第 12 卷第 4 期 2014 年 12 月
1672-6553/2014/12(4)/327-8

动力学与控制学报
JOURNAL OF DYNAMICS AND CONTROL

Vol. 12 No. 4
Dec. 2014

含双圆孔 Mindlin 板弹性波散射与动应力集中*

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摘要　基于 Mindlin 平板理论,采用波函数展开法和局部坐标系方法,对含双圆孔平板结构中弹性波散射与动应力集中问题进行了研究. 文中对 Mindlin 板含双圆孔时的开孔动弯矩集中系数做了数值计算,并分析了开孔间距对动弯矩分布的影响. 结果表明:与单圆孔情况相比,由于开孔之间的相互影响,双圆孔间的动弯矩分布会发生比较复杂的变化. 孔间距相互作用有时会使动应力集中得到缓解,而有时会使动应力集中加剧. 在低频和平板较薄的情况下,平板开孔动弯矩互不影响间距较小;在较高频率和平板较厚的情况下,平板开孔动弯矩互不影响间距较大. 所有这些现象都与入射波波长与孔径等特征尺度有关. 因此,在工程结构动力学分析与强度设计中,应对不同波长和特征尺度下的动应力作具体的分析计算,而不是简单地套用静载强度设计标准或规范.

关键词　Mindlin 板理论,　弹性波散射与动应力集中,　双圆孔,　局部坐标系

DOI:　10.6052/1672-6553-2014-026

引言

平板结构在航空航天、土木建筑等工程中广泛应用. 为满足工程操作需要有时必须要在平板上开排孔,由于开孔间的相互影响,弹性波在平板内传播过程中将会发生复杂的散射现象,导致开孔附近出现复杂的动应力集中现象[1]. 由于经典薄板理论的局限性,当分析如开孔应力集中、弹性动力学问题时,会产生一定的误差. Reissner[2] 首次提出了考虑剪切变形影响的平板理论的静力学方程. 而后,Mindlin[3] 考虑了横向剪切变形和转动惯量的影响,给出了平板弯曲动力学方程. 基于 Mindlin 板理论分析计算平板开孔动应力集中问题,其结果更接近工程实际.

Pao 等[4-5] 首次研究了含单个圆孔 Mindlin 中厚板弹性波的散射与动应力集中问题,给出了问题的解析解. 力学文献[6-7]对相关结构的动力学问题进行了讨论. 文献[8-9]采用刘氏复变函数方法对平板中单个非圆孔的弹性波散射与动应力集中问题进行了研究,并给出了数值结果. 刘氏复变函数方法是继前苏联力学家 Mushkhelishvili[10] 提出求解二维静应力集中问题的复变函数方法后,又一次提出的求解二维动应力集中问题的有效方法. 可是,对于 Mindlin 板中含多个开孔情况,由于问题求解复杂,目前还无人进行分析求解.

本文将采用刘氏[11]复变函数方法和保角映射技术,对 Mindlin 中两个开孔弹性波散射与动应力集中问题进行研究,给出具体的数值算例,并对数值计算结果做分析讨论.

1　Mindlin 板弯曲波动方程及其一般解

Pao 给出了直角坐标系中 Mindlin 板由稳态弯曲波所决定的位移分量:

$$
\begin{aligned}
u_x &= -z\mathrm{Re}[\Psi_x(x,y)\exp(-i\omega t)] \\
u_y &= -z\mathrm{Re}[\Psi_y(x,y)\exp(-i\omega t)] \\
u_z &= w = \mathrm{Re}[W(x,y)\exp(-i\omega t)]
\end{aligned} \tag{1}
$$

其中,Ψ_x,Ψ_y 为广义位移分量,ω 为弯曲波的圆频率.

引入位移势函数 W_1,W_2 和 F,则广义位移分量 Ψ_x,Ψ_y 及 W 可表示为:

2013-03-21 收到第 1 稿,2013-03-29 收到修改稿.
* 国家自然科学基金资助项目(51378451,51276129)
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$$\Psi_x(x,y) = -(\sigma_1 -1)\frac{\partial W_1}{\partial x} - (\sigma_2 -1)\frac{\partial W_2}{\partial x} - \frac{\partial F}{\partial y}$$

$$\Psi_y(x,y) = -(\sigma_1 -1)\frac{\partial W_1}{\partial y} - (\sigma_2 -1)\frac{\partial W_2}{\partial y} + \frac{\partial F}{\partial x} \tag{2}$$

其中，$\sigma_1 = \frac{2}{1-v}(\frac{k_2}{\beta})^2$，$\sigma_2 = -\frac{2}{1-v}(\frac{k_1}{\beta})^2$.

