系数方程得2-阶余量谐波近似为:

$$\begin{aligned}&\omega_{(2)}=\sqrt{\omega_0^2+\omega_1+\omega_2},\\&v_{(2)}(\tau)=v_{(0)}(\tau)+v_1(\tau)+v_2(\tau)\\&\qquad=(A+a_{11}+a_{21}+a_{22})\cos(\tau)-\\&\qquad\quad(a_{11}+a_{21})\cos(3\tau)-a_{22}\cos(5\tau),\\&\tau=\omega_{(1)}t\end{aligned} \tag{25}$$

2.4 高阶余量谐波近似

类似于上述求解过程.一般地,k-阶余量谐波近似:

$$\begin{aligned}&\omega_{(k)}=\sqrt{\omega_{(k-1)}^2+\omega_k},k=2,3,4,\cdots,\\&v_{(k)}(\tau)=v_{(k-1)}(\tau)+v_k(\tau),\\&v_{(k-1)}(\tau)=v_{(k-2)}(\tau)+v_{k-1}(\tau),\\&\omega_{(k-1)}=\sqrt{\omega_{(k-2)}^2+\omega_{k-1}},\\&v_k(\tau)=\sum_{i=1}^{k}a_{ki}[\cos(\tau)-\cos((2i+1)\tau)],\\&v_{(0)}=A\cos(\tau),\omega_{(0)}=\omega_0,k=2,3,4,\cdots.\end{aligned} \tag{26}$$

2.5 稳态下的高阶近似响应

一方面,将方程(5),(6)相加得:

$$m(\ddot{x}_1+\ddot{x}_2)+k(x_1+x_2)=0 \tag{27}$$

令$\rho(t)=x_1(t)+x_2(t)$,方程(27)变为一个二阶线性微分方程:

$$m\ddot{\rho}+k\rho=0 \tag{28}$$

初值条件为:

$$\rho(0)=x_{10}+x_{20},\ \dot{\rho}(0)=0 \tag{29}$$

其解为:

$$\rho\triangleq x_1+x_2=(x_{10}+x_{20})\cos(\sqrt{\frac{k}{m}}t) \tag{30}$$

另一方面,根据上述余量谐波平衡解程序,已获得变量$v(t)$的各阶近似响应.如:2-阶余量谐波平衡解为:

$$\begin{aligned}x_1-x_2=&(x_{10}-x_{20})\cos(\omega_{(2)}t)+a_{11}[\cos(\omega_{(2)}t)-\\&\cos(3\omega_{(2)}t)]+a_{21}[\cos(\omega_{(2)}t)-\\&\cos(3\omega_{(2)}t)]+a_{22}[\cos(\omega_{(2)}t)-\\&\cos(5\omega_{(2)}t)]\end{aligned} \tag{31}$$

由(30),(31)得模型(4)的 2-阶近似响应为:

$$x_1(t)=\frac{x_{10}+x_{20}}{2}\cos(\sqrt{\frac{k}{m}}t)+\frac{(x_{10}-x_{20})}{2}\cos(\omega_{(2)}t)+\frac{a_{11}}{2}[\cos(\omega_{(2)}t)-\cos(3\omega_{(2)}t)]+\frac{a_{21}}{2}[\cos(\omega_{(2)}t)-\cos(3\omega_{(2)}t)]+\frac{a_{22}}{2}[\cos(\omega_{(2)}t)-\cos(5\omega_{(2)}t)] \quad (32)$$

$$x_2(t)=\frac{x_{10}+x_{20}}{2}\cos(\sqrt{\frac{k}{m}}t)-\frac{(x_{10}-x_{20})}{2}\cos(\omega_{(2)}t)-\frac{a_{11}}{2}[\cos(\omega_{(2)}t)-\cos(3\omega_{(2)}t)]-\frac{a_{21}}{2}[\cos(\omega_{(2)}t)-\cos(3\omega_{(2)}t)]-\frac{a_{22}}{2}[\cos(\omega_{(2)}t)-\cos(5\omega_{(2)}t)] \quad (33)$$

同样地,模型系统(2)及(4)的高阶近似响应易由(26),(30)~(33)获得.

