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平面框架结构折线型弹塑性动力学非传统 Hamilton 型增量变分原理*

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摘要 根据古典阴阳互补和现代对偶互补的基本思想,通过罗恩提出的一条简单而统一的新途径,系统地建立了平面框架结构折线型弹塑性动力学的各类非传统 Hamilton 型变分原理. 文中首先给出平面框架结构折线型弹塑性动力学的广义虚功原理的表式,然后从该式出发,不仅能得到平面框架结构折线型弹塑性动力学的虚功原理,而且通过所给出的广义 Legendre 变换,还能系统地成对导出平面框架结构折线型弹塑性动力学的 5 类变量分原理的互补泛函,以及 1 类变量和相空间非传统 Hamilton 型变分原理的泛函. 同时,通过这条新途径还能清楚地阐明这些原理的内在联系.

关键词 框架结构, 弹塑性动力学, 相空间, 非传统 Hamilton 型变分原理, 初值-边值问题

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引言

结构的弹塑性分析一般都相当复杂,因此,目前大多数都将非线性的本构关系简化为分段线性弹性的本构关系. 对于平面框架折线型弹塑性动力学,可建立相应的增量变分原理,然后采用基于增量变分原理的增量有限元法进行分析.

对于静力学增量变分原理的研究,1976 年 Horrigmoe 和 Bergan[1] 在传统变分原理的基础上通过考虑增量变形来解决非线性问题,并与有限元结合,放松单元的连续性要求,修正了传统的增量变分原理;同年,Pian[2] 给出了小位移问题增量变分原理和放松连续性要求的修正的增量变分原理;1980 年 Mason[3] 给出了固体力学和壳体理论中的增量能量方法,明确提出了增量变分原理的概念,并给出了两条得到增量变分原理的途径;1984 年刘正兴[4] 建立了基于增量变分原理的柔韧梁和柔韧板单元;钟万勰[5] 给出了小位移弹性问题中的各类增量变分原理;1993 年 Grzegorz[6] 给出了线弹性摩擦接触问题的增量变分原理;2007 年 Lahellec 和 Suquet[7] 给出了非线性非弹性复合材料的增量变分原理. 2000 年贺国京和陈大鹏[8] 提出了结构非线性振动的杂交混合幅值增量变分原理. 但是,有关框架结构的一些重要的增量型基本原理,例如增量虚功原理和能反映其初值-边值问题的全部特征各类非传统 Hamilton 型增量变分原理至今国内外还没有系统建立.

本文根据文献[9]提出的一条简单而统一的新途径,系统地建立了框架结构折线型弹塑性动力学的广义增量虚功原理和增量虚功原理、以及各类非传统 Hamilton 型增量变分原理和相空间非传统 Hamilton 型增量变分原理. 这种新的增量变分原理能反映这种动力学初值-边值问题的全部特征.

1 平面框架折线型弹塑性动力学基本方程及条件

(1) 速度位移关系

$$j\text{ 梁}:\quad v_j^i + \Delta v_j = \dot{u}_j^i + \Delta \dot{u}_j \quad (j = 1, 2, \cdots, ml) \tag{1a}$$

$$k\text{ 柱}:\quad v_k^i + \Delta v_k = \dot{u}_k^i + \Delta \dot{u}_k \quad (k = 1, 2, \cdots, mz) \tag{1b}$$

式中 $v = [v_s, v_r]^T$, $u = [u_s, u_r]^T$.

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(2)动量速度关系

j 梁： $\Delta p_j = \rho_j A_j \Delta v_j \quad (j = 1, 2, \cdots, ml)$ (2a)

k 柱： $\Delta p_k = \rho_k A_k \Delta v_k \quad (k = 1, 2, \cdots, mz)$ (2b)

式中 $p = [p_s, p_r]^T$.

(3)运动方程

j 梁：

$$B(Q_j^i + \Delta Q_j) - \dot{p}_j^i - \Delta\dot{p}_j + f_j^i + \Delta f_j - c\dot{u}_j^i - c\Delta\dot{u}_j = 0 \quad (3a)$$

$$B(Q_j^i + \Delta Q_j) - \rho_j A_j(\ddot{u}_j^i + \Delta\ddot{u}_j) + f_j + \Delta f_j - c\dot{u}_j^i - c\Delta\dot{u}_j = 0 \quad (j = 1, 2, \cdots, ml) \quad (3b)$$

k 柱：

$$B(Q_k^i + \Delta Q_k) - \dot{p}_k^i - \Delta\dot{p}_k + f_k^i + \Delta f_k - c\dot{u}_k^i - c\Delta\dot{u}_k = 0 \quad (3c)$$

$$B(Q_k^i + \Delta Q_k) - \rho_k A_k(\ddot{u}_k^i + \Delta\ddot{u}_k) + f_k^i + \Delta f_k - c\dot{u}_k^i - c\Delta\dot{u}_k = 0 \quad (k = 1, 2, \cdots, mz) \quad (3d)$$

式中 $Q = [N_s, M]^T$, $f = [f_s, f_r]^T$, $0 = [0, 0]^T$, B 是微分算子矩阵, $B = \begin{bmatrix} d/ds & 0 \\ 0 & -d^2/ds^2 \end{bmatrix}$.

(4)广义应变与位移关系

j 梁： $\kappa_j^i + \Delta\kappa_j = B(u_j^i + \Delta u_j) \quad (j = 1, 2, \cdots, ml)$ (4a)

k 柱： $\kappa_k^i + \Delta\kappa_k = B(u_k^i + \Delta u_k) \quad (k = 1, 2, \cdots, mz)$ (4b)

式中 $= [\varepsilon_s, \kappa]^T$.

(5)广义内力和广义应变关系

对于一般弹塑性问题，有

$$Q = \partial\Phi(\kappa)/\partial\kappa, \ \kappa = \partial\Psi(Q)/\partial Q$$

式中，$\Phi(\kappa)$ 和 $\Psi(Q)$ 分别为应变能密度和余应变能密度.

将增量段的广义内力和广义应变关系线性化，一般弹塑性问题简化为折线型弹塑性问题后，有：

j 梁： $\Delta\boldsymbol{Q}_j = [D]_j^i \Delta\boldsymbol{\kappa}_j$ (5a)

或 $\Delta\boldsymbol{\kappa}_j = [D]_j^{i-1} \Delta\boldsymbol{Q}_j \quad (j = 1, 2, \cdots, ml)$ (5b)

k 柱： $\Delta\boldsymbol{Q}_k = [D]_k^i \Delta\boldsymbol{\kappa}_k$ (5c)

或 $\Delta\boldsymbol{\kappa}_k = [D]_k^{i-1} \Delta\boldsymbol{Q}_k \quad (k = 1, 2, \cdots, mz)$ (5d)

式中 $[D]^i = \begin{bmatrix} E^i A^i & 0 \\ 0 & E^i I^i \end{bmatrix}$, $[D]^i$ 为割线弹塑性矩阵.

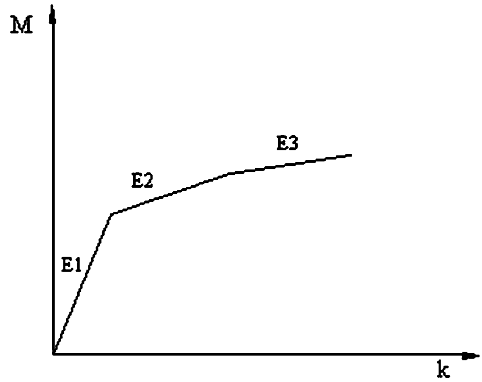

图 1　折线型弯矩曲率关系

Fig. 1　Piecewise linear relationship of moment-curvature

(6)结构边界条件

对于结构的整体直角坐标系 x, z，给定节点外力的节点条件和给定节点位移的支撑节点条件分别为：

$$T_{l2}^i + \Delta T_{l2} = [T_{xl2}^i + \Delta T_{xl2}, T_{zl2}^i + \Delta T_{zl2}]^T = \bar{T}_{l2}^i + \Delta\bar{T}_{l2}$$

($l2 = 1, 2, \cdots, mf$ 为给定节点外力的节点数) (6a)

$$u_{l1}^i + \Delta u_{l1} = [u_{xl1}^i + \Delta u_{xl1}, u_{zl1}^i + \Delta u_{zl1}]^T = \bar{u}_{l1}^i + \Delta\bar{u}_{l1}$$

($l1 = 1, 2, \cdots, ms$ 为给定节点支撑的节点数) (6b)

(7)初始条件

$$u_{0j}^i(s) + \Delta u_{0j}(s) = u_j^i(s,0) + \Delta u_j(s,0) = [u_{sj}^i(s,0) + \Delta u_{sj}(s,0), u_{rj}^i(s,0) + \Delta u_{rj}(s,0)] = \tilde{u}_{0j}^i(s) + \Delta\tilde{u}_{0j}(s) = [\tilde{u}_{s0j}^i(s) + \Delta\tilde{u}_{s0j}(s), \tilde{u}_{r0j}^i(s) + \Delta\tilde{u}_{r0j}(s)] \quad (7a)$$

$$p_{0j}^i(s) + \Delta p_{0j}(s) = p_j^i(s,0) + \Delta p_j(s,0) = [p_{sj}^i(s,0) + \Delta p_{sj}(s,0), p_{rj}^i(s,0) + \Delta p_{rj}(s,0)] = \tilde{p}_{0j}^i(s) + \Delta\tilde{p}_{0j}(s) = [\tilde{p}_{s0j}^i(s) + \Delta\tilde{p}_{s0j}(s), \tilde{p}_{r0j}^i(s) + \Delta\tilde{p}_{r0j}(s)] \quad (7b)$$

$$u_{0k}^i(s) + \Delta u_{0k}(s) = u_k^i(s,0) + \Delta u_k(s,0) = [u_{sk}^i(s,0) + \Delta u_{sk}^i(s,0), u_{rk}^i(s,0) + \Delta u_{rk}(s,0)] = \tilde{u}_{0k}^i(s) + \Delta\tilde{u}_{0k}(s) = [\tilde{u}_{s0k}^i(s) + \Delta\tilde{u}_{s0k}(s), \tilde{u}_{r0k}^i(s) + \Delta\tilde{u}_{r0k}(s)] \quad (7c)$$

$$p_{0k}^i(s) + \Delta p_{0k}(s) = p_k^i(s,0) + \Delta p_k(s,0) = [p_{sk}^i(s,0) + \Delta p_{sk}(s,0), p_{rk}^i(s,0) + \Delta p_{rk}(s,0)] = \tilde{p}_{0k}^i(s) + \Delta\tilde{p}_{0k}(s) = [\tilde{p}_{s0k}^i(s) + \Delta\tilde{p}_{s0k}(s), \tilde{p}_{r0k}^i(s) + \Delta\tilde{p}_{r0k}(s)] \quad (7d)$$

式中 $\tilde{u}_{0j}^{i}(s),\Delta\tilde{u}_{0j}(s),\tilde{p}_{0j}^{i}(s),\Delta\tilde{p}_{0j}(s),\tilde{u}_{0k}^{i}(s),\Delta\tilde{u}_{0k}(s),\tilde{p}_{0k}^{i}(s),\Delta\tilde{p}_{0k}(s)$ 为已知初始值.

