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分数阶 Birkhoff 系统基于 Caputo 导数的 Noether 对称性与守恒量*

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摘要　在 Caputo 分数阶导数下研究分数阶 Birkhoff 系统的 Noether 对称性与守恒量. 首先, 定义 Caputo 分数阶导数下的分数阶 Pfaff 作用量, 建立分数阶 Birkhoff 方程及其相应的横截性条件; 其次, 基于 Pfaff 作用量在无限小变换下的不变性, 分别在时间不变和时间变化的无限小变换下, 给出了不变性条件. 基于 Frederico 和 Torres 的分数阶守恒量概念, 建立了分数阶 Birkhoff 系统的 Noether 定理, 揭示了分数阶 Noether 对称性与分数阶守恒量之间的内在联系.

关键词　分数阶 Birkhoff 系统,　分数阶 Noether 对称性,　分数阶守恒量,　分数阶 Pfaff 作用量,　Caputo 分数阶导数

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引言

动力学系统对称性的研究一直是分析力学的一个重要发展方向. 1918 年 Noether[1] 研究了 Hamilton 作用量在无限小变换下的不变性质, 揭示了力学系统的守恒量与其内在的动力学对称性之间的关系. Djukić 和 Vujanović[2] 将 Noether 定理推广到完整非保守系统, 李子平[3], Bahar[4], 刘端[5] 进一步将 Noether 定理推广到非完整非保守系统. 梅凤翔[6] 通过引进 r 参数变换群的无限小群变换的广义准对称性概念, 建立了 Birkhoff 系统的 Noether 理论. 近年来, 对 Noether 对称性的研究已经取得了一系列重要成果[7-10].

分数阶微积分的概念最早出现在 L' Hospital 于 1695 年写给 Leibniz 的信中, 但是直到 1974 年第一本关于分数阶微积分理论的著作才问世[11]. 近 20 年来, 随着分数阶微积分应用领域的不断拓展, 分数阶微积分及其应用研究有了很大的发展. 1996 年, Riewe[12-13] 首次将分数阶微积分应用于非保守系统动力学建模, 提出并初步研究了分数阶变分问题. 之后, Agrawal[14-15], Baleanu[16-17], Atanacković[18-19], El-Nabulsi[20-22] 等对分数阶变分问题进行了深入研究. Frederico 和 Torres 最早开展了分数阶 Noether 对称性与守恒量的研究[23-25], 基于 Riemann-Liouville 分数阶导数定义[23], Caputo 分数阶导数定义[24], Riesz-Caputo 分数阶导数定义[25], 分别考虑时间不变和时间变化的无限小变换作用, 得到了分数阶 Noether 定理. 在此基础上, Frederico 和 Torres 进一步给出了 Hamilton 系统的分数阶 Noether 定理[26]. 此外, Frederico 和 Torres 基于 El-Nabulsi 动力学模型研究了类分数阶作用变分的不变性问题[27-28]. 近年来, 约束力学系统基于分数阶模型的 Noether 对称性与守恒量的研究已经取得了一些重要成果[29-34]. 但是, 研究主要限于分数阶 Lagrange 系统和分数阶 Hamilton 系统.

本文基于 Caputo 分数阶导数的定义, 研究分数阶 Birkhoff 系统的分数阶 Noether 对称性. 从 Pfaff 作用量在无限小变换下的不变性出发, 分别在时间不变和时间变化的无限小变换下, 研究了分数阶 Pfaff 作用量的不变性, 建立了分数阶 Birkhoff 系统的 Noether 定理.

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1　分数阶导数

本节列出研究所涉及的 Riemann-Liouville 分数阶导数和 Caputo 分数阶导数的定义及相关性质,详细的证明和讨论可参见[35-36].

Riemann-Liouville 分数阶左导数定义为

$$ {}_{t_1}D_t^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_{t_1}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau \tag{1} $$

Riemann-Liouville 分数阶右导数为

$$ {}_{t}D_{t_2}^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_{t}^{t2} \frac{f(\tau)}{(\tau-t)^{\alpha-m+1}}\mathrm{d}\tau \tag{2} $$

Caputo 分数阶左导数定义为

$$ {}_{t_1}^{C}D_t^{\alpha}f(t) = \frac{1}{\Gamma(m-\alpha)}\int_{t_1}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau \tag{3} $$

Caputo 分数阶右导数定义为

$$ {}_{t}^{C}D_{t_2}^{\alpha}f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)}\int_{t}^{t2} \frac{f^{(m)}(\tau)}{(\tau-t)^{\alpha-m+1}}\mathrm{d}\tau \tag{4} $$

其中 $\Gamma(*)$ 是 Euler-Gamma 函数, α 是导数的阶,且 $m-1\leqslant\alpha<m$, m 为正整数. 如果 α 是整数,上述分数阶导数成为整数阶导数,有

$$ {}_{t_1}D_t^{\alpha}f(t) = {}_{t_1}^{C}D_t^{\alpha}f(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha}f(t) , $$

$$ {}_{t}D_{t_2}^{\alpha}f(t) = {}_{t}^{C}D_{t_2}^{\alpha}f(t) = \left(-\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\alpha}f(t). \tag{5} $$