利用位移势函数给出的 Mindlin 板方程有如下形式：

$$(\nabla^2 + k_1^2)W_1 = 0$$
$$(\nabla^2 - k_2^2)W_2 = 0 \tag{3}$$
$$(\nabla^2 - k_3^2)F = 0$$

式中，$k_{1,2}^2 = \frac{1}{2}(\frac{\omega}{\omega_0})^2[(\frac{1}{R^2} - \frac{1}{S^2})^2 + \frac{4}{RS}(\frac{\omega_0}{\omega})^2 \pm (\frac{1}{R^2} + \frac{1}{S^2})]$，$\beta = \frac{\pi}{h}\sqrt{1-(\frac{\omega}{\omega_0})^2}$，$R = \frac{h^2}{12}$，$S = \frac{2h^2}{\pi^2(1-v)}$，$\frac{\omega}{\omega_0} = \frac{k^2h^2}{\sqrt{6(1-v)}\pi}$.

其中，h 为板的厚度，$k = (\rho h\omega^2/D)^{1/4}$ 为波数；

式(3)表明位移势函数 W_1、W_2 和 F 分别表示三种不同的弹性波. W_1 表示速度较慢且能在平板中传播的弯曲波；W_2 表示速度较快的弯曲波，且它是衰减的波；F 表示厚板中沿厚度方向的剪切波，它同样是衰减的波.

在直角坐标系中，Mindlin 板弯曲时的广义内力为

$$M_x = D[(\sigma_1 -1)(\frac{\partial^2 W_1}{\partial x^2} + v\frac{\partial^2 W_1}{\partial y^2}) + (\sigma_2 -1)(\frac{\partial^2 W_2}{\partial x^2} + v\frac{\partial^2 W_2}{\partial y^2}) + (1-v)\frac{\partial^2 F}{\partial x\partial y}]$$

$$M_y = D[(\sigma_1 -1)(v\frac{\partial^2 W_1}{\partial x^2} + \frac{\partial^2 W_1}{\partial y^2}) + (\sigma_2 -1)(v\frac{\partial^2 W_2}{\partial x^2} + \frac{\partial^2 W_2}{\partial y^2}) - (1-v)\frac{\partial^2 F}{\partial x\partial y}]$$

$$M_{xy} = D(1-v)[(\sigma_1 -1)\frac{\partial^2 W_1}{\partial x\partial y} + (\sigma_2 -1)\frac{\partial^2 W_2}{\partial x\partial y} - \frac{1}{2}(\frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial y^2})]$$

$$Q_x = \frac{\pi^2}{12}Gh(\sigma_1\frac{\partial W_1}{\partial x} + \sigma_2\frac{\partial W_2}{\partial x} + \frac{\partial F}{\partial y})$$

$$Q_y = \frac{\pi^2}{12}Gh(\sigma_1\frac{\partial W_1}{\partial y} + \sigma_2\frac{\partial W_2}{\partial y} - \frac{\partial F}{\partial x}) \tag{4}$$

采用复变函数方法，引进复变量 $\zeta = x+iy$ 和 $\bar{\zeta} = x - iy$，则不难验证厚板中广义内力的复变量的复合表达式为

$$M_x + M_y = D(1+v)[(\sigma_1 -1)\nabla^2 W_1 + (\sigma_2 -1)\nabla^2 W_2]$$

$$M_y - M_x + 2iM_{xy} = -4D(1-v)[(\sigma_1 -1)\frac{\partial^2 W_1}{\partial \zeta^2} + (\sigma_2 -1)\frac{\partial^2 W_2}{\partial \zeta^2} + \frac{\partial^2 F}{\partial \zeta^2}]$$

$$Q_x - iQ_y = 2\kappa^2 Gh(\sigma_1\frac{\partial W_1}{\partial \zeta} + \sigma_2\frac{\partial W_2}{\partial \zeta} + i\frac{\partial F}{\partial \zeta}) \tag{5}$$