3　结果分析及讨论

本部分结合实例分析,图解了模型系统(2)、(4)的各阶余量谐波平衡解频率以及与已有文献的结果比较.系统(4)的精确解频率[4,15]为:

$$\omega_{Exact}(x_{10},x_{20})=\frac{\pi\sqrt{\frac{k+2k_1}{m}+\frac{k_2}{m}(x_{20}-x_{10})^2}}{2}\cdot\left(\int_0^{\frac{\pi}{2}}\frac{dt}{1-\Delta\sin^2 t}\right)^{-1} \quad (34)$$

其中 $\Delta=\frac{\frac{k_2}{m}(x_{20}-x_{10})^2}{2[\frac{k+2k_1}{m}+\frac{k_2}{m}(x_{20}-x_{10})^2]}$.

针对系统(2)、(4),表 1~2 给出了各阶近似余量谐波平衡解频率及本文 2-阶余量谐波近似结果与已有文献结果的比较. 其中相对误差定义为

$$(\text{相对误差}\%)=\frac{\omega-\omega_{Exact}}{\omega_{Exact}}\times 100\% \quad (35)$$

从表 1a 和 2a 可以看出,模型系统(2)和(4)在稳态下的近似解与精确频率的误差随着余量谐波平衡解阶数的增大而大大减小,这表明余量谐波平衡解程序的收敛性.本文给出的 2 阶近似振动频率结果与已有文献相比:本文的结果比改进的振幅-频率公式[4],初始同伦分析解[8],Pade 同伦近似[8],3-阶线性谐波平衡解[14],3-阶牛顿谐波平衡解[15]等在各类参数下结果更加精确,与精确解的

表 1a　模型系统(2)的各阶余量谐波平衡解频率

Table 1a　Various-orders residue harmonic balance solution frequencies of model system (2)

Parameters m	k_1	k_2	x_{10}	x_{20}	Initial harmonic approximation	1-order residue harmonic approximation	2-order residue harmonic approximation	Exact frequency
1	5	5	5	1	11.4018 (1.8736%)	11.1976 (0.04914%)	11.1916 (−0.004467%)	11.1921
1	1	1	10	−5	18.4255 (2.1924%)	18.0422 (0.06656%)	18.0292 (−0.005546%)	18.0302
5	10	10	20	30	17.4356 (2.1585%)	17.0782 (0.06445%)	17.0663 (−0.005273%)	17.0672
10	50	−0.01	−20	40	2.14476 (3.1383%)	2.08228 (0.13369%)	2.07979 (0.009618%)	2.07950
100	200	300	400	200	424.2687 (2.2204%)	415.3362 (0.06823%)	415.02943 (−0.005679%)	415.0530

表 1b　模型系统(2)的近似解频率结果比较

Table 1b　Comparisons of approximate frequency for model system (2)

Parameters m	k_1	k_2	x_{10}	x_{20}	ω_{IAFF} [4]	ω_{0HAM} [8]	$\omega_{[1,1]pade'}$ [8]	ω_{3LHB} [15]	ω_{3NHB} [15]	ω_{2RHB}	ω_{Exact}
1	5	5	5	1	11.4018 (1.874%)	11.4017 (1.873%)	11.1975 (0.048%)	11.1927 (0.005%)	11.1926 (0.004%)	11.1916 (−0.004%)	11.1921
1	1	1	10	−5	18.4255 (2.192%)	18.4255 (2.192%)	18.0422 (0.067%)	18.0316 (0.008%)	18.0314 (0.007%)	18.0292 (−0.0055%)	18.0302
5	10	10	20	30	17.4356 (2.159%)	17.4356 (2.159%)	17.0782 (0.064%)	17.0685 (0.008%)	17.0683 (0.007%)	17.0663 (−0.0053%)	17.0672
10	50	−0.01	−20	40	2.14480 (3.140%)	2.14048 (2.93%)	2.0822 (0.13%)	2.0799 (0.019%)	2.0798 (0.018%)	2.07979 (0.014%)	2.07950
100	200	300	400	200	424.2688 (2.220%)	424.2688 (2.220%)	415.3372 (0.068%)	415.1521 (0.008%)	415.0823 (0.007%)	415.0294 (−0.0057%)	415.0530

表 2a 模型系统(4)的各阶余量谐波平衡解频率

Table 2a Various-orders residue harmonic balance solution frequencies of model system (4)