(8)梁柱交结点的联结条件

①梁柱交结点的位移协调条件:

$$\begin{cases} u_{l3+0}^{i}+\Delta u_{l3+0}=[u_{x,l3+0}^{i}+\Delta u_{x,l3+0},u_{z,l3+0}^{i}+\Delta u_{z,l3+0}]^{T}=u_{l3+1-0}^{i}+\Delta u_{l3+1-0}= \\ [u_{x,l3+1-0}^{i}+\Delta u_{x,l3+1-0},u_{z,l3+1-0}^{i}+\Delta u_{z,l3+1-0}]^{T}=u_{j1,k1}^{i}+\Delta u_{j1,k1} \qquad (8a) \\ u_{l3-0}^{i}+\Delta u_{l3-0}=[u_{x,l3-0}^{i}+\Delta u_{x,l3-0},u_{z,l3-0}^{i}+\Delta u_{z,l3-0}]^{T}=u_{l3-1+0}^{i}+\Delta u_{l3-1+0}= \\ [u_{x,l3-1+0}^{i}+\Delta u_{x,l3-1+0},u_{z,l3-1+0}^{i}+\Delta u_{z,l3-1+0}]^{T}=u_{j1-1,k1}^{i}+\Delta u_{j1-1,k1} \\ \begin{pmatrix} l3\neq\wp,\ l3-1\neq\wp,\ m,2m,\cdots,(n-1)m \\ l3+1\neq\wp,\ m+1,2m+1,\cdots,(n-1)m+1 \end{pmatrix} \end{cases} \tag{8b}$$

$$\begin{cases} u_{l4+0}^{i}+\Delta u_{l4+0}=[u_{x,l4+0}^{i}+\Delta u_{x,l4+0},u_{z,l4+0}^{i}+\Delta u_{z,l4+0}]^{T}=u_{l4+1-0}^{i}+\Delta u_{l4+1-0}= \\ [u_{x,l4+1-0}^{i}+\Delta u_{x,l4+1-0},u_{z,l4+1-0}^{i}+\Delta u_{z,l4+1-0}]^{T}=u_{j1,k1}^{i}+\Delta u_{j1,k1} \qquad (8c) \\ u_{l4-0}^{i}+\Delta u_{l4-0}=[u_{x,l4-0}^{i}+\Delta u_{x,l4-0},u_{z,l4-0}^{i}+\Delta u_{z,l4-0}]^{T}=u_{l4-1+0}^{i}+\Delta u_{l4-1+0}= \\ [u_{x,l4-1+0}^{i}+\Delta u_{x,l4-1+0},u_{z,l4-1+0}^{i}+\Delta u_{z,l4-1+0}]^{T}=u_{j1-1,k1}^{i}+\Delta u_{j1-1,k1} \\ \begin{pmatrix} l4\neq\aleph,l4-1\neq\aleph,(m+1)n,(m+2)n,\cdots,(2m-1)n \\ l4+1\neq\aleph,(m+1)n+1,(m+2)n+1,\cdots,(2m-1)n+1 \end{pmatrix} \end{cases} \tag{8d}$$

②梁柱交结点的力平衡条件:

$$\begin{cases} T_{l3+0}^{i}+\Delta T_{l3+0}=[T_{x,l3+0}^{i}+\Delta T_{x,l3+0},T_{z,l3+0}^{i}+\Delta T_{z,l3+0}]^{T}=T_{l3+1-0}^{i}+\Delta T_{l3+1-0}= \\ [T_{x,l3+1-0}^{i}+\Delta T_{x,l3+1-0},T_{z,l3+1-0}^{i}+\Delta T_{z,l3+1-0}]^{T}=T_{j1,k1}^{i}+\Delta T_{j1,k1} \qquad (8e) \\ T_{l3-0}^{i}+\Delta T_{l3-0}=[T_{x,l3-0}^{i}+\Delta T_{x,l3-0},T_{z,l3-0}^{i}+\Delta T_{z,l3-0}]^{T}=T_{l3-1+0}^{i}+\Delta T_{l3-1+0}= \\ [T_{x,l3-1+0}^{i}+\Delta T_{x,l3-1+0},T_{z,l3-1+0}^{i}+\Delta T_{z,l3-1+0}]^{T}=T_{j1-1,k1}^{i}+\Delta T_{j1-1,k1} \\ \begin{pmatrix} l3\neq\wp,l3-1\neq\wp,m,2m,\cdots,(n-1)m \\ l3+1\neq\wp,m+1,2m+1,\cdots,(n-1)m+1 \end{pmatrix} \end{cases} \tag{8f}$$

$$\begin{cases} T_{l4+0}^{i}+\Delta T_{l4+0}=[T_{x,l4+0}^{i}+\Delta T_{x,l4+0},T_{z,l4+0}^{i}+\Delta T_{z,l4+0}]^{T}=T_{l4+1-0}^{i}+\Delta T_{l4+1-0}= \\ [T_{x,l4+1-0}^{i}+\Delta T_{x,l4+1-0},T_{z,l4+1-0}^{i}+\Delta T_{z,l4+1-0}]^{T}=T_{j1,k1}^{i}+\Delta T_{j1,k1} \qquad (8g) \\ T_{l4-0}^{i}+\Delta T_{l4-0}=[T_{x,l4-0}^{i}+\Delta T_{x,l4-0},T_{z,l4-0}^{i}+\Delta T_{z,l4-0}]^{T}=T_{l4-1+0}^{i}+\Delta T_{l4-1+0}= \\ [T_{x,l4-1+0}^{i}+\Delta T_{x,l4-1+0},T_{z,l4-1+0}^{i}+\Delta T_{z,l4-1+0}]^{T}=T_{j1-1,k1}^{i}+\Delta T_{j1-1,k1} \\ \begin{pmatrix} l4\neq\aleph,l4-1\neq\aleph,(m+1)n,(m+2)n,\cdots,(2m-1)n \\ l4+1\neq\aleph,(m+1)n+1,(m+2)n+1,\cdots,(2m-1)n+1 \end{pmatrix} \end{cases} \tag{8h}$$

2 增量广义虚功原理、增量虚功原理

可以证明,对于任意无关的 p,Q,u,有下列积分关系式成立:

$$\sum_{j=1}^{ml}\Big\{\int_{0}^{t_1}\int_{l_j}[p_j^T\dot{u}_j-Q_j^T(Bu_j)]dsdt+\int_{0}^{t_1}\int_{l_j}[\dot{p}_j^Tu_j-(BQ_j)^Tu_j]dsdt-\int_{l_j}[u_j^T(s,t_1)p_j(s,t_1)-u_j^T(s,0)p_j(s,0)]ds\Big\}+\sum_{k=1}^{mz}\Big\{\int_{0}^{t_1}\int_{l_k}[p_k^T\dot{u}_k-Q_k^T(Bu_k)]dsdt+\int_{0}^{t_1}\int_{l_k}[\dot{p}_k^Tu_k-(BQ_k)^Tu_k]dsdt-\int_{l_k}[u_k^T(s,t_1)p_k(s,t_1)-u_k^T(s,0)p_k(s,0)]ds\Big\}+\int_{0}^{t_1}[\sum_{l2=1}^{mf}T_{l2}^Tu_{l2}+\sum_{l1=1}^{ms}T_{l1}^Tu_{l1}]dt+\int_{0}^{t_1}\Big\{\sum_{l3=1}^{mn}[(T_{l3+0}^Tu_{l3+0}-T_{l3+1-0}^Tu_{l3+1-0})+(T_{l3-0}^Tu_{l3-0}-T_{l3-1+0}^Tu_{l3-1+0})]+\sum_{l4=mn+1}^{(2m+1)n}[(T_{l4+0}^Tu_{l4+0}-T_{l4+1-0}^Tu_{l4+1-0})+(T_{l4-0}^Tu_{l4-0}-T_{l4-1+0}^Tu_{l4-1+0})]\Big\}dt=0 \tag{9}$$

上式即为平面框架结构动力学的广义虚功原理.

根据(9)式,对于互不相关的任意函数 p^i,Q^i,u^i 可得

$$\sum_{j=1}^{ml}\Big\{\int_{0}^{t_1}\int_{l_j}[p_j^{iT}\dot{u}_j^i-Q_j^{iT}(Bu_j^i)+\dot{p}_j^{iT}u_j^i-(BQ_j^i)^Tu_j^i]dsdt-\int_{l_j}[u_j^{iT}(s,t_1)p_j^i(s,t_1)-u_j^{iT}(s,0)p_j^i(s,0)]ds\Big\}+\sum_{k=1}^{mz}\Big\{\int_{0}^{t_1}\int_{l_k}[p_k^{iT}\dot{u}_k^i-$$

$$
\begin{aligned}
& Q_k^{i\,T}(Bu_k^i) + \dot{p}_k^{i\,T}u_k^i - (BQ_k^i)^T u_k^i\,]dsdt - \\
& \int_{l_k}[u_k^{i\,T}(s,t_1)p_k^i(s,t_1) - u_k^{i\,T}(s,0)p_k^i(s,0)]ds\Big\} + \\
& \int_0^{t_1}\Big[\sum_{l2=1}^{mf}T_{l2}^{i\,T}u_{l2}^i + \sum_{l1=1}^{ms}T_{l1}^{i\,T}u_{l1}^i\Big]dt + \\
& \int_0^{t_1}\Big\{\sum_{l3=1}^{mn}\big[(T_{l3+0}^i)^T u_{l3+0} - (T_{l3+1-0}^i)^T u_{l3+1-0} + \\
& (T_{l3-0}^i)^T u_{l3-0} - (T_{l3-1+0}^i)^T u_{l3-1+0}\big] + \\
& \sum_{l4=mn+1}^{(2m+1)n}\big[(T_{l4+0}^i)^T u_{l4+0} - (T_{l4+1-0}^i)^T u_{l4+1-0} + \\
& (T_{l4-0}^i)^T u_{l4-0} - (T_{l4-1+0}^i)^T u_{l4-1+0}\big]\Big\}dt = 0
\end{aligned}
\tag{10}
$$