且方程(3)有如下形式

$$\frac{\partial^2 W_1}{\partial\zeta\partial\bar{\zeta}} + \frac{k_1^2}{4}W_1 = 0$$
$$\frac{\partial^2 W_2}{\partial\zeta\partial\bar{\zeta}} - \frac{k_2^2}{4}W_2 = 0 \tag{6}$$
$$\frac{\partial^2 F}{\partial\zeta\partial\bar{\zeta}} - \frac{\beta}{4}F = 0$$

在求解 Mindlin 板中任意形状开孔附近的动应力集中问题时，可使用保角映射法，映射函数具有如下形式

$$\zeta = \Omega(\eta) = c\eta + \Phi(\eta) \tag{7}$$

其中一般为复常数；$\Phi(\eta)$ 为全纯函数为保证映射函数的单值性，在研究域内 $\Omega'(\eta)$ 不能为零.

于是，在 η 平面上式(6)可以写成

$$\frac{\partial^2 W_1}{\partial\eta\partial\bar{\eta}} + \frac{k_1^2}{4}\Omega'(\eta)\overline{\Omega'(\eta)}W_1 = 0$$
$$\frac{\partial^2 W_2}{\partial\eta\partial\bar{\eta}} - \frac{k_2^2}{4}\Omega'(\eta)\overline{\Omega'(\eta)}W_2 = 0 \tag{8}$$
$$\frac{\partial^2 F}{\partial\eta\partial\bar{\eta}} - \frac{\beta^2}{4}\Omega'(\eta)\overline{\Omega'(\eta)}F = 0$$

方程(8)所决定的 Mindlin 板中每个开孔散射波的一般解为

$$W_1^{(s)} = \sum_{n=-\infty}^{\infty} A_n^m H_n^{(1)}(k_1|\Omega(\eta)|)\left\{\frac{\Omega(\eta)}{|\Omega(\eta)|}\right\}^n$$
$$W_2^{(s)} = \sum_{n=-\infty}^{\infty} B_n^m K_n(k_2|\Omega(\eta)|)\left\{\frac{\Omega(\eta)}{|\Omega(\eta)|}\right\}^n \tag{9}$$
$$F^{(s)} = \sum_{n=-\infty}^{\infty} C_n^m K_n(\beta|\Omega(\eta)|)\left\{\frac{\Omega(\eta)}{|\Omega(\eta)|}\right\}^n$$

式中，A_n^m、B_n^m、C_n^m 为第 m 个开孔产生的散射波的模式系数，$H_n^{(1)}(.)$ 和 $K_n(.)$ 分别为 n 阶第一类 Hankel 函数和第二类修正的 Bessel 函数.

不失一般性，设无穷远处有一入射波沿 x 轴正方向传播. 略去时间因子，在极坐标系中入射波可

描述为:

$$W_1^{(i)}=\mathrm{Re}\left[W_0e^{i(k_1x-\omega t)}\right]=\mathrm{Re}\left[W_0e^{i(k_1r\cos\theta-\omega t)}\right]$$
$$W_2^{(i)}=0 \qquad (10)$$
$$F^{(i)}=0$$

求解平板开孔边值问题时,平板弯曲波的总波场应由入射场与散射场迭加而成,即 Mindlin 板的位移函数及广义位移函数的形式为

$$W_1^{(t)}=W_1^{(i)}+\sum_{m=1}^{2}W_1^{m(s)}$$
$$W_2^{(t)}=\sum_{m=1}^{2}W_2^{m(s)} \qquad (11)$$
$$F^{(t)}=\sum_{m=1}^{2}F^{m(s)}$$

在分析计算时,需要将在各局部极坐标系中的广义内力分量转换到待计算的极坐标系中去.

2　边界条件及模式系数

在 η 平面上,研究自由边界条件,根据 Mindlin 理论可以给出三个边界条件:

$$M_\rho\Big|_{\rho=1}=0$$
$$M_{\rho\theta}\Big|_{\rho=1}=0 \qquad (12)$$
$$Q_\rho\Big|_{\rho=1}=0$$

将式(9)~(11)代入满足开孔的边