Parameters						Initial harmonic	1-order residue harmonic	2-order residue harmonic	Exact
m	k	k_1	k_2	x_{10}	x_{20}	approximation	approximation	approximation	frequency
1	1	1	1	5	1	5.1962 (1.7307%)	5.1099 (0.04111%)	5.1076 (−0.003916%)	5.1078
10	25	20	−0.05	−10	10	1.8708 (1.6021%)	1.8420 (0.03802%)	1.8413 (0.00220%)	1.8413
5	10	20	30	−10	10	60.0833 (2.2075%)	58.8253 (0.06753%)	58.7823 (−0.005614%)	58.7856
10	50	70	90	20	−40	220.4971 (2.2186%)	215.8583 (0.06815%)	215.6991 (−0.005656%)	215.7113
100	200	300	400	−50	50	244.9653 (2.2196%)	239.8090 (0.06802%)	239.6319 (−0.005884%)	239.646

表 2b 模型系统(4)的近似解频率结果比较

Table 2b Comparisons of approximate frequency for model system (4)

Parameters						ω_{IAFF} [4]	ω_{0HAM} [8]	$\omega_{[1,1]pade'}$ [8]	ω_{3LHB} [15]	ω_{3NHB} [15]	ω_{2RHB}	ω_{Exact}
m	k	k_1	k_2	x_{10}	x_{20}							
1	1	1	1	5	1	5.1961 (1.73%)	5.1961 (1.73%)	5.1099 (0.041%)	5.1080 (0.004%)	5.1080 (0.039%)	5.1076 (−0.039%)	5.1078
10	25	20	−0.05	−10	10	1.8708 (1.602%)	1.8708 (1.602%)	1.8419 (0.033%)	1.8414 (0.005%)	1.8414 (0.0054%)	1.84134 (0.0022%)	1.8413
5	10	20	30	−10	10	60.0833 (2.21%)	60.0832 (2.21%)	58.8254 (0.068%)	58.7904 (0.008%)	58.7897 (0.0070%)	58.7823 (−0.0056%)	58.7856
10	50	70	90	20	−40	220.4972 (2.22%)	220.4972 (2.22%)	215.8588 (0.068%)	215.7290 (0.008%)	215.7265 (0.0071%)	215.6991 (−0.0057%)	215.7113
100	200	300	400	−50	50	244.9653 (2.22%)	244.9653 (2.22%)	239.8095 (0.068%)	239.6652 (0.008%)	239.6624 (0.0068%)	239.6319 (−0.0059%)	239.646

相对误差已降低.诸如,对于系统(2),在参数 $m=10$, $k_1=50$, $k_2=-0.01$, $x_{10}=-20$, $x_{20}=40$ 下,上述文献方法的解相对误差依次为 3.14%, 2.93%, 0.13%, 0.019%, 0.018%,本文的 2 阶-余量谐波平衡解频率与精确解间相对误差为:0.14%.对于系统(4),在参数 $m=10$, $k=50$, $k_1=70$, $k_2=90$, $x_{10}=20$, $x_{20}=-40$下,上述文献方法的解相对误差依次为 2.22%, 2.22%, 0.068%, 0.008%, 0.0071%,本文的 2 阶-余量谐波平衡解频率与精确解的相对误差为:−0.0057%.

为进一步图解本文获得响应的有效性,图 3~5 分别显示了系统(2)和(4)在不同参数下系统的时域振幅曲线比较.

图 3 在系统(2)参数 $m=5$, $k_1=10$, $k_2=10$, $x_{10}=20$, $x_{20}=30$ 下的近似解析响应为:

$$x_1(t)=25-4.7796\cos(\omega_{(2)}t)-0.2117\cos(3\omega_{(2)}t)-0.0087\cos(5\omega_{(2)}t),\ \omega_{(2)}=17.0663,$$

$$x_2(t)=25+4.7796\cos(\omega_{(2)}t)+0.2117\cos(3\omega_{(2)}t)+0.0087\cos(5\omega_{(2)}t),\ \omega_{(2)}=17.0663.$$

图 4 在系统(4)参数 $m=1$, $k=1$, $k_1=1$, $k_2=1$, $x_{10}=5$, $x_{20}=1$ 下的近似解析响应为:

$$x_1(t)=3\cos(t)+1.9129\cos(\omega_{(2)}t)+0.0837\cos(3\omega_{(2)}t)+0.0034\cos(5\omega_{(2)}t),\ \omega_{(2)}=10.8684,$$

$$x_2(t)=3\cos(t)-1.9129\cos(\omega_{(2)}t)-0.0837\cos(3\omega_{(2)}t)-0.0034\cos(5\omega_{(2)}t),\ \omega_{(2)}=10.8684.$$

图 5 在系统(4)参数 $m=50$, $k=1$, $k_1=1$, $k_2=1$, $x_{10}=10$, $x_{20}=5$ 下的近似解析响应为:

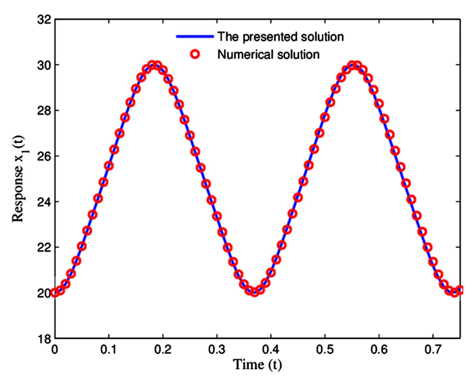

图 3a 解析近似响应 $x_1(t)$ 与数值解比较,当 $m=5$, $k_1=10$, $k_2=10$, $x_{10}=20$, $x_{20}=30$

Fig.3a Comparison of analytical solution $x_1(t)$ with the numerical one for case $m=5$, $k_1=10$, $k_2=10$, $x_{10}=20$, $x_{20}=30$

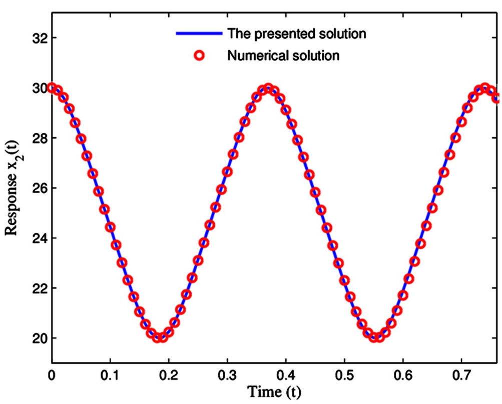

图 3b　解析近似响应 $x_2(t)$ 与数值解比较，当 $m=5,k_1=10,k_2=10,x_{10}=20,x_{20}=30.$

Fig.3b　Comparison of analytical approximate solution $x_2(t)$ with the numerical one for the case $m=5,k_1=10,k_2=10,x_{10}=20,x_{20}=30$

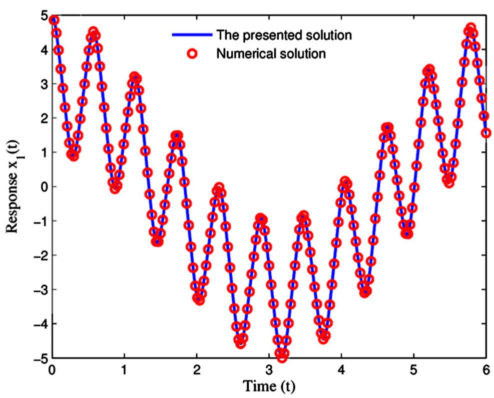

图 4a　解析近似响应 $x_1(t)$ 与数值解比较，当 $m=1,k=1,k_1=1,k_2=1,x_{10}=5,x_{20}=1$

Fig.4a　Comparison of analytical approximate solution $x_1(t)$ with the numerical one for the case $m=1,k=1,k_1=1,k_2=1,x_{10}=5,x_{20}=1$

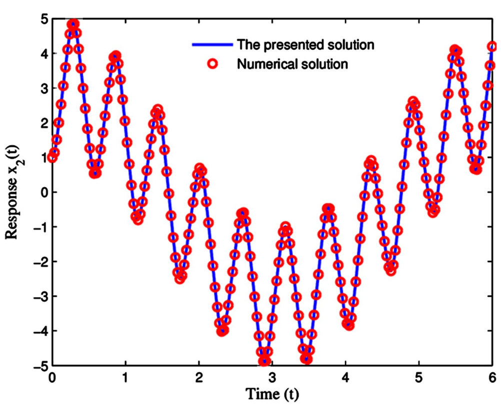

图 4b　解析近似响应 $x_2(t)$ 与数值解比较，当 $m=1,k=1,k_1=1,k_2=1,x_{10}=5,x_{20}=1$

Fig.4b　Comparison of analytical approximate solution $x_2(t)$ with the numerical one for the case $m=1,k=1,k_1=1,k_2=1,x_{10}=5,x_{20}=1$