对于互不相关的任意函数 $p^i + \Delta p, Q^i + \Delta Q, u^i + \Delta u$ 可得

$$
\begin{aligned}
& \sum_{j=1}^{ml}\Big\{\int_0^{t_1}\int_{l_j}\{(p_j^i + \Delta p_j)^T(\dot{u}_j^i + \Delta\dot{u}_j) - \\
& (Q_j^i + \Delta Q_j)^T[(Bu_j^i) + (B\Delta u_j)] + \\
& [\dot{p}_j^i + \Delta\dot{p}_j - (BQ_j^i) - (B\Delta Q_j)]^T(u_j^i + \\
& \Delta u_j)\}dsdt - \int_{l_j}\{[u_j^i(s,t_1) + \Delta u_j(s,t_1)]^T\cdot \\
& [p_j^i(s,t_1) + \Delta p_j(s,t_1)] - [u_j^i(s,0) + \\
& \Delta u_j(s,0)]^T[p_j^i(s,0) + \Delta p_j(s,0)]\}ds\Big\} + \\
& \sum_{k=1}^{mz}\Big\{\int_0^{t_1}\int_{l_k}\{(p_k^i + \Delta p_k)^T(\dot{u}_k^i + \Delta\dot{u}_k) - \\
& (Q_k^i + \Delta Q_k)^T[(Bu_k^i) + (B\Delta u_k)] + [\dot{p}_k^i + \\
& \Delta\dot{p}_k - (BQ_k^i) - (B\Delta Q_k)]^T(u_k^i + \Delta u_k)\}dsdt - \\
& \int_{l_k}\{[u_k^i(s,t_1) + \Delta u_k(s,t_1)]^T[p_k^i(s,t_1) + \\
& \Delta p_k(s,t_1)] - [u_k^i(s,0) + \Delta u_k(s,0)]^T\cdot \\
& [p_k^i(s,0) + \Delta p_k(s,0)]\}ds\Big\} + \\
& \int_0^{t_1}\Big\{\sum_{l2=1}^{mf}[(T_{l2}^i + \Delta T_{l2})^T(u_{l2}^i + \Delta u_{l2})] + \\
& \sum_{l1=1}^{ms}[(T_{l1}^i + \Delta T_{l1})^T(u_{l1}^i + \Delta u_{l1})]\Big\}dt + \\
& \int_0^{t_1}\Big\{\sum_{l3=1}^{mn}[(T_{l3+0}^i + \Delta T_{l3+0})^T(u_{l3+0}^i + \Delta u_{l3+0}) - \\
& (T_{l3+1-0}^i + \Delta T_{l3+1-0})^T(u_{l3+1-0}^i + \Delta u_{l3+1-0}) + \\
& (T_{l3-0}^i + \Delta T_{l3-0})^T(u_{l3-0}^i + \Delta u_{l3-0}) - \\
& (T_{l3-1+0}^i + \Delta T_{l3-1+0})^T(u_{l3-1+0}^i + \Delta u_{l3-1+0})] + \\
& \sum_{l4=mn+1}^{(2m+1)n}[(T_{l4+0}^i + \Delta T_{l4+0})^T(u_{l4+0}^i + \Delta u_{l4+0}) - \\
& (T_{l4+1-0}^i + \Delta T_{l4+1-0})^T(u_{l4+1-0}^i + \Delta u_{l4+1-0}) + \\
& (T_{l4-0}^i + \Delta T_{l4-0})^T(u_{l4-0}^i + \Delta u_{l4-0}) - \\
& (T_{l4-1+0}^i + \Delta T_{l4-1+0})^T(u_{l4-1+0}^i + \\
& \Delta u_{l4-1+0})]\Big\}dt = 0
\end{aligned}
\tag{11}
$$

将式(11)和式(10)相减,可得

$$
\begin{aligned}
& \sum_{j=1}^{ml}\int_0^{t_1}\int_{l_j}[(p_j^i + \Delta p_j)^T\Delta\dot{u}_j + \Delta p_j^T\dot{u}_j^i - \\
& (Q_j^i + \Delta Q_j)^T(B\Delta u_j) - \Delta Q_j^T(Bu_j^i)]dsdt + \\
& \sum_{j=1}^{ml}\int_0^{t_1}\int_{l_j}\{(\dot{p}_j^i + \Delta\dot{p}_j)^T\Delta u_j + \Delta\dot{p}_j^T u_j^i - \\
& [(BQ_j^i) + (B\Delta Q_j)]^T\Delta u_j - (B\Delta Q_j)^T u_j^i\}dsdt - \\
& \sum_{j=1}^{ml}\int_0^{t_1}\int_{l_j}\{[u_j^i(s,t_1) + \Delta u_j(s,t_1)]^T\Delta p_j(s,t_1) + \\
& \Delta u_j^T(s,t_1)p_j^i(s,t_1) - [u_j^i(s,0) + \Delta u_j(s,0)]^T\cdot \\
& \Delta p_j(s,0) - \Delta u_j^T(s,0)p_j^i(s,0)\}ds + \\
& \sum_{k=1}^{mz}\int_0^{t_1}\int_{l_k}[(p_k^i + \Delta p_k)^T\Delta\dot{u}_k + \Delta p_k^T\dot{u}_k^i - \\
& (Q_k^i + \Delta Q_k)^T(B\Delta u_k) - \Delta Q_k^T(Bu_k^i)]dsdt + \\
& \sum_{k=1}^{mz}\int_0^{t_1}\int_{l_k}\{(\dot{p}_k^i + \Delta\dot{p}_k)^T\Delta u_k + \Delta\dot{p}_k^T u_k^i - \\
& [(BQ_k^i) - (B\Delta Q_k)]^T\Delta u_k - (B\Delta Q_k)^T u_k^i\}dsdt - \\
& \sum_{k=1}^{mz}\int_0^{t_1}\int_{l_k}\{[u_k^i(s,t_1) + \Delta u_k(s,t_1)]^T\Delta p_k(s,t_1) + \\
& \Delta u_k^T(s,t_1)p_k^i(s,t_1) - [u_k^i(s,0) + \Delta u_k(s,0)]^T\cdot \\
& \Delta p_k(s,0) - \Delta u_k^T(s,0)p_k^i(s,0)\}ds + \\
& \int_0^{t_1}\Big\{\sum_{l2=1}^{mf}[(T_{l2}^i + \Delta T_{l2})^T\Delta u_{l2} + \Delta T_{l2}^T u_{l2}^i] + \\
& \sum_{l1=1}^{ms}[(T_{l1}^i + \Delta T_{l1})^T\Delta u_{l1} + \Delta T_{l1}^T u_{l1}^i]\Big\}dt + \\
& \int_0^{t_1}\Big\{\sum_{l3=1}^{mn}[(T_{l3+0}^i + \Delta T_{l3+0})^T\Delta u_{l3+0} + \\
& (\Delta T_{l3+0})^T u_{l3+0}^i - (T_{l3+1-0}^i + \Delta T_{l3+1-0})^T\Delta u_{l3+1-0} - \\
& (\Delta T_{l3+1-0})^T u_{l3+1-0}^i + (T_{l3-0}^i + \Delta T_{l3-0})^T\Delta u_{l3-0}^i + \\
& (\Delta T_{l3-0})^T u_{l3-0}^i - (T_{l3-1+0}^i + \Delta T_{l3-1+0})^T\Delta u_{l3-1+0} - \\
& (\Delta T_{l3-1+0})^T u_{l3-1+0}^i + \sum_{l4=mn+1}^{(2m+1)n}[(T_{l4+0}^i + \\
& \Delta T_{l4+0})^T u_{l4+0}^i + (\Delta T_{l4+0})^T u_{l4+0}^i - (T_{l4+1-0}^i + \\
& (\Delta T_{l4+1-0})^T\Delta u_{l4+1-0} - \Delta T_{l4+1-0})^T u_{l4+1-0}^i + \\
& (T_{l4-0}^i + \Delta T_{l4-0})^T\Delta u_{l4-0} + (\Delta T_{l4-0})^T u_{l4-0}^i - \\
& (T_{l4-1+0}^i + \Delta T_{l4-1+0})^T\Delta u_{l4-1+0} - (\Delta T_{l4-1+0})^T\cdot \\
& u_{l4-1+0}^i]\Big\}dt = T_1 + T_2 - T_3 + T_4 + T_5 - \\
& T_6 + T_7 + T_8 = 0
\end{aligned}
\tag{12}
$$

式(12)是本文给出的一个重要关系式,在力学上可称为平面框架结构动力学增量广义虚功原理. 从该式出发,不仅能系统建立增量虚功原理和平面框架结构材料非线性动力学的非传统 Hamilton 型增量变分原理,而且能清楚地阐明这些原理之间的内在联系.

当 $\Delta p, \Delta Q$ 满足方程(3a,c)和条件(6b),(7b,d),(8e-h);Δu 满足方程(1a,b),(4a,b)和条件(6a), (7a,c),(8a－d)时,由式(12)可得

$$
\begin{aligned}
&\sum_{j=1}^{ml}\int_0^{t_1}\int_{l_j}[(Q_j^i+\Delta Q_j)^T\Delta\boldsymbol{\kappa}_j+\Delta Q_j^T\boldsymbol{\kappa}_j^i-\\
&(p_j^i+\Delta p_j)^T\Delta v_j-\Delta p_j^Tv_j^i]dsdt+\\
&\sum_{k=1}^{mz}\int_0^{t_1}\int_{l_k}[(Q_k^i+\Delta Q_k)^T\Delta\boldsymbol{\kappa}_k+\Delta Q_k^T\boldsymbol{\kappa}_k^i-(p_k^i+\\
&\Delta p_k)^T\Delta v_k-\Delta p_k^Tv_k^i]dsdt=\sum_{j=1}^{ml}\int_0^{t_1}\int_{l_j}[(f_j^i+\\
&\Delta f_j-c\dot{u}_j^i-c\Delta\dot{u}_j)^T\Delta u_j+({}^{-}_{\Delta}u_j^i]dsdt+\\
&\sum_{k=1}^{mz}\int_0^{t_1}\int_{l_k}[(f_k^i+\Delta f_k-c\dot{u}_k^i-c\Delta\dot{u}_k]^T\Delta u_k+\\
&(\Delta f_k-c\Delta\dot{u}_k)^Tu_k^i]dsdt+\\
&\int_0^{t_1}\{\sum_{l2=1}^{mf}[(T_{l2}^i+\Delta T_{l2})^T\Delta u_{l2}+\Delta T_{l2}^Tu_{l2}^i]+\\
&\sum_{l1=1}^{ms}[(T_{l1}^i+\Delta T_{l1})^T\Delta u_{l1}+\Delta T_{l1}^Tu_{l1}^i]\}dt-\\
&\int_{l_j}\{[u_j^i(s,t_1)+\Delta u_j(s,t_1)]^T\Delta p_j(s,t_1)+\\
&\Delta u_j^T(s,t_1)p_j^i(s,t_1)-[\tilde{u}_{0j}^i(s)+\Delta\tilde{u}_{0j}(s)]^T\cdot\\
&\Delta\tilde{p}_{0j}(s)-\Delta u_{0j}^T(s)\tilde{p}_{0j}^i(s)\}ds\}-\\
&\int_{l_k}\{[u_k^i(s,t_1)+\Delta u_k(s,t_1)]^T\Delta p_k(s,t_1)+\\
&\Delta u_k^T(s,t_1)p_k^i(s,t_1)-\tilde{u}_{0k}^i(s)+\Delta\tilde{u}_{0k}(s)]^T\cdot\\
&\Delta\tilde{p}_{0k}(s)-\Delta\tilde{u}_{0k}^T(s)\tilde{p}_{0k}^i(s)\}ds\}
\end{aligned}
\tag{13}
$$

式(13)可以看成是平面框架结构动力学增量虚功原理的表式,它反映广义动力可能状态与广义运动可能状态之间的最一般关系,或者说,它反映阴变量 $\Delta u,\Delta v,\Delta\boldsymbol{\kappa}$ 与阳变量 $\Delta f,\Delta p,\Delta Q$ 这两组对偶变量之间的最一般关系.

3 平面框架结构折线型弹塑性动力学增量变分原理

3.1 5 类变量增量变分原理

对于互不相关的任意函数 p 和 v,有下列关系式:

$$p^Tv=K(v)+K^*(p)-B(v,p) \tag{14a}$$

式中,$K(v)=\rho Av^Tv/2$, $K^*(p)=p^Tp/2\rho A$, $B(v,p)=(\rho Av-p)^T(\rho Av-p)/2\rho A$, $K(v)$ 和 $K^*(p)$ 分别为动能密度和余动能密度.

根据式(14),对于增量段,可得

$$p^{iT}v^i=K(v^i)+K^*(p^i)-B(v^i,p^i) \tag{14b}$$

$$
\begin{aligned}
&(p^i+\Delta p)^T(v^i+\Delta v)=K(v^i+\Delta v)+K^*(p^i+\Delta p)-\\
&B(v^i+\Delta v,p^i+\Delta p)
\end{aligned}
\tag{14c}
$$

将式(14c)和式(14b)相减,可得

$$
\begin{aligned}
&(p^i+\Delta p)^T\Delta v+\Delta p^Tv^i=K(v^i+\Delta v)-K(v^i)+\\
&K^*(p^i+\Delta p)-K^*(p^i)-[B(v^i+\Delta v,p^i+\\
&\Delta p)-B(v^i,p^i)]
\end{aligned}
\tag{15a}
$$

式中

$$
\begin{aligned}
&K(v^i+\Delta v)-K(v^i)=\rho A(v^i+\Delta v)^T(v^i+\\
&\Delta v)/2-\rho Av^{iT}v^i/2=p^{iT}\Delta v+K^{\Delta}(\Delta v)
\end{aligned}
\tag{15b}
$$

$$
\begin{aligned}
&K^*(p^i+\Delta p)-K^*(p^i)=\\
&(p^i+\Delta p)^T(p^i+\Delta p)/2\rho A-p^{iT}p^i/2\rho A=\\
&v^{iT}\Delta p+K^{\Delta *}(\Delta p)
\end{aligned}
\tag{15c}
$$

$$
\begin{aligned}
&B(v^i+\Delta v,p^i+\Delta p)-B(v^i,p^i)=[(\rho Av^i+\\
&\rho A\Delta v-p^i-\Delta p)^T(\rho Av^i+\rho A\Delta v-p^i-\Delta p)-\\
&(\rho Av^i-p^i)^T(\rho Av^i-p^i)]/2\rho A=B^{\Delta}(\Delta v,\Delta p)
\end{aligned}
\tag{15d}
$$

其中

$$
\begin{aligned}
&B^{\Delta}(\Delta v,\Delta p)=(\rho Av^i-p^i)^T(\rho A\Delta v-\Delta p)/\rho A+\\
&(\rho A\Delta v-\Delta p)^T(\rho A\Delta v-\Delta p)/2\rho A,
\end{aligned}
$$

$K^{\Delta}(\Delta v)=\rho A\Delta v^T\Delta v/2$, $K^{\Delta *}(\Delta p)=\Delta p^T\Delta p/2\rho A$

由(3.2),(3.2a－c)式可得

$$
\begin{aligned}
&(p^i+\Delta p)^T\Delta v+\Delta p^Tv^i=p^{iT}\Delta v+K^{\Delta}(\Delta v)+\\
&\Delta p^Tv^i+K^{\Delta *}(\Delta p)-B^{\Delta}(\Delta v,\Delta p)
\end{aligned}
\tag{16}
$$

当且仅当 $\Delta p,\Delta v$ 之间满足(2a,b)式时,才有

$$
\begin{aligned}
&(p^i+\Delta p)^T\Delta v+\Delta p^Tv^i=p^{iT}\Delta v+K^{\Delta}(\Delta v)+\\
&\Delta p^Tv^i+K^{\Delta *}(\Delta p)
\end{aligned}
\tag{17}
$$

当 Q 与 $\boldsymbol{\kappa}$ 分别是互不相关的任意函数时,可以得到下列关系式

$$Q^T\boldsymbol{\kappa}=\Phi(\boldsymbol{\kappa})+\Psi(Q)+A(Q,\boldsymbol{\kappa}) \tag{18a}$$

式中

$$A(Q,\boldsymbol{\kappa})=(Q-[D]\boldsymbol{\kappa})^T(\boldsymbol{\kappa}-[D]^{-1}Q)/2 \tag{18b}$$

根据式(18a),对于增量段,可得

$$Q^{iT}\boldsymbol{\kappa}^i=\Phi(\boldsymbol{\kappa}^i)+\Psi(Q^i)+A(\boldsymbol{\kappa}^i,Q^i) \tag{19a}$$

$$
\begin{aligned}
&(Q^i+\Delta Q)^T(\boldsymbol{\kappa}^i+\Delta\boldsymbol{\kappa})=\Phi(\boldsymbol{\kappa}^i+\Delta\boldsymbol{\kappa})+\\
&\Psi(Q^i+\Delta Q)+A(\boldsymbol{\kappa}^i+\Delta\boldsymbol{\kappa},Q^i+\Delta Q)
\end{aligned}
\tag{19b}
$$

将式(19b)和式(19a)相减,可得

$$
\begin{aligned}
&(Q^i+\Delta Q)^T(\boldsymbol{\kappa}^i+\Delta\boldsymbol{\kappa})-Q^{iT}\boldsymbol{\kappa}^i=\Phi(\boldsymbol{\kappa}^i+\\
&\Delta\boldsymbol{\kappa})-\Phi(\boldsymbol{\kappa}^i)+\Psi(Q^i+\Delta Q)-\Psi(Q^i)+\\
&A(\boldsymbol{\kappa}^i+\Delta\boldsymbol{\kappa},Q^i+\Delta Q)-A(\boldsymbol{\kappa}^i,Q^i)
\end{aligned}
\tag{20}
$$

考虑到增量段线性化的广义内力与广义应变关系,于是有

$$
\begin{aligned}
&\Phi(\boldsymbol{\kappa}^i+\Delta\boldsymbol{\kappa})-\Phi(\boldsymbol{\kappa}^i)=[(\boldsymbol{\kappa}^i+\Delta\boldsymbol{\kappa})^T[D]^i(\boldsymbol{\kappa}^i+\\
&\Delta\boldsymbol{\kappa})-\boldsymbol{\kappa}^{iT}[D]^i\boldsymbol{\kappa}^i]/2=Q^{iT}\Delta\boldsymbol{\kappa}+\Phi^{\Delta}(\Delta\boldsymbol{\kappa})
\end{aligned}
\tag{21}
$$

式中

$$\Phi^{\Delta}(\Delta\boldsymbol{\kappa})=\Delta\boldsymbol{\kappa}^T[D]^i\Delta\boldsymbol{\kappa}/2 \tag{22}$$

同理

$$
\begin{aligned}
&\Psi(Q^i+\Delta Q)-\Psi(Q^i)=[(Q^i+\Delta Q)^T[D]^{i-1}(Q^i+\\
&\Delta Q)-Q^{iT}[D]^{i-1}Q^i]/2=\boldsymbol{\kappa}^{iT}\Delta Q+\Psi^{\Delta}(\Delta Q)
\end{aligned}
\tag{23}
$$

式中

$$\Psi^{\Delta}(\Delta Q)=\Delta Q^{T}[D]^{i-1}\Delta Q/2 \tag{24}$$

而

$$\begin{aligned}A(\kappa^{i}+\Delta\kappa,Q^{i}+\Delta Q)-A(\kappa^{i},Q^{i})=\{&[Q^{i}+\Delta Q-[D]^{i}(\kappa^{i}+\Delta\kappa)]^{T}[\kappa^{i}+\Delta\kappa-[D]^{i-1}(Q^{i}+\\&\Delta Q)]-[(Q^{i}-[D]^{i}\kappa^{i})^{T}(\kappa^{i}-[D]^{i-1}Q^{i})]\}/2=A^{\Delta}(\Delta\kappa,\Delta Q)\end{aligned} \tag{25}$$

式中

$$\begin{aligned}A^{\Delta}(\Delta\kappa,\Delta Q)=&(Q^{i}-[D]^{i}\kappa^{i})^{T}(\Delta\kappa-[D]^{i-1}\Delta Q)+\\&(\Delta Q-[D]^{i}\Delta\kappa)^{T}(\Delta\kappa-[D]^{i-1}\Delta Q)/2\end{aligned} \tag{26}$$

由式(19)～(26)可得

$$\begin{aligned}(Q^{i}+\Delta Q)^{T}\Delta\kappa+\Delta Q^{T}\kappa^{i}=&Q^{iT}\Delta\kappa+\Phi^{\Delta}(\Delta\kappa)+\\&\kappa^{iT}\Delta Q+\Psi^{\Delta}(\Delta Q)+A^{\Delta}(\Delta\kappa,\Delta Q)\end{aligned} \tag{27}$$

当且仅当 ΔQ 和 $\Delta\kappa$ 之间满足式(5)时,才有

$$\begin{aligned}(Q^{i}+\Delta Q)^{T}\Delta\kappa+\Delta Q^{T}\kappa^{i}=&Q^{iT}\Delta\kappa+\Phi^{\Delta}(\Delta\kappa)+\\&\kappa^{iT}\Delta Q+\Psi^{\Delta}(\Delta Q)\end{aligned} \tag{28a}$$

上述的(17)和(27)式是本文给出的广义 Legendre 变换式.

因此,式(12)中 $(p_j^{i}+\Delta p_j)^{T}\Delta\dot{u}_j+\Delta p_j^{T}\dot{u}_j^{i}$ 和 $(p_k^{i}+\Delta p_k)^{T}\Delta\dot{u}_k+\Delta p_k^{T}\dot{u}_k^{i}$ 可分别变换为

$$\begin{aligned}(p_j^{i}+\Delta p_j)^{T}\Delta\dot{u}_j+\Delta p_j^{T}\dot{u}_j^{i}=&p_j^{iT}\Delta v_j+K_j^{\Delta}(\Delta v_j)+\\&v_j^{iT}\Delta p_j+K_j^{\Delta *}(\Delta p_j)-B_j^{\Delta}(\Delta p_j,\Delta v_j)-\\&\Delta p_j^{iT}(v_j^{i}+\Delta v_j-\dot{u}_j^{i}-\Delta\dot{u}_j)-p_j^{iT}(\Delta v_j-\Delta\dot{u}_j)\end{aligned} \tag{28b}$$

$$\begin{aligned}(p_k^{i}+\Delta p_k)^{T}\Delta\dot{u}_k+\Delta p_k^{T}\dot{u}_k^{i}=&p_k^{iT}\Delta v_k+K_k^{\Delta}(\Delta v_k^{i})+\\&v_k^{iT}\Delta p_k+K_k^{\Delta *}(\Delta p_k)-B_k^{\Delta}(\Delta p_k,\Delta v_k)-\\&\Delta p_k^{iT}(v_k^{i}+\Delta v_k-\dot{u}_k^{i}-\Delta\dot{u}_k)-p_k^{iT}(\Delta v_k-\Delta\dot{u}_k)\end{aligned} \tag{28c}$$

式(12)的 $(Q_j^{i}+\Delta Q_j)^{T}(B\Delta u_j)+\Delta Q_j^{T}(Bu_j^{i})$ 和 $(Q_k^{i}+\Delta Q_k)^{T}(B\Delta u_k)+\Delta Q_k^{T}(Bu_k^{i})$ 可以变换为

$$\begin{aligned}(Q_j^{i}+\Delta Q_j)^{T}(B\Delta u_j)+\Delta Q_j^{T}(Bu_j^{i})=&Q_j^{iT}\Delta\kappa_j+\\&\Phi_j^{\Delta}(\Delta\kappa_j)+\kappa_j^{iT}\Delta Q_j+A_j^{\Delta}(\Delta\kappa_j,\Delta Q_j)+\\&\Psi_j^{\Delta}(\Delta Q_j)-\Delta Q_j^{iT}[\kappa_j^{i}+\Delta\kappa_j-(Bu_j^{i})-\\&(B\Delta u_j)]-Q_j^{iT}(\Delta\kappa_j-B\Delta u_j)\end{aligned} \tag{29a}$$

$$\begin{aligned}-(Q_k^{i}+\Delta Q_k)^{T}(B\Delta u_k)+\Delta Q_k^{T}(Bu_k^{i})=&Q_k^{iT}\Delta\kappa_k+\\&\Phi_k^{\Delta}(\Delta\kappa_k)+\kappa_k^{iT}\Delta Q_k+A_k^{\Delta}(\Delta\kappa_k,\Delta Q_k)+\\&\Psi_k^{\Delta}(\Delta Q_k)-\Delta Q_k^{iT}[\kappa_k^{i}+\Delta\kappa_k-(Bu_k^{i})-\\&(B\Delta u_k)]-Q_k^{iT}(\Delta\kappa_k-B\Delta u_k)\end{aligned} \tag{29b}$$

将式(12)的 $T_2-T_3+T_5-T_6+T_7+T_8$ 变换为

$$\begin{aligned}T_2-T_3+T_5-T_6+T_7+T_8=&\\&\sum_{j=1}^{ml}\int_0^{t_1}\int_{l_j}[(\dot{p}_j^{i}-BQ_j^{i}-f_j^{i}+c\,\mathring{u}_j^{i})^{T}\Delta u_j+(\Delta\dot{p}_j-\\&B\Delta Q_j-\Delta f_j+c\,\Delta\mathring{u}_j)^{T}(u_j^{i}+\Delta u_j)]dsdt+\\&\sum_{j=1}^{ml}\int_0^{t_1}\int_{l_j}[(f_j^{i}-c\,\mathring{u}_j^{i})^{T}\Delta u_j+\\&(\Delta f_j-c\Delta\mathring{u}_j)^{T}(u_j^{i}+\Delta u_j)]dsdt+\\&\sum_{k=1}^{mz}\Big\{\int_0^{t_1}\int_{l_k}[(\dot{p}_k^{i}-BQ_k^{i}-f_k^{i}+c\,\mathring{u}_j^{i})^{T}\Delta u_k+\\&(\Delta\dot{p}_k-B\Delta Q_k-\Delta f_k+c\Delta\mathring{u}_k)^{T}(u_k^{i}+\\&\Delta u_k)]dsdt+\sum_{k=1}^{mz}\int_0^{t_1}\int_{l_k}[(f_k^{i}-c\,\mathring{u}_k^{i})^{T}\Delta u_k+\\&(\Delta f_k-c\Delta\mathring{u}_k)^{T}(u_k^{i}+\Delta u_k)]dsdt+\\&\Pi_{IB}^{\Delta}+\Pi_{IC}^{\Delta}+\mathring{\Pi}^{\Delta}+\Gamma_{IB}^{\Delta}+\Gamma_{IC}^{\Delta}+\mathring{\Gamma}^{\Delta}\end{aligned} \tag{30}$$

式中

$$\begin{aligned}\mathring{\Pi}^{\Delta}=&-\sum_{j=1}^{ml}\int_{l_j}[(\mathring{p}_{1j}^{i}+\Delta\mathring{p}_{1j})^{T}\Delta u_{1j}+\Delta\mathring{p}_{1j}^{T}u_{1j}^{i}+\\&(\mathring{p}_{0j}^{i}+\Delta\mathring{p}_{0j})^{T}\Delta u_{1j}+\Delta\mathring{p}_{0j}^{T}u_{1j}^{i}]ds-\\&\sum_{k=1}^{mz}\int_{l_k}[(\mathring{p}_{1k}^{i}+\Delta\mathring{p}_{1k})^{T}\Delta u_{1k}+\Delta\mathring{p}_{1k}^{T}u_{1k}^{i}+\\&(\mathring{p}_{0k}^{i}+\Delta\mathring{p}_{0k}]^{T}\Delta u_{1k}+\Delta\mathring{p}_{0k}^{T}u_{1k}^{i}]ds\end{aligned}$$

$$\begin{aligned}\mathring{\Gamma}^{\Delta}=&-\sum_{j=1}^{ml}\int_{l_j}[(\mathring{u}_{1j}^{i}+\Delta\mathring{u}_{1j})^{T}\Delta p_{1j}+\Delta\mathring{u}_{1j}^{T}p_{1j}^{i}-\\&(\mathring{p}_{0j}^{i}+\Delta\mathring{p}_{0j})^{T}\Delta u_{1j}-\Delta\mathring{p}_{0j}^{T}u_{1j}^{i}]ds-\\&\sum_{k=1}^{mz}\int_{l_k}[(\mathring{u}_{1k}^{i}+\Delta\mathring{u}_{1k})^{T}\Delta p_{1k}+\Delta\mathring{u}_{0k}^{T}p_{1k}^{i}-\\&(\mathring{p}_{0k}^{i}+\Delta\mathring{p}_{0k})^{T}\Delta u_{1k}-\Delta\mathring{p}_{0k}^{T}u_{1k}^{i}]ds\end{aligned}$$

$$\begin{aligned}\Pi_{IB}^{\Delta}=&\int_0^{t_1}\{\sum_{l2=1}^{mf}[(\bar{T}_{l2}^{i}+\Delta\bar{T}_{l2})^{T}\Delta u_{l2}+\Delta\bar{T}_{l2}^{T}u_{l2}^{i}]+\\&\sum_{l1=1}^{ms}[(T_{l1}^{i}+\Delta T_{l1})^{T}(\Delta u_{l1}-\Delta\bar{u}_{l1})+\Delta T_{l1}^{T}(u_{l1}^{i}-\\&\bar{u}_{l1}^{i})]\}dt+\sum_{j=1}^{ml}\int_{l_j}[(\tilde{p}_{0j}^{i}+\Delta\tilde{p}_{0j})^{T}\Delta u_{1j}+\\&\Delta\tilde{p}_{0j}^{T}u_{1j}^{i}-(\tilde{u}_{0j}^{i}+\Delta\tilde{u}_{0j}-u_{0j}^{i}-\Delta u_{0j})^{T}\Delta p_{0j}-\\&(\Delta\tilde{u}_{0j}-\Delta u_{0j})^{T}p_{0j}^{i}]ds+\sum_{k=1}^{mz}\int_{l_k}[(\tilde{p}_{0k}^{i}+\\&\Delta\tilde{p}_{0k})^{T}\Delta u_{1k}+\Delta\tilde{p}_{0k}^{T}u_{1k}^{i}-(\tilde{u}_{0k}^{i}+\Delta\tilde{u}_{0k}-\\&u_{0k}^{i}-\Delta u_{0k})^{T}\Delta p_{0k}-(\Delta\tilde{u}_{0k}-\Delta u_{0k})^{T}p_{0k}^{i}]ds\end{aligned}$$

$$\begin{aligned}\Gamma_{IB}^{\Delta}=&\int_0^{t_1}\{\sum_{l2=1}^{mf}[(T_{l2}^{i}+\Delta T_{l2}-\bar{T}_{l2}^{i}-\\&\Delta\bar{T}_{l2})^{T}\Delta u_{l2}+(\Delta T_{l2}-\Delta\bar{T}_{l2})^{T}u_{l2}^{i}]+\\&\sum_{l1=1}^{ms}[(T_{l1}^{i}+\Delta T_{l1})^{T}\Delta\bar{u}_{l1}+\Delta T_{l1}^{T}\bar{u}_{l1}^{i}]\}dt+\\&\sum_{j=1}^{ml}\int_{l_j}[(\tilde{u}_{0j}^{i}+\Delta\tilde{u}_{0j})^{T}\Delta p_{0j}+\Delta\tilde{u}_{0j}^{T}p_{0j}^{i}-\\&(\tilde{p}_{0j}^{i}+\Delta\tilde{p}_{0j})^{T}\Delta u_{1j}-\Delta\tilde{p}_{0j}^{T}u_{1j}^{i}]ds+\\&\sum_{k=1}^{mz}\int_{l_k}[(\tilde{u}_{0k}^{i}+\Delta\tilde{u}_{0k})^{T}\Delta p_{0k}+\Delta\tilde{u}_{0k}^{T}p_{0k}^{i}-\\&(\tilde{p}_{0k}^{i}+\Delta\tilde{p}_{0k})^{T}\Delta u_{1k}-\Delta\tilde{p}_{0k}^{T}u_{1k}^{i}]ds\end{aligned}$$

$$\Pi_{IC}^{\Delta}=\int_0^{t_1}\{\sum_{l3=1}^{mn}[\Delta T_{l3+0}^{T}(u_{l3+0}^{i}+\Delta u_{l3+0}-\mathring{u}_{j1,k1}^{i}-$$

$$
\begin{aligned}
&\Delta\mathring{u}_{j1,k1}) + (T^{i}_{l3+0})^{T}(\Delta u_{l3+0} - \Delta\mathring{u}_{j1,k1}) - \\
&\Delta T^{T}_{l3+1-0}(u^{i}_{l3+1-0} + \Delta u_{l3+1-0} - \mathring{u}^{i}_{j1,k1} - \Delta\mathring{u}_{j1,k1}) - \\
&(T^{i}_{l3+1-0})^{T}(\Delta u_{l3+1-0} - \Delta\mathring{u}_{j1,k1}) - (T^{i}_{l3+1-0})^{T}\cdot \\
&(\Delta u_{l3+1-0} - \Delta\mathring{u}_{j1,k1}) - (\mathring{T}^{i}_{j1,k1} + \Delta\mathring{T}^{i}_{j1,k1})^{T}\Delta u_{l3+0} - \\
&\Delta\mathring{T}^{T}_{j1,k1}u^{i}_{l3+0} + (\mathring{T}^{i}_{j1,k1} + \Delta\mathring{T}_{j1,k1})^{T}\Delta u_{l3+1-0} + \\
&\Delta\mathring{T}^{T}_{j1,k1}u^{i}_{l3+1-0} + \Delta T^{T}_{l3-0}(u^{i}_{l3-0} + \Delta u_{l3-0} - \mathring{u}^{i}_{j1-1,k1} - \\
&\Delta\mathring{u}_{j1-1,k1}) + (T^{i}_{l3-0})^{T}(\Delta u_{l3-0} - \Delta\mathring{u}_{j1-1,k1}) - \\
&\Delta T^{T}_{l3-1+0}(u^{i}_{l3-1+0} + \Delta u_{l3-1+0} - \mathring{u}^{i}_{j1-1,k1} - \Delta\mathring{u}_{j1-1,k1}) - \\
&(T^{i}_{l3-1+0})^{T}(\Delta u_{l3-1+0} - \Delta\mathring{u}_{j1-1,k1}) - (T^{i}_{j1-1,k1} + \\
&\Delta\mathring{T}^{i}_{j1-1,k1})^{T}\Delta u_{l3-0} - \Delta\mathring{T}^{T}_{j1-1,k1}u^{i}_{l3-0} + (T^{i}_{j1-1,k1} + \\
&\Delta\mathring{T}^{i}_{j1-1,k1})^{T}\Delta u_{l3-1+0} + \Delta\mathring{T}^{T}_{j1-1,k1}u^{i}_{l3-1+0}] + \\
&\sum_{l4=mn+1}^{(2m+1)n}[\Delta T^{T}_{l4+0}(u^{i}_{l4+0} + \Delta u_{l4+0} - \mathring{u}^{i}_{j1,k1} - \Delta\mathring{u}_{j1,k1}) + \\
&(T^{i}_{l4+0})^{T}(\Delta u_{l4+0} - \Delta\mathring{u}_{j1,k1}) - \Delta T^{T}_{l4+1-0}(u^{i}_{l4+1-0} + \\
&\Delta u_{l4+1-0} - \mathring{u}^{i}_{j1,k1} - \Delta\mathring{u}_{j1,k1}) + (T^{i}_{l4+1-0})^{T}(\Delta u_{l4+1-0} - \\
&\Delta\mathring{u}_{j1,k1}) - (T^{i}_{j1,k1} + \Delta\mathring{T}^{i}_{j1,k1})^{T}\Delta u_{l4+0} - \\
&\Delta\mathring{T}^{T}_{j1,k1}u^{i}_{l4+0} + (\mathring{T}^{i}_{j1,k1} + \Delta\mathring{T}_{j1,k1})^{T}\Delta u_{l4+1-0} + \\
&\Delta\mathring{T}^{T}_{j1,k1}u^{i}_{l4+1-0} + \Delta T^{T}_{l4-0}(u^{i}_{l4-0} + \Delta u_{l4-0} - \mathring{u}^{i}_{j1,k1-1} - \\
&\Delta\mathring{u}_{j1,k1-1}) + (T^{i}_{l4-0})^{T}(\Delta u_{l4-0} - \Delta\mathring{u}_{j1,k1-1}) - \\
&\Delta T^{T}_{l4-1+0}(u^{i}_{l4-1+0} + \Delta u_{l4-1+0} - \mathring{u}^{i}_{j1,k1-1} - \Delta\mathring{u}_{j1,k1-1}) - \\
&(T^{i}_{l4-1+0})^{T}(\Delta u_{l4-1+0} - \Delta\mathring{u}_{j1,k1-1}) - (T^{i}_{j1,k1+1} + \\
&\Delta\mathring{T}_{j1,k1+1})^{T}\Delta u_{l4-0} - \Delta\mathring{T}^{T}_{j1,k1-1}u^{i}_{l4-0} + (T^{i}_{j1,k1-1} + \\
&\Delta\mathring{T}_{j1,k1-1})^{T}\Delta u_{l4-1+0} + \Delta\mathring{T}^{T}_{j1,k1-1}u^{i}_{l4-1+0}]\}dt
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\Delta}_{IC} = &\int_{0}^{t_1}\{\sum_{l3=1}^{mn}[(T^{i}_{l3+0} + \Delta T_{l3+0} - \mathring{T}^{i}_{j1,k1} - \\
&\Delta\mathring{T}_{j1,k1})^{T}\Delta u_{l3+0} + (\Delta T_{l3+0} - \mathring{T}^{i}_{j1,k1})^{T}u^{i}_{l3+0} - \\
&(T^{i}_{l3+1-0} + \Delta T_{l3+1-0} - \mathring{T}^{i}_{j1,k1} - \Delta\mathring{T}_{j1,k1})^{T}\Delta u_{l3+1-0} - \\
&(\Delta T_{l3+1-0} - \mathring{T}^{i}_{j1,k1})^{T}\Delta u^{i}_{l3+1-0} - \Delta T^{T}_{l3+0}(\mathring{u}^{i}_{j1,k1} + \\
&\Delta\mathring{u}_{j1,k1}) - (T^{i}_{l3+0})^{T}\Delta\mathring{u}_{j1,k1} + \Delta T^{T}_{l3+1-0}(\mathring{u}^{i}_{j1,k1} + \\
&\Delta\mathring{u}_{j1,k1}) + (T^{i}_{l3+1-0})^{T}\Delta\mathring{u}_{j1,k1} + (T^{i}_{l3-0} + \\
&\Delta T_{l3-0} - T^{i}_{j1-1,k1} - \Delta T_{j1-1,k1})^{T}\Delta u_{l3-0} + \\
&(\Delta T_{l3-0} - \Delta\mathring{T}_{j1-1,k1})^{T}u^{i}_{l3-0} - \Delta T^{i}_{l3-0}(\mathring{u}^{i}_{j1,k1} + \\
&\Delta u_{j1-1,k1}) + (\Delta T_{l3-1+0} - \Delta\mathring{T}_{j1-1,k1})^{T}u^{i}_{l3-1+0} + \\
&(T^{i}_{l3-1+0})^{T}\Delta\mathring{u}_{j1-1,k1}] + \sum_{l4=mn+1}^{(2m+1)n}[(T^{i}_{l4+0} + \\
&\Delta T_{l4+0} - \mathring{T}^{i}_{j1,k1} - \Delta\mathring{T}_{j1,k1})^{T}\Delta u_{l4+0} + (\Delta T_{l4+0} - \\
&\mathring{T}^{i}_{j1,k1})^{T}u^{i}_{l4+0} - (T^{i}_{l4+1-0} + \Delta T_{l4+1-0} - \mathring{T}^{i}_{j1,k1} - \\
&\mathring{T}_{j1,k1})^{T}u_{l4+1-0} - (\Delta T_{l4+1-0} - \mathring{T}^{i}_{j1,k1})^{T}u^{i}_{l4+1-0} - \\
&\Delta T^{T}_{l4+0}(\mathring{u}^{i}_{j1,k1} + \Delta\mathring{u}_{j1,k1}) - (T^{i}_{l4+0})^{T}\Delta\mathring{u}_{j1,k1} + \\
&\Delta T^{T}_{l4+1-0}(\mathring{u}^{i}_{j1,k1} + \Delta\mathring{u}_{j1,k1}) + (T^{i}_{l4+1-0})^{T}\Delta\mathring{u}_{j1,k1} + \\
&(T^{i}_{l4-0} + \Delta T_{l4-0} - T^{i}_{j1,k1-1} - \Delta\mathring{T}_{j1,k1-1})^{T}\Delta u_{l4-0} + \\
&(\Delta T_{l4-0} - \Delta T_{j1,k1-1})^{T}u^{i}_{l4-0} - (T^{i}_{l4-1+0} + \\
&\Delta T_{l4-1+0} - T^{i}_{j1,k1-1} - \Delta\mathring{T}_{j1,k1-1})^{T}\Delta u_{l4-1+0} - \\
&(\Delta T_{l4-0} - \Delta\mathring{T}_{j1,k1-1})^{T}u^{i}_{l4-0} - \Delta T^{T}_{l4-0}(\mathring{u}^{i}_{j1,k1-1} + \\
&\Delta\mathring{u}_{j1,k1-1}) + (\Delta T_{l4-1+0} - \Delta\mathring{T}_{j1,k1-1})^{T}u^{i}_{l4-1+0} + \\
&(T^{i}_{l4-1+0})^{T}\Delta\mathring{u}_{j1,k1-1}]\}dt
\end{aligned}
$$

其中带上标°的量为限制变分量.

将(28a,b),(29a,b)和(30)式代入(12)式中,经整理后可得

$$
\begin{aligned}
&\Pi^{\Delta}_{L5}(\Delta p,\Delta v,\Delta Q,\Delta\boldsymbol{\kappa},\Delta u) + \\
&\Gamma^{\Delta}_{L5}(\Delta p,\Delta v,\Delta Q,\Delta\boldsymbol{\kappa},\Delta u) = 0 \qquad (31a)
\end{aligned}
$$

而泛函 Π^{Δ}_{L5} 和 Γ^{Δ}_{L5} 分别为

$$
\begin{aligned}
\Pi^{\Delta}_{L5} = &\sum_{j=1}^{ml}\int_{0}^{t_1}\int_{l_j}\{K^{\Delta}_{j}(\Delta v_j) + p^{i\,T}_{j}\Delta\dot{u}_j - \Delta p^{T}_{j}(v^{i}_{j} + \\
&\Delta v_j - \dot{u}^{i}_{j} - \Delta\dot{u}_j) - Q^{i\,T}_{j}(B\Delta u_j) - \Phi^{\Delta}_{j}(\Delta\boldsymbol{\kappa}_j) + \\
&\Delta Q^{T}_{j}[\boldsymbol{\kappa}^{i}_{j} + \Delta\boldsymbol{\kappa}_j - (Bu^{i}_{j}) - (B\Delta u_j)] + \\
&(f^{i}_{j} - c\dot{u}^{i}_{j})^{T}\Delta u_j + (\Delta f_j - c\dot{u}_j)^{T}(u^{i}_{j} + \Delta u_j)\}dsdt + \\
&\sum_{k=1}^{mz}\int_{0}^{t_1}\int_{l_k}\{K^{\Delta}_{k}(\Delta v_k) + p^{i\,T}_{k}\Delta\dot{u}_k - \Delta p^{T}_{k}(v^{i}_{k} + \\
&\Delta v_k - \dot{u}^{i}_{k} - \Delta\dot{u}_k) - \Phi^{\Delta}_{k}(\Delta\boldsymbol{\kappa}_k) - Q^{i\,T}_{k}(B\Delta u_k) + \\
&\Delta Q^{T}_{k}[\boldsymbol{\kappa}^{i}_{k} + \Delta\boldsymbol{\kappa}_k - (Bu^{i}_{k}) - (B\Delta u_k)] + \\
&(f^{i}_{k} - c\,\dot{\mathring{u}}^{i}_{k})^{T}\Delta u_k + (\Delta f_k - c\,\dot{\mathring{u}}^{i}_{k})^{T}(u^{i}_{k} + \\
&\Delta u_k)\}dsdt + \Pi^{\Delta}_{IB} + \Pi^{\Delta}_{IC} + \mathring{\Pi}^{\Delta} \qquad (31b)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^{\Delta}_{L5} = &\sum_{j=1}^{ml}\int_{0}^{t_1}\int_{l_j}[v^{i\,T}_{j}\Delta p_j + K^{\Delta *}_{j}(\Delta p_j) - \boldsymbol{\kappa}^{i\,T}_{j}\Delta Q_j - \\
&\Psi^{\Delta}_{j}(\Delta Q_j) - B^{\Delta}_{j}(\Delta v_j,\Delta p_j) - A^{\Delta}_{j}(\Delta\boldsymbol{\kappa}_j,\Delta Q_j) - \\
&(BQ^{i}_{j} + f^{i}_{j} - \dot{p}^{i}_{j} - c\,\dot{\mathring{u}}^{i}_{j})^{T}\Delta u_j - (B\Delta Q_j + \\
&\Delta f_j - \Delta\dot{p}^{j} - c\Delta\,\dot{\mathring{u}}_j)^{T}(u^{i}_{j} + \Delta u_j)]dsdt + \\
&\sum_{k=1}^{mz}\int_{0}^{t_1}\int_{l_k}[v^{i\,T}_{k}\Delta p_k + K^{\Delta *}_{k}(\Delta p_k) - \boldsymbol{\kappa}^{i\,T}_{k}\Delta Q_k - \\
&\Psi^{\Delta}_{k}(\Delta Q_k) - B^{\Delta}_{k}(\Delta v_k,\Delta p_k) - A^{\Delta}_{k}(\Delta\boldsymbol{\kappa}_k,\Delta Q_k) - \\
&(BQ^{i}_{k} + f^{i}_{k} - \dot{p}^{i}_{k} - c\,\dot{\mathring{u}}^{i}_{k})^{T}\Delta u_k - (B\Delta Q_k + \\
&\Delta f_k - \Delta\dot{p}^{k} - c\Delta\,\dot{\mathring{u}}_k)^{T}(u^{i}_{k} + \Delta u_k)]dsdt + \\
&\Gamma_{IC} + \Gamma^{\Delta}_{IB} + \mathring{\Gamma}^{\Delta} \qquad (31c)
\end{aligned}
$$

定理 1 当且仅当 $\Delta\boldsymbol{p},\Delta\boldsymbol{v},\Delta\boldsymbol{Q},\Delta\boldsymbol{\kappa},\Delta\boldsymbol{u}$ 是混合问题(1a,b),(2a,b),(3a,c),(4a,b),(5a,c),(6a,b),(7a-d),(8a-h)式的解,则必定满足下列变分式

$$
\delta\Pi^{\Delta}_{L5} = 0 \quad 或 \quad \delta\Gamma^{\Delta}_{L5} = 0 \qquad (32)
$$

证明 将(31b)式对自变函数 $\Delta p,\Delta v,\Delta Q,\Delta\boldsymbol{\kappa},\Delta u$ 变分,可得

$$
\begin{aligned}
\delta\Pi^{\Delta}_{L5} = &\sum_{j=1}^{ml}\{\int_{0}^{t_1}\int_{l_j}\{(\rho_j A_j\Delta v_j - \Delta p_j)^{T}\delta\Delta v_j - \\
&\delta\Delta p^{T}_{j}(v^{i}_{j} + \Delta v_j - \dot{u}^{i}_{j} - \Delta\dot{u}_j) + \delta\Delta Q^{T}_{j}[\boldsymbol{\kappa}^{i}_{j} +
\end{aligned}
$$

$$
\begin{aligned}
&\Delta\boldsymbol{\kappa}_j - B(u_j^i + \Delta u_j)] - ([D]_j^i\Delta\boldsymbol{\kappa}_j - \\
&\Delta Q_j)^T\delta\Delta\boldsymbol{\kappa}_j + [BQ_j^i + B\Delta Q_j - \dot{p}_j^i - \Delta\dot{p}_j + \\
&f_j^i + \Delta f_j - c\,\mathring{\dot{u}}_j^i - c\Delta\,\mathring{\dot{u}}_j]^T\delta\Delta u_j\}\,dsdt - \\
&\int_{l_j}[(\mathring{p}_{0j}^i + \Delta\mathring{p}_{0j} - \tilde{p}_{0j}^i - \Delta\tilde{p}_{0j})^T\delta\Delta u_{1j} - \\
&(u_{0j}^i + \Delta u_{0j} - \tilde{u}_{0j}^i - \Delta\tilde{u}_{0j})^T\delta\Delta p_{0j}]\,ds\} + \\
&\sum_{k=1}^{mz}\{\int_0^{t_1}\int_{l_k}\{(\rho_k A_k\Delta v_k - \Delta p_k)^T\delta\Delta v_k - \\
&\delta\Delta p_k^T(v_k^i + \Delta v_k - \dot{u}_k^i - \Delta\dot{u}_k) + \delta\Delta Q_k^T[\boldsymbol{\kappa}_k^i + \\
&\Delta\boldsymbol{\kappa}_k - (B(u_k^i + \Delta u_k)] - ([D]_k^i\Delta\boldsymbol{\kappa}_k - \\
&\Delta Q_k)^T\delta\Delta\boldsymbol{\kappa}_k + [BQ_k^i + B\Delta Q_k - \dot{p}_k^i - \Delta\dot{p}_k + \\
&f_k^i + \Delta f_k - c\,\mathring{\dot{u}}_k^i - c\Delta\,\mathring{\dot{u}}_k]^T\delta\Delta u_k\}\,dsdt - \\
&\int_{l_k}[(\mathring{p}_{0k}^i + \Delta\mathring{p}_{0k} - \tilde{p}_{0k}^i - \Delta\tilde{p}_{0k})^T\delta\Delta u_{1k} - \\
&(u_{0k}^i + \Delta u_{0k} - \tilde{u}_{0k}^i - \Delta\tilde{u}_{0k})^T\delta\Delta p_{0k}]\,ds\} + \\
&\int_0^{t_1}\{\sum_{l3=1}^{mn}[\delta\Delta T_{l3+0}^T(u_{l3+0}^i + \Delta u_{l3+0} - u_{j1,k1}^i - \\
&\Delta u_{j1,k1}) - \delta\Delta T_{l3+1-0}^T(u_{l3+1-0}^i + \Delta u_{l3+1-0} - \\
&u_{j1,k1}^i - \Delta u_{j1,k1}) + (T_{l3+0}^i + \Delta T_{l3+0} - T_{j1,k1}^i - \\
&\Delta T_{j1,k1})^T\delta\Delta u_{l3+0} - (T_{l3+1-0}^i + \Delta T_{l3+1-0} - \\
&T_{j1,k1}^i - \Delta T_{j1,k1})^T\delta\Delta u_{l3+1-0} + \delta\Delta T_{l3-0}^T(u_{l3-0}^i + \\
&\Delta u_{l3-0} - u_{j1,k1-1}^i - \Delta u_{j1,k1-1}) - \\
&\delta\Delta T_{l3-1+0}^T(u_{l3-1+0}^i + \Delta u_{l3-1+0} - u_{j1,k1-1}^i - \\
&\Delta u_{j1,k1-1}) + (T_{l3-0}^i + \Delta T_{l3-0} - T_{j1-1,k1}^i - \\
&\Delta T_{j1-1,k1})^T\delta\Delta u_{l3-0} - (T_{l3-1+0}^i + \Delta T_{l3-1+0} - \\
&T_{j1-1,k1}^i - \Delta T_{j1-1,k1})^T\delta\Delta u_{l3-1+0} + \\
&\int_0^{t_1}\{\sum_{l4=mn+1}^{(2m+1)n}[\delta\Delta T_{l4+0}^T(u_{l4+0}^i + \Delta u_{l4+0} - u_{j1,k1}^i - \\
&\Delta u_{j1,k1}) - \delta\Delta T_{l4+1-0}^T(u_{l4+1-0}^i + \Delta u_{l4+1-0} - \\
&u_{j1,k1}^i - \Delta u_{j1,k1}) + (T_{l4+0}^i + \Delta T_{l4+0} - T_{j1,k1}^i - \\
&\Delta T_{j1,k1})^T\delta\Delta u_{l4+0} - (T_{l4+1-0}^i + \Delta T_{l4+1-0} - \\
&T_{j1,k1}^i - \Delta T_{j1,k1})^T\delta\Delta u_{l4+1-0} + \\
&\delta\Delta T_{l4-0}^T(u_{l4-0}^i + \Delta u_{l4-0} - u_{j1,k1-1}^i - \Delta u_{j1,k1-1}) - \\
&\delta\Delta T_{l4-1+0}^T(u_{l4-1+0}^i + \Delta u_{l4-1+0} - u_{j1,k1-1}^i - \\
&\Delta u_{j1,k1-1}) + (T_{l4-0}^i + \Delta T_{l4-0} - T_{j1,k1-1}^i - \\
&\Delta T_{j1,k1-1})^T\delta\Delta u_{l4-0} + (T_{l4-1+0}^i + \Delta T_{l4-1+0} - \\
&T_{j1,k1-1}^i - \Delta T_{j1,k1-1})^T\delta\Delta u_{l4-1+0} + \\
&\int_0^{t_1}[\sum_{l1=1}^{ms}(u_{l1}^i + \Delta u_{l1} - \bar{u}_{l1}^i - \Delta\bar{u}_{l1})^T\delta\Delta T_{l1} - \\
&\sum_{l2=1}^{mf}(T_{l2}^i + \Delta T_{l2} - \bar{T}_{l2}^i - \Delta\bar{T}_{l2})^T\delta\Delta u_{l2}]\,dt
\end{aligned}
\tag{33}
$$

充分性　若 $\Delta\boldsymbol{p},\Delta\boldsymbol{v},\Delta\boldsymbol{Q},\Delta\boldsymbol{\kappa},\Delta\boldsymbol{u}$ 是混合问题(1a,b)～(8a－h)式的解，则(33)式就变成为 $\delta\Pi_{L5}^{\Delta}=0$，即(32)式成立.

必要性　若(32)式成立，即 $\delta\Pi_{L5}^{\Delta}=0$，注意到(33)式，由于 $\delta\Delta\boldsymbol{p},\delta\Delta\boldsymbol{v},\delta\Delta\boldsymbol{Q},\delta\Delta\boldsymbol{\kappa},\delta\Delta\boldsymbol{u}$ 的任意性，并根据变分法的有关引理，故由此可得(1a,b)，(2a,b)，(3a,c)，(4a,b)，(5a,c)，(6a,b)，(7a－d)，(8a－h)式，即 $\Delta\boldsymbol{p},\Delta\boldsymbol{v},\Delta\boldsymbol{Q},\Delta\boldsymbol{\kappa},\Delta\boldsymbol{u}$ 是混合问题(1a,b)～(8a－h)式的解.

Π_{L5}^{Δ} 和 Γ_{L5}^{Δ} 分别是 5 类变量平面框架结构折线型弹塑性动力学非传统 Hamilton 型增量变分原理的势能形式和余能形式的泛函. 对于任意无关的 $\Delta p,\Delta v,\Delta Q,\Delta\kappa,\Delta u$ 它们之间存在互补关系(31a).

3.2　1 类变量增量变分原理

当 $\Delta\boldsymbol{p},\Delta\boldsymbol{v},\Delta\boldsymbol{u}$ 满足(1a,b)和(2a,b)式，$\Delta\boldsymbol{\kappa}$，$\Delta\boldsymbol{u}$ 满足(4a,b)式时，泛函 Π_{L5}^{Δ} 就变成为

$$
\begin{aligned}
\Pi_{L1}^{\Delta} = &\sum_{j=1}^{ml}\int_0^{t_1}\int_{l_j}\{K_j^{\Delta}(\Delta\dot{u}_j) + \rho_j A_j\dot{u}_j^{iT}\Delta\dot{u}_j - \\
&[B(u_j^i + \Delta u_j/2)]^T[D]_j^i(B\Delta u_j) + (f_j^i - \\
&c\,\mathring{\dot{u}}_j^i)^T\Delta u_j + (\Delta f_j - c\Delta\,\mathring{\dot{u}}_j)^T(u_j^i + \Delta u_j)\}\,dsdt - \\
&\sum_{j=1}^{ml}\int_{l_j}\rho_j A_j[(\mathring{\dot{u}}_{1j}^i + \Delta\,\mathring{\dot{u}}_{1j} + \mathring{\dot{u}}_{0j}^i + \Delta\,\mathring{\dot{u}}_{0j})^T\Delta u_{1j} + \\
&(\Delta\,\mathring{\dot{u}}_{1j}^T + \Delta\,\mathring{\dot{u}}_{0j}^T)^T u_{1j}^i]\,ds + \sum_{k=1}^{mz}\int_0^{t_1}\int_{l_k}\{K_k^{\Delta}(\Delta\dot{u}_k) + \\
&\rho_k A_k\dot{u}_k^{iT}\Delta\dot{u}_k - [B(u_k^i + \Delta u_k/2)]^T\cdot \\
&[D]_k^i(B\Delta u_k) + (f_k^i - c\,\mathring{\dot{u}}_k^i)^T\Delta u_k + \\
&(\Delta f_k - c\Delta\,\mathring{\dot{u}}_k)^T(u_k^i + \Delta u_k)\}\,dsdt - \\
&\sum_{k=1}^{mz}\int_{l_k}\rho_k A_k[(\Delta\,\mathring{\dot{u}}_{1k}^T + \Delta\,\mathring{\dot{u}}_{0k}^T)^T u_{1k}^i + (\mathring{\dot{u}}_{1k}^i + \\
&\Delta\,\mathring{\dot{u}}_{1k} + \mathring{\dot{u}}_{0k}^i + \Delta\,\mathring{\dot{u}}_{0k})^T\Delta u_{1k}]\,ds + \Pi_{IB}^{\Delta} + \Pi_{IC}^{\Delta}
\end{aligned}
\tag{34}
$$

定理 2　当且仅当 $\Delta\boldsymbol{u}$ 是混合问题(6a,b)，(7a－d)，(8a－h)及下式

$$
\begin{cases}
B\{[D]_j^i[(Bu_j^i) + (B\Delta u_j)]\} - \rho_j A_j(\ddot{u}_j^i + \\
\quad\Delta\ddot{u}_j) + f_j^i + \Delta f_j - c\dot{u}_j^i - c\Delta\dot{u}_j = 0 \\
B\{[D]_j^i[(Bu_k^i) + (B\Delta u_k)]\} + \rho_k A_k(\ddot{u}_k^i + \\
\quad\Delta\ddot{u}_k) - f_k^i - \Delta f_k - c\dot{u}_k^i - c\Delta\dot{u}_k = 0
\end{cases}
\tag{35}
$$

的解，则必定满足变分式 $\delta\Pi_{L1}^{\Delta}=0$.

Π_{L1}^{Δ} 是 1 类变量平面框架结构折线型弹塑性动力学非传统 Hamilton 型增量变分原理势能形式的泛函.

3.3　相空间非传统 Hamilton 型增量变分原理

当 $\Delta\boldsymbol{p}$ 与 $\Delta\boldsymbol{v}$ 和 $\Delta\boldsymbol{\kappa}$ 与 $\Delta\boldsymbol{u}$ 分别满足(2a,b)式和(4a,b)式时，泛函 Π_{L5}^{Δ} 就变为

$$\widetilde{\Pi}_{I2}^{\Delta}(\Delta p,\Delta u) = \sum_{j=1}^{ml}\int_0^{t_1}\int_{l_j}[(p_j^i+\Delta p_j)\Delta\dot{u}_j + \Delta p_j\dot{u}^i - H_j^i(\Delta p,\Delta u)]dsdt + \sum_{k=1}^{mz}\int_0^{t_1}\int_{l_k}[(p_k^i+\Delta p_j)\Delta\dot{u}_k - H_k^i(\Delta p_k,\Delta u_k)]dsdt + \Pi_{IB}^{\Delta}+\Pi_{IC}^{\Delta}+\mathring{\Pi}^{\Delta} \quad (36)$$

式中 Hamilton 函数 $H(\Delta p,\Delta u)=K^{\Delta *}(\Delta p)+\Delta p^T p/\rho A+\Phi^{\Delta}(B\Delta u)+(Bu)^T[D](B\Delta u)-(f-c\,\mathring{\dot{u}})^T\Delta u-(\Delta f-c\Delta\,\mathring{\dot{u}})^T(u+\Delta u)$

由 $\delta\widetilde{\Pi}_{I2}^{\Delta}(\Delta u,\Delta p)=0$，可以推导出 Hamilton 正则方程

$$\begin{cases}\dot{u}_j^i+\Delta\dot{u}_j=\partial H_j^i(\Delta u_j,\Delta p_j)/\partial\Delta p_j=H_{j\Delta p_j}^i,\\ \dot{u}_k^i+\Delta\dot{u}_k=\partial H_k^i(\Delta u_k,\Delta p_k)/\partial\Delta p_k=H_{k\Delta p_k}^i\\ \dot{p}_j^i+\Delta\dot{p}_j=-\partial H_j^i(\Delta u_j,\Delta p_j)/\partial\Delta u_j=-H_{j\Delta u_j},\\ \dot{p}_k^i+\Delta\dot{p}_k=-\partial H_k^i(\Delta u_k,\Delta p_k)/\partial\Delta u_k=-H_{k\Delta u_k}^i\end{cases} \quad (37)$$

或者

$$\begin{cases}\dot{u}_j^i+\Delta\dot{u}_j=(p_j^i+\Delta p_j)/\rho_jA_j,\\ \dot{u}_k^i+\Delta\dot{u}_k=(p_k^i+\Delta p_k)/\rho_kA_k\\ \dot{p}_j^i+\Delta\dot{p}_j-B\{[D]_j^i[B(u_j^i+\Delta u_j)]\}=\\ \quad f_j^i+\Delta f_j-c\,\mathring{\dot{u}}_j^i-c\Delta\,\mathring{\dot{u}}_j\\ \dot{p}_k^i+\Delta\dot{p}_k-B\{[D]_j^i[B(u_k^i+\Delta u_k)]\}=\\ \quad f_k^i+\Delta f_k-c\,\mathring{\dot{u}}_k^i-c\Delta\,\mathring{\dot{u}}_k\end{cases} \quad (38)$$

和边界条件(6a,b)与初始条件(7a－d)及联结条件(8a－h).

定理 3　当且仅当 $\boldsymbol{\Delta u},\boldsymbol{\Delta p}$ 是混合问题(36),(6a,b),(7a－d)与式(8a－h)的解,则必定满足变分式 $\delta\widetilde{\Pi}_{I2}^{\Delta}(\boldsymbol{\Delta u},\boldsymbol{\Delta p})=0$.

$\widetilde{\Pi}_{I2}^{\Delta}$是 2 类变量平面框架结构折线型弹塑性动力学相空间非传统 Hamilton 型增量变分原理的泛函.

为了揭示 Hamilton 正则方程的数学结构,就要打破传统概念的限制,引进新概念. 为此,将(37)式写成矩阵形式

$$[\dot{u}_j^i+\Delta\dot{u}_j,\dot{u}_k^i+\Delta\dot{u}_k,\dot{p}_j^i+\Delta\dot{p}_j,\dot{p}_k^i+\Delta\dot{p}_k]^T = J[H_{ju_j}^i,H_{ku_k}^i,H_{j\Delta p_j}^i,H_{k\Delta p_k}^i]^T \quad (39)$$

式中, $J=\begin{bmatrix}0 & I_4\\ -I_4 & 0\end{bmatrix}$, I_4 为 4 阶单位阵,方阵 J 是辛几何的度量矩阵,它是辛矩阵.

式(39)揭示了 Hamilton 正则方程和相应的相空间非传统 Hamilton 变分原理都具有自然辛结构.

4　结语

本文所建立的平面框架结构折线型弹塑性动力学各类变量非传统 Hamilton 型增量变分原理都是限制变分原理,它们能反映空间框架结构弹性动力学初值－边值问题的全部特征. 文中所建立的这些新的增量变分原理,不仅在平面框架结构动力学理论及建立有关工程实用理论方面有重要的意义,而且为建立基于增量变分原理的直接解法,如有限元法等提供了重要的理论基础. 因篇幅所限,有关这些变分原理的应用研究,将另文阐述.

参 考 文 献

1　Horrigmoe G, Bergan P G. Incremental variational principles and finite element models for nonlinear problems. *Computer Methods in Applied Mechanics and Engineering*, 1976,7(2):201～217

2　Pian T H H. Variational principles for incremental finite element methods. *Journal of the Franklin Institute*, 1976,302(5－6):473～488

3　Mason J. Variational incremental and energy Methods in solid mechanics and shell theory. Amsterdam: Elsevier Scientific Publisher Company, 1980

4　刘正兴. 基于增量变分原理的柔韧梁和柔韧板单元. 计算结构力学及其应用,1984,1(4):49～59 (Liu Z X. Flexible beam and fexible plate element based on incremental variational principle. *Computational Structural Mechanics and Application*,1984,1(4):49～59(in Chinese))

5　钟万勰. 数值计算方法. 北京:中国建筑工业出版社,1991:15～20 (Zhong W X. Calculating numerical method. Beijing: China Architecture Building Press,1991:15～20 (in Chinese))

6　Grzegorz Z. The incremental variational principle and finite element displacement approximation for the frictional contact problem of linear elasticity. *International Journal of Non-Linear Mechanics*, 1993,28(1):13～28

7　Lahellec N, Suquet P. On the effective behavior of nonlinear inelastic composites: I. Incremental variational principles. *Journal of the Mechanics and Physics of Solids*, 2007, 21(5):85～89

8　贺国京,陈大鹏. 结构非线性振动的杂交/混合幅值增量变分原理. 铁道学报,2000,22(2):60～66 (He G W, Chen D P. Hybrid/mixed amplitude incremental variational principle for nonlinear vibration of structures. *Journal of the*

China Railway Society, 2000, 22 (2): 60 ~ 66 (in Chinese))

9 罗恩. 关于线弹性动力学中各种 Gurtin 型变分原理. 中国科学(A 辑), 1987(9):936 ~ 948 (Luo E. On the Gurtin – type variational principles in linear elastodynamics. *Science in China* (*Series A*), 1987(9):936 ~ 948 (in Chinese))

10 姜凤华,罗恩. 分段线性弹性薄板动力学非传统 Hamilton 型变分原理. 振动与冲击,2008,27(11):109 ~ 114 (Jiang F H, Luo E. The unconventional Hamilton-type incremental variational principles for piecewise linear elstodynamics of thin plates. *Journal of Vibration and Shock*, 2008,27(11):109 ~ 114 (in Chinese))

THE UNCONVENTIONAL HAMILTON-TYPE INCREMENTAL VARIATIONAL PRINCIPLES FOR PIECEWISE LINEAR ELASTODYNAMICS OF FRAME STRUCTURE *

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Abstract　According to the basic idea of classical yin-yang complementarity and modern dual-complementarity, in a simple and unified new way proposed by Luo, the unconventional Hamilton-type incremental variational principles for broken line elasto-plastic dynamics of frame structure can be established systematically. The unconventional Hamilton-type incremental variational principle can fully characterize the initial-boundary-value problem of broken line elasto-plastic dynamics of frame structure. In this paper, an important integral relation was given, which can be considered as the expression of the generalized principle of virtual work for broken line elasto-plastic dynamics of frame structure. Based on this relation, it is possible to derive systematically the complementary functionals for five-field, and the functional for one-field unconventional Hamilton-type incremental variational principles and the unconventional Hamilton-type incremental variational principle in phase space by the generalized Legendre transformations were also given. Furthermore, with this new approach, the intrinsic relationship among various principles can be explained clearly.

Key words　frame structure,　elasto-plastic dynamics,　phase space,　unconventional Hamilton-type incremental variational principle,　initial-boundary-value problem

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