设 f 和 g 是区间 $[t_1,t_2]$ 上的光滑函数,则 Caputo 导数下的分数阶分部积分公式为

$$ \int_{t_1}^{t2} g(t)\,{}_{t_1}^{C}D_t^{\alpha}f(t)\,\mathrm{d}t = \int_{t_1}^{t2} f(t)\,{}_{t}D_{t_2}^{\alpha}g(t)\,\mathrm{d}t + \sum_{k=0}^{m-1} {}_{t}D_{t_2}^{\alpha+k-m}g(t)\frac{\mathrm{d}^{m-1-k}f(t)}{\mathrm{d}t^{m-1-k}}\Bigg|_{t_1}^{t_2} \tag{6} $$

和

$$ \int_{t_1}^{t2} g(t)\,{}_{t}^{C}D_{t_2}^{\alpha}f(t)\,\mathrm{d}t = \int_{t_1}^{t2} f(t)\,{}_{t_1}D_t^{\alpha}g(t)\,\mathrm{d}t + \sum_{k=0}^{m-1} (-1)^{m+k}\,{}_{t_1}D_t^{\alpha+k-m}g(t)\frac{\mathrm{d}^{m-1-k}f(t)}{\mathrm{d}t^{m-1-k}}\Bigg|_{t_1}^{t_2} \tag{7} $$

在 Caputo 导数下,定义算子 ${}^{C}D_t^{\gamma}(f,g)$ 为

$$ {}^{C}D_t^{\gamma}(f,g) = -g\,{}_{t}D_{t_2}^{\gamma}f + f\,{}_{t_1}^{C}D_t^{\gamma}g , \tag{8} $$

或

$$ {}^{C}D_t^{\gamma}(f,g) = -g\,{}_{t}^{C}D_{t_2}^{\gamma}f + f\,{}_{t_1}D_t^{\gamma}g . \tag{9} $$

当 $\gamma=1$ 时,公式(8)和(9)成为

$$ {}^{C}D_t^{1}(f,g) = -g\,{}_{t}D_{t_2}^{1}f + f\,{}_{t_1}^{C}D_t^{1}g = \dot{f}g + f\dot{g} = \frac{\mathrm{d}}{\mathrm{d}t}(fg) , \tag{10} $$

$$ {}^{C}D_t^{1}(f,g) = -g\,{}_{t}^{C}D_{t_2}^{1}f + f\,{}_{t_1}D_t^{1}g = \dot{f}g + f\dot{g} = \frac{\mathrm{d}}{\mathrm{d}t}(fg) \tag{11} $$

此时 ${}^{C}D_t^{1}(f,g) = {}^{C}D_t^{1}(g,f)$,但是一般情况下 ${}^{C}D_t^{1}(f,g) \neq {}^{C}D_t^{1}(g,f)$.

2　分数阶 Birkhofff 方程

考虑由 $2n$ 个 Birkhoff 变量 $a^{\mu}(\mu=1,2,\cdots,2n)$ 来描述的 Birkhoff 系统. 假设系统的 Birkhoff 函数 $B=B(t,a^{\nu})$,Birkhoff 函数组为 $R_{\mu}=R_{\mu}(t,a^{\nu})$,分数阶导数的阶为 α ,且 $0<\alpha<1$. 积分泛函

$$ S(a^{\mu}(\cdot)) = \int_{t_1}^{t2}\Big\{\sum_{\nu=1}^{2n} R_{\nu}^{\alpha}(t,a^{\mu})\,{}_{t_1}^{C}D_t^{\alpha}a^{\nu} + \sum_{\nu=1}^{2n} R_{\nu}^{\beta}(t,a^{\mu})\,{}_{t}^{C}D_{t_2}^{\beta}a^{\nu} - B(t,a^{\mu})\Big\}\mathrm{d}t \tag{12} $$

称为基于 Caputo 导数的分数阶 Pfaff 作用量. 等时变分原理

$$ \delta S = 0 \tag{13} $$

带有交换关系

$$ {}_{t_1}^{C}D_t^{\alpha}\delta a^{\nu} = \delta\,{}_{t_1}^{C}D_t^{\alpha}a^{\nu},\quad {}_{t}^{C}D_{t_2}^{\beta}\delta a^{\nu} = \delta\,{}_{t}^{C}D_{t_2}^{\beta}a^{\nu},\quad (\nu=1,2,\cdots,2n) \tag{14} $$

以及端点条件

$$ \delta a^{\nu}\big|_{t=t_1} = \delta a^{\nu}\big|_{t=t_2} = 0,\quad (\nu=1,2,\cdots,2n) \tag{15} $$

称为基于 Caputo 导数的分数阶 Pfaff – Birkhoff 原理.

由分数阶 Pfaff – Birkhoff 原理(13) – (15)容易导出如下方程[37]

$$ \sum_{\nu=1}^{2n}\left(\frac{\partial R_{\nu}^{\alpha}}{\partial a^{\mu}}\,{}_{t_1}^{C}D_t^{\alpha}a^{\nu} + \frac{\partial R_{\nu}^{\beta}}{\partial a^{\mu}}\,{}_{t}^{C}D_{t_2}^{\beta}a^{\nu}\right) - \frac{\partial B}{\partial a^{\mu}} + {}_{t}D_{t_2}^{\alpha}R_{\mu}^{\alpha} + {}_{t_1}D_t^{\beta}R_{\mu}^{\beta} = 0,\quad (\mu=1,2,\cdots,2n) \tag{16} $$

以及相应的横截性条件

$$ \sum_{\nu=1}^{2n}\left[\left({}_{t_1}D_t^{\alpha-1}R_{\nu}^{\alpha}\right)\delta a^{\nu}\right]\Big|_{t_1}^{t_2} - $$

$$\sum_{\nu=1}^{2n}[({}_tD_{t_2}^{\beta-1}R_\nu^\beta)\delta a^\nu]\Big|_{t_1}^{t_2}=0 \qquad (17)$$

由端点条件(15)可得横截性条件(17)恒成立. 方程(16)称为基于 Caputo 分数阶导数的分数阶 Birkhoff 方程.

由分数阶 Birkhoff 方程(16)可以得到经典 Birkhoff 方程. 实际上,令分数阶 Pfaff 作用量(12)中不含 Caputo 右导数,即

$$S(a^\mu(\cdot))=\int_{t_1}^{t2}\{\sum_{\nu=1}^{2n}R_\nu^\alpha(t,a^\mu){}_{t_1}^CD_t^\alpha a^\nu-B(t,a^\mu)\}\mathrm{d}t \qquad (18)$$

则分数阶 Birkhoff 方程(16)成为

$$\sum_{\nu=1}^{2n}\frac{\partial R_\nu^\alpha}{\partial a^\mu}{}_{t_1}^CD_t^\alpha a^\nu-\frac{\partial B}{\partial a^\mu}+{}_tD_{t_2}^\alpha R_\mu^\alpha=0,\quad(\mu=1,2,\cdots,2n) \qquad (19)$$

当 $\alpha\to1$ 时,方程(19)为

$$\sum_{\nu=1}^{2n}(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu})\dot a^\nu-(\frac{\partial B}{\partial a^\mu}+\frac{\partial R_\mu}{\partial t})=0,\quad(\mu=1,2,\cdots,2n) \qquad (20)$$

方程(20)为经典的 Birkhoff 方程. 因此,经典的整数阶 Birkhoff 方程是分数阶 Birkhoff 方程(16)的特例.

3 分数阶 Birkhoff 系统的 Noether 定理

首先,引入 Frederico 和 Torres 提出的分数阶守恒量概念[23].

定义 1 $I(t,a^\nu,{}_{t_1}^CD_t^\alpha a^\nu,{}_t^CD_{t_2}^\beta a^\nu)$ 是分数阶守恒量当且仅当沿着分数阶 Birkhoff 方程(16)的解曲线,有

$$I(t,a^\nu,{}_{t_1}^CD_t^\alpha a^\nu,{}_t^CD_{t_2}^\beta a^\nu)=\sum_{i=1}^{m}I_i^1(t,a^\nu,{}_{t_1}^CD_t^\alpha a^\nu,{}_t^CD_{t_2}^\beta a^\nu)\times I_i^2(t,a^\nu,{}_{t_1}^CD_t^\alpha a^\nu,{}_t^CD_{t_2}^\beta a^\nu) \qquad (21)$$

其中 m 是任意整数,对于每一组函数 I_i^1 和 I_i^2 $(i=1,2,\cdots,m)$,满足

$${}^CD_t^{\gamma_i}(I_i^{j_i^1}(t,a^\nu,{}_{t_1}^CD_t^\alpha a^\nu,{}_t^CD_{t_2}^\beta a^\nu),I_i^{j_i^2}(t,a^\nu,{}_{t_1}^CD_t^\alpha a^\nu,{}_t^CD_{t_2}^\beta a^\nu))=0 \qquad (22)$$

其中 $\gamma_i\in\{\alpha,\beta\}$,$j_i^1=1$ 和 $j_i^2=2$(或者 $j_i^1=2$ 和 $j_i^2=1$).

其次,引进时间不变的单参数无限小变换群

$$\bar a^\mu(t)=a^\mu(t)+\varepsilon\xi_\mu(t,a^\nu)+o(\varepsilon),\quad(\mu=1,2,\cdots,2n) \qquad (23)$$

我们来定义分数阶 Birkhoff 系统在无限小变换(23)下的 Noether 对称性,并给出相应的分数阶守恒量.

定义 2 如果分数阶 Pfaff 作用量(12)在无限小变换(23)作用下,对于任意的子区间 $[T_1,T_2]\subseteq(t_1,t_2)$,成立

$$\int_{T_1}^{T2}\{\sum_{\mu=1}^{2n}R_\mu^\alpha(t,a^\nu){}_{t_1}^CD_t^\alpha a^\mu+\sum_{\mu=1}^{2n}R_\mu^\beta(t,a^\nu){}_t^CD_{t_2}^\beta a^\mu-B(t,a^\nu)\}\mathrm{d}t=\int_{T_1}^{T2}\{\sum_{\mu=1}^{2n}R_\mu^\alpha(t,\bar a^\nu){}_{t_1}^CD_t^\alpha\bar a^\mu+\sum_{\mu=1}^{2n}R_\mu^\beta(t,\bar a^\nu){}_t^CD_{t_2}^\beta\bar a^\mu-B(t,\bar a^\nu)\}\mathrm{d}t \qquad (24)$$

则称这种不变性为分数阶 Birkhoff 系统在时间不变的无限小变换下的 Noether 对称性.

定理 1 如果分数阶 Pfaff 作用量(12)在变换(23)作用下保持不变,那么

$$\sum_{\mu=1}^{2n}(\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\alpha(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)){}_{t_1}^CD_t^\alpha a^\mu+\sum_{\mu=1}^{2n}(\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\beta(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)){}_t^CD_{t_2}^\beta a^\mu+\sum_{\mu=1}^{2n}R_\mu^\alpha(t,a^\nu){}_{t_1}^CD_t^\alpha\xi_\mu(t,a^\nu)+\sum_{\mu=1}^{2n}R_\mu^\beta(t,a^\nu){}_t^CD_{t_2}^\beta\xi_\mu(t,a^\nu)-\sum_{\nu=1}^{2n}\frac{\partial B(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)=0 \qquad (25)$$

成立.

证明 由积分区间 $[T_1,T_2]$ 的任意性,由(24)式可得

$$\sum_{\mu=1}^{2n}R_\mu^\alpha(t,a^\nu){}_{t_1}^CD_t^\alpha a^\mu+\sum_{\mu=1}^{2n}R_\mu^\beta(t,a^\nu){}_t^CD_{t_2}^\beta a^\mu-B(t,a^\nu)=\sum_{\mu=1}^{2n}R_\mu^\alpha(t,\bar a^\nu){}_{t_1}^CD_t^\alpha\bar a^\mu+\sum_{\mu=1}^{2n}R_\mu^\beta(t,\bar a^\nu){}_t^CD_{t_2}^\beta\bar a^\mu-B(t,\bar a^\nu) \qquad (26)$$

式(26)两边对 ε 求导,然后令 $\varepsilon=0$,有

$$0=\sum_{\mu=1}^{2n}(\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\alpha(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)){}_{t_1}^CD_t^\alpha a^\mu+\sum_{\mu=1}^{2n}(\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\beta(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)){}_t^CD_{t_2}^\beta a^\mu+$$

$$\sum_{\mu=1}^{2n} R_\mu^\alpha(t,a^\nu)\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big[\frac{1}{\Gamma(m-\alpha)}\times \int_{t_1}^{t}(t-\tau)^{m-\alpha-1}\frac{\mathrm{d}^m(a^\mu(\tau))}{\mathrm{d}\tau^m}\mathrm{d}\tau+ \frac{\varepsilon}{\Gamma(m-\alpha)}\int_{t_1}^{t}(t-\tau)^{m-\alpha-1}\times \frac{\mathrm{d}^m\xi(\tau,a^\mu)}{\mathrm{d}\tau^m}\mathrm{d}\tau\Big]_{\varepsilon=0}+\sum_{\mu=1}^{2n}R_\mu^\beta(t,a^\nu)\times \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big[\frac{(-1)^m}{\Gamma(m-\beta)}\int_t^{t2}(\tau-t)^{m-\beta-1}\frac{\mathrm{d}^m a^\mu(\tau)}{\mathrm{d}\tau^m}\mathrm{d}\tau+ \frac{(-1)^m\varepsilon}{\Gamma(m-\beta)}\int_t^{t2}(\tau-t)^{m-\beta-1}\frac{\mathrm{d}^m\xi(\tau,a^\mu)}{\mathrm{d}\tau^m}\mathrm{d}\tau\Big]_{\varepsilon=0}- \sum_{\nu=1}^{2n}\frac{\partial B(t,a^\mu)}{\partial a^\nu}\xi_\nu(t,a^\mu)=$$
$$\sum_{\mu=1}^{2n}\Big(\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\alpha(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)\Big){}_{t_1}^{C}D_t^{\alpha}a^\mu+ \sum_{\mu=1}^{2n}\Big(\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\beta(t,a^\nu)}{\partial a^\nu}\xi_\nu(t,a^\mu)\Big){}_{t}^{C}D_{t_2}^{\beta}a^\mu+ \sum_{\mu=1}^{2n}R_\mu^\alpha(t,a^\nu)\frac{1}{\Gamma(m-\alpha)}\times \int_{t_1}^{t}(t-\tau)^{m-\alpha-1}\frac{\mathrm{d}^m\xi(\tau,a^\mu)}{\mathrm{d}\tau^m}\mathrm{d}\tau+ \sum_{\mu=1}^{2n}R_\mu^\beta(t,a^\nu)\frac{(-1)^m}{\Gamma(m-\beta)}\times \int_t^{t2}(\tau-t)^{m-\beta-1}\frac{\mathrm{d}^m\xi(\tau,a^\mu)}{\mathrm{d}\tau^m}\mathrm{d}\tau- \sum_{\nu=1}^{2n}\frac{\partial B(t,a^\mu)}{\partial a^\nu}\xi_\nu(t,a^\mu) \tag{27}$$

此时(27)式即为(25)式,证毕.

定理 2　如果分数阶 Pfaff 作用量(12)在定义 2 下保持不变,那么

$$I(t,a^\nu,{}_{t_1}^{C}D_t^{\alpha}a^\nu,{}_{t}^{C}D_{t_2}^{\beta}a^\nu)=\sum_{\mu=1}^{2n}(R_\mu^\alpha(t,a^\nu)-R_\mu^\beta(t,a^\nu))\xi_\mu(t,a^\nu) \tag{28}$$

是分数阶 Birkhoff 系统(16)的分数阶守恒量.

证明　由分数阶 Birkhoff 方程(16)可得

$$\frac{\partial B}{\partial a^\mu}=\sum_{\nu=1}^{2n}\Big(\frac{\partial R_\nu^\alpha}{\partial a^\mu}{}_{t_1}^{C}D_t^{\alpha}a^\nu+ \frac{\partial R_\nu^\beta}{\partial a^\mu}{}_{t}^{C}D_{t_2}^{\beta}a^\nu\Big)+ {}_tD_{t_2}^{\alpha}R_\mu^\alpha+{}_{t_1}D_t^{\beta}R_\mu^\beta \tag{29}$$

由于分数阶 Pfaff 作用量(12)在定义 2 下保持不变,故将(29)式代入(25)式,得

$$\sum_{\mu=1}^{2n}\Big(\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\alpha}{\partial a^\nu}\xi_\nu\Big){}_{t_1}^{C}D_t^{\alpha}a^\mu+ \sum_{\mu=1}^{2n}\Big(\sum_{\nu=1}^{2n}\frac{\partial R_\mu^\beta}{\partial a^\nu}\xi_\nu\Big){}_{t}^{C}D_{t_2}^{\beta}a^\mu+ \sum_{\mu=1}^{2n}R_\mu^\alpha{}_{t_1}^{C}D_t^{\alpha}\xi_\mu+\sum_{\mu=1}^{2n}R_\mu^\beta{}_{t}^{C}D_{t_2}^{\beta}\xi_\mu- \sum_{\mu=1}^{2n}\sum_{\nu=1}^{2n}\Big(\frac{\partial R_\nu^\alpha}{\partial a^\mu}{}_{t_1}^{C}D_t^{\alpha}a^\nu+ \frac{\partial R_\nu^\beta}{\partial a^\mu}{}_{t}^{C}D_{t_2}^{\beta}a^\nu\Big)\xi_\mu- \sum_{\mu=1}^{2n}\xi_\mu{}_tD_{t_2}^{\alpha}R_\mu^\alpha-\sum_{\mu=1}^{2n}\xi_\mu{}_{t_1}D_t^{\beta}R_\mu^\beta=0 \tag{30}$$

化简得

$$\sum_{\mu=1}^{2n}R_\mu^\alpha{}_{t_1}^{C}D_t^{\alpha}\xi_\mu+\sum_{\mu=1}^{2n}R_\mu^\beta{}_{t}^{C}D_{t_2}^{\beta}\xi_\mu- \sum_{\mu=1}^{2n}\xi_\mu{}_tD_{t_2}^{\alpha}R_\mu^\alpha-\sum_{\mu=1}^{2n}\xi_\mu{}_{t_1}D_t^{\beta}R_\mu^\beta=0 \tag{31}$$

即

$$\sum_{\mu=1}^{2n}\{{}^{C}D_t^{\alpha}(R_\mu^\alpha,\xi_\mu)-{}^{C}D_t^{\beta}(\xi_\mu,R_\mu^\beta)\}=0. \tag{32}$$

从而,由分数阶守恒量的定义 1 可知(28)式是该情形下的分数阶守恒量.

最后,引进时间变化的单参数无限小变换群

$$\bar t=t+\varepsilon\zeta(t,a^\nu)+o(\varepsilon),$$
$$\bar a^\mu(t)=a^\mu(t)+\varepsilon\xi_\mu(t,a^\nu)+o(\varepsilon),$$
$$(\mu=1,2,\cdots,2n) \tag{33}$$

我们来定义分数阶 Birkhoff 系统在无限小变换(33)下的 Noether 对称性,并给出相应的分数阶守恒量.

定义 3　如果分数阶 Pfaff 作用量(12)在无限小变换(33)作用下,对于任意的子区间 $[T_1,T_2]\subseteq(t_1,t_2)$,成立

$$\int_{T_1}^{T_2}\Big\{\sum_{\mu=1}^{2n}R_\mu^\alpha(t,a^\nu){}_{t_1}^{C}D_t^{\alpha}a^\mu+ \sum_{\mu=1}^{2n}R_\mu^\beta(t,a^\nu){}_{t}^{C}D_{t_2}^{\beta}a^\mu-B(t,a^\nu)\Big\}\mathrm{d}t= \int_{\bar T_1}^{\bar T_2}\Big\{\sum_{\mu=1}^{2n}R_\mu^\alpha(\bar t,\bar a^\nu){}_{\bar t_1}^{C}D_{\bar t}^{\alpha}\bar a^\mu+ \sum_{\mu=1}^{2n}R_\mu^\beta(\bar t,\bar a^\nu){}_{\bar t}^{C}D_{\bar t_2}^{\beta}\bar a^\mu-B(\bar t,\bar a^\nu)\Big\}\mathrm{d}\bar t \tag{34}$$

则称这种不变性为分数阶 Birkhoff 系统在时间变化的无限小变换下的 Noether 对称性.

定理 3　如果分数阶 Pfaff 作用量(12)在定义 3 下保持不变,那么

$$I(t,a^\nu,{}^C_{t_1}D^\alpha_t a^\nu,{}^C_tD^\beta_{t_2}a^\nu)=\sum_{\mu=1}^{2n}(R^\alpha_\mu-R^\beta_\mu)\xi_\mu+[(1-\alpha)\sum_{\mu=1}^{2n}R^\alpha_\mu{}^C_{t_1}D^\alpha_t a^\mu+(1-\beta)\sum_{\mu=1}^{2n}R^\beta_\mu{}^C_tD^\beta_{t_2}a^\mu-B]\zeta \tag{35}$$

是分数阶 Birkhoff 系统(16)的分数阶守恒量.

证明 取关于时间 t (t 是独立变量)的李普希兹变换

$$t\in[t_1,t_2]\mapsto\sigma f(\lambda)\in[\sigma_1,\sigma_2] \tag{36}$$

当 $\lambda=0$ 时,满足 $t'_\sigma=\frac{\mathrm{d}t(\sigma)}{\mathrm{d}\sigma}=f(\lambda)=1$. 在变换(36)作用下,分数阶 Pfaff 作用量(12)为

$$\bar{S}(t(\cdot),a^\nu(\cdot))=\int_{\sigma_1}^{\sigma_2}\{\sum_{\mu=1}^{2n}R^\alpha_\mu(t(\sigma),a^\nu(t(\sigma)))\times{}^C_{\sigma_1}D^\alpha_{t(\sigma)}a^\mu(t(\sigma))+\sum_{\mu=1}^{2n}R^\beta_\mu(t(\sigma),a^\nu(t(\sigma))){}^C_{t(\sigma)}D^\beta_{\sigma_2}a^\mu(t(\sigma))-B(t(\sigma),a^\nu(t(\sigma)))\}t'_\sigma\mathrm{d}\sigma \tag{37}$$

其中 $t(\sigma_1)=t_1$, $t(\sigma_2)=t_2$,

$${}^C_{\sigma_1}D^\alpha_{t(\sigma)}a^\mu(t(\sigma))=\frac{1}{\Gamma(m-\alpha)}\times\int_{\frac{t_1}{f(\lambda)}}^{\sigma f(\lambda)}(\sigma f(\lambda)-\tau)^{m-\alpha-1}\frac{\mathrm{d}^m a^\mu(\tau f^{-1}(\lambda))}{\mathrm{d}\tau^m}\mathrm{d}\tau=\frac{(t'_\sigma)^{-\alpha}}{\Gamma(m-\alpha)}\int_{\frac{t_1}{(t'_\sigma)^2}}^{\sigma}(\sigma-s)^{m-\alpha-1}\frac{\mathrm{d}^m a^\mu(s)}{\mathrm{d}s^m}\mathrm{d}s=(t'_\sigma)^{-\alpha}{}^C_{t_1/(t'_\sigma)^2}D^\alpha_\sigma a^\mu(\sigma) \tag{38}$$

其中 $s=\tau f^{-1}(\lambda)$. 同样可得

$${}^C_{t(\sigma)}D^\alpha_{\sigma_2}a^\mu(t(\sigma))=(t'_\sigma)^{-\beta}{}^C_\sigma D^\beta_{t_2/(t'_\sigma)^2}a^\mu(\sigma) \tag{39}$$

将(38)式和(39)式代入(37)式,得

$$\bar{S}(t(\cdot),a^\nu(\cdot))=\int_{\sigma_1}^{\sigma_2}\{\sum_{\mu=1}^{2n}R^\alpha_\mu(t(\sigma),a^\nu(t(\sigma)))\times(t'_\sigma)^{-\alpha}{}^C_{t_1/(t'_\sigma)^2}D^\alpha_\sigma a^\mu(\sigma)+\sum_{\mu=1}^{2n}R^\beta_\mu(t(\sigma),a^\nu(t(\sigma)))(t'_\sigma)^{-\beta}{}^C_\sigma D^\beta_{t_2/(t'_\sigma)^2}a^\mu(\sigma)-B(t(\sigma),a^\nu(t(\sigma)))\}t'_\sigma\mathrm{d}\sigma\doteq\int_{\sigma_1}^{\sigma_2}\bar{B}_f(t(\sigma),a^\nu(t(\sigma)),t'_\sigma,{}^C_{t_1/(t'_\sigma)^2}D^\alpha_\sigma a^\mu(\sigma),{}^C_\sigma D^\beta_{t_2/(t'_\sigma)^2}a^\mu(\sigma))\mathrm{d}\sigma=\int_{t_1}^{t_2}\{\sum_{\mu=1}^{2n}R^\alpha_\mu(t,a^\nu){}^C_{t_1}D^\alpha_t a^\mu+\sum_{\mu=1}^{2n}R^\beta_\mu(t,a^\nu){}^C_tD^\beta_{t_2}a^\mu-B(t,a^\nu)\}\mathrm{d}t=S(a^\nu(\cdot)) \tag{40}$$

如果分数阶 Pfaff 作用量(12)在定义 3 下保持不变,那么分数阶 Pfaff 作用量(37)在定义 2 下保持不变,由定理 2,我们得到

$$I_f(t(\sigma),a^\nu(t(\sigma)),t'_\sigma,{}^C_{t_1/(t'_\sigma)^2}D^\alpha_\sigma a^\mu(\sigma),{}^C_\sigma D^\beta_{t_2/(t'_\sigma)^2}a^\mu(\sigma))=\sum_{\mu=1}^{2n}(\frac{\partial\bar{B}_f}{\partial{}^C_{t_1/(t'_\sigma)^2}D^\alpha_\sigma a^\mu(\sigma)}-\frac{\partial\bar{B}_f}{\partial{}^C_\sigma D^\beta_{t_2/(t'_\sigma)^2}a^\mu(\sigma)})\xi_\mu+\frac{\partial\bar{B}_f}{\partial t'_\sigma}\zeta \tag{41}$$

式(41)是系统的分数阶守恒量. 当 $\lambda=0$ 时,有

$${}^C_{t_1/(t'_\sigma)^2}D^\alpha_\sigma a^\mu(\sigma)={}^C_{t_1}D^\alpha_t a^\mu(t),\quad {}^C_\sigma D^\beta_{t_2/(t'_\sigma)^2}a^\mu(\sigma)={}^C_tD^\alpha_{t_2}a^\mu(t). \tag{42}$$

因此,我们得到

$$\frac{\partial\bar{B}_f}{\partial{}^C_{t_1/(t'_\sigma)^2}D^\alpha_\sigma a^\mu(\sigma)}-\frac{\partial\bar{B}_f}{\partial{}^C_\sigma D^\beta_{t_2/(t'_\sigma)^2}a^\mu(\sigma)}=R^\alpha_\mu-R^\beta_\mu \tag{43}$$

而

$$\frac{\partial\bar{B}_f}{\partial t'_\sigma}=\frac{\partial}{\partial t'_\sigma}[\sum_{\mu=1}^{2n}R^\alpha_\mu(t(\sigma),a^\nu(t(\sigma)))\times\frac{(t'_\sigma)^{-\alpha}}{\Gamma(m-\alpha)}\int_{\frac{t_1}{(t'_\sigma)^2}}^{\sigma}(\sigma-s)^{m-\alpha-1}\frac{\mathrm{d}^m a^\mu(s)}{\mathrm{d}s^m}\mathrm{d}s]t'_\sigma+\frac{\partial}{\partial t'_\sigma}\left[\sum_{\mu=1}^{2n}R^\beta_\mu(t(\sigma),a^\nu(t(\sigma)))\frac{(t'_\sigma)^{-\beta}}{\Gamma(m-\alpha)}\times\int_{\sigma}^{\frac{t_2}{(t'_\sigma)^2}}(\sigma-s)^{m-\alpha-1}\frac{\mathrm{d}^m a^\mu(s)}{\mathrm{d}s^m}\mathrm{d}s\right]t'_\sigma+\sum_{\mu=1}^{2n}R^\alpha_\mu(t(\sigma),a^\nu(t(\sigma)))(t'_\sigma)^{-\alpha}{}^C_{t_1/(t'_\sigma)^2}D^\alpha_\sigma a^\mu(\sigma)+\sum_{\mu=1}^{2n}R^\beta_\mu(t(\sigma),a^\nu(t(\sigma)))(t'_\sigma)^{-\beta}{}^C_\sigma D^\beta_{t_2/(t'_\sigma)^2}a^\mu(\sigma)-B(t(\sigma),a^\nu(t(\sigma)))=\sum_{\mu=1}^{2n}(-\alpha R^\alpha_\mu{}^C_{t_1}D^\alpha_t a^\mu-\beta R^\beta_\mu{}^C_tD^\beta_{t_2}a^\mu+R^\alpha_\mu{}^C_{t_1}D^\alpha_t a^\mu+R^\beta_\mu{}^C_tD^\beta_{t_2}a^\mu)-B \tag{44}$$

将式(43)和(44)代入式(41),我们得到守恒量(35).证毕.

定理2 和定理3 称为分数阶 Birkhoff 系统在 Caputo 导数下的分数阶 Noether 定理,它们揭示了分数阶 Noether 对称性与分数阶守恒量之间的关系. 利用分数阶 Noether 定理,可由分数阶 Birkhoff 系统的 Noether 对称性找到相应的分数阶守恒量.

4　算例

下面举例说明结果的应用.

例 考虑二阶 Birkhoff 系统,其分数阶 Pfaff 作用量为

$$S = \int_{t_1}^{t2} \{ a^2 {}_{t_1}^{C}D_t^{\alpha} a^1 + a^2 {}_{t}^{C}D_{t_2}^{\beta} a^1 - \frac{1}{2}[(a^1)^2 + (a^2)^2] \} \mathrm{d}t \tag{45}$$

试研究该 Birkhoff 系统的分数阶 Noether 对称性与分数阶守恒量.

由作用量(45)可知,系统的 Birkhoff 函数和 Birkhoff 函数组分别为

$$B = \frac{1}{2}[(a^1)^2 + (a^2)^2],$$

$$R_1 = a^2, R_2 = 0 \tag{46}$$

显然,存在如下 Noether 对称变换

$$(\zeta, \xi_1, \xi_2) = (-1, 0, 0) \tag{47}$$

使得分数阶 Pfaff 作用量(45)在定义 3 意义下不变,故由定理 3 该系统的分数阶守恒量为

$$I = -(1-\alpha) a^2 {}_{t_1}^{C}D_t^{\alpha} a^1 - (1-\beta) a^2 {}_{t}^{C}D_{t_2}^{\beta} a^1 + \frac{1}{2}[(a^1)^2 + (a^2)^2] \tag{48}$$

如果作用量(45)中只含左导数,令 $\alpha \to 1$,则守恒量(48)给出

$$I = \frac{1}{2}[(a^1)^2 + (a^2)^2] \tag{49}$$

式(49)是整数阶模型下 Birkhoff 系统(46)的守恒量.

5　结论

近 20 年来,分数阶微积分被成功地广泛应用于科学和工程的各个领域. 分数阶微积分也被用于非保守系统或耗散系统的动力学建模,从而可以解决用经典的整数阶导数下的方法难以解决的问题. 本文的主要工作:一是基于 Caputo 分数阶导数提出分数阶 Pfaff 变分问题,建立了分数阶力学系统的分数阶 Birkhoff 方程(16);二是基于分数阶 Pfaff 作用量在无限小变换下的不变性,定义了分数阶 Birkhoff 系统的 Noether 对称性,依据 Frederico 和 Torres 提出的分数阶守恒量概念,给出了分数阶 Birkhoff 系统的守恒量,建立了分数阶 Noether 定理,从而揭示了分数阶对称性与分数阶守恒量的内在联系. 经典的 Birkhoff 系统是本文之特例,因此本文结果具有普遍意义.

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NOETHER SYMMETRY AND CONSERVED QUANTITY FOR FRACTIONAL BIRKHOFFIAN SYSTEMS IN TERMS OF CAPUTO DERIVATIVES*

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Abstract This paper studies the Noether symmetry and corresponding conserved quantity for fractional Birkhoffian systems in terms of Caputo fractional derivatives. Firstly, the fractional Pfaff action is defined within Caputo fractional derivatives. The fractional Birkhoff's equations and corresponding transversality conditions are also established. Secondly, based on the invariance of the Pfaff action under the infinitesimal transformations, the conditions of invariance are given under a special one-parameter group of infinitesimal transformations without transforming the time as well as a general one-parameter group with transforming the time, respectively. Finally, according to the notion of fractional conserved quantity presented by Frederico and Torres, the Noether theorem for the fractional Birkhoffian systems is constructed, which states the relationship between a fractional Noether symmetry and a fractional conserved quantity.

Key words fractional Birkhoffian system, fractional Noether symmetry, fractional conserved quantity, fractional Pfaff action, Caputo fractional derivative

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