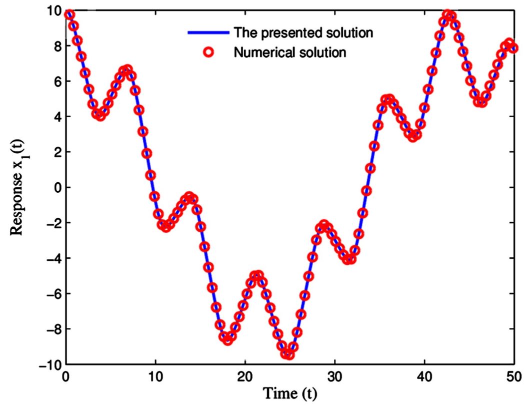

图 5a　解析近似响应 $x_1(t)$ 与数值解比较，当 $m=50,k=1,k_1=1,k_2=1,x_{10}=10,x_{20}=5$

Fig.5a　Comparison of analytical approximated solution $x_1(t)$ with the numerical one for the case $m=50,k=1,k_1=1,k_2=1,x_{10}=10,x_{20}=5$

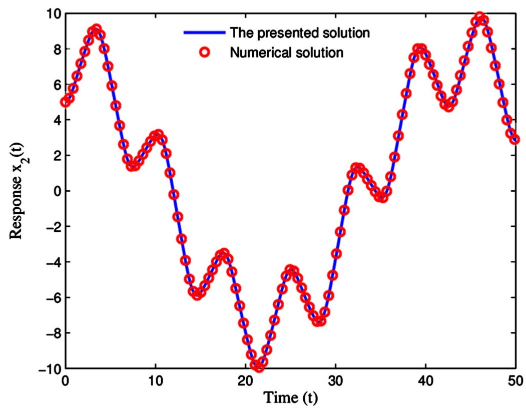

图 5b　解析近似响应 $x_2(t)$ 与数值解比较，当 $m=50,k=1,k_1=1,k_2=1,x_{10}=10,x_{20}=5$

Fig.5b　Comparison of analytical approximate solution $x_2(t)$ with the numerical one for the case $m=50,k=1,k_1=1,k_2=1,x_{10}=10,x_{20}=5$

$$
\begin{aligned}
x_1(t)= & 7.5\cos\left(\sqrt{\frac{1}{50}}t\right)+2.3971\cos(\omega_{(2)}t)+ \\
& 0.0990\cos(3\omega_{(2)}t)+0.0038\cos(5\omega_{(2)}t), \\
& \omega_{(2)}=0.883305, \\
x_2(t)= & 7.5\cos\left(\sqrt{\frac{1}{50}}t\right)-2.3971\cos(\omega_{(2)}t)- \\
& 0.0990\cos(3\omega_{(2)}t)-0.0038\cos(5\omega_{(2)}t), \\
& \omega_{(2)}=0.883305.
\end{aligned}
$$

从图 3~5 不难看出,本文给出的近似响应与数值解在系统(2)、(4)不同参数下都吻合得相当好.

4　结论

基于谐波平衡,通过融合同伦思想优势,构建

了不含小参数,适用于求解双自由度非线性两质点振动系统的高阶近似余量谐波平衡解程序.不同于线性谐波平衡与牛顿谐波平衡方法,本文解程序在每一阶近似中均消除了上一阶的谐波余量,高阶振动频率近似表达仅需初始谐波近似,不需根据前一阶近似进行调整.理论上,任何精度的高阶近似均能依次获得.相应地,系统的高阶近似响应表达易通过求解、整合获得.结果显示,本文获得2-阶余量谐波平衡近似振动频率比已有的在改进振幅-频率公式法、初始同伦分析法、3-阶线性谐波平衡法、3-阶牛顿谐波平衡法等方面更加精确,与精确解的相对误差在不同参数下均降低了.相应地,本文获得的2-阶解析近似响应与数值解吻合得相当好.

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ANALYSIS ON VIBRATION FREQUENCY AND RESPONSE OF TWO-MASS DYNAMIC SYSTEMS BASED ON RESIDUE HARMONIC BALANCE *

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Abstract To fully understand and accurately predict the motion characteristics of two mass dynamic systems, this paper constructs a solution procedure of residue harmonic balance for the study of two-degree-of-freedom motion systems, taking two-mass dynamic system connected with fixed boundaries as an example. This solution procedure combines the advantages of harmonic balance and homotopy method, and the higher-order approximations only depend on the initial harmonic approximation, and they don't need to be adjusted according to the previous approximations. The results show that the frequencies based on the presented second-order residue harmonic balance are more accurate than the results obtained by other existing methods, and the relative errors are reduced in different degrees. Moreover, the presented steady responses are in good agreement with the numerical ones. It is shown this method can be applied widely to other nonlinear oscillation problems.

Key words two-degree-of-freedom system, residue harmonic balance, higher-order approximation, frequency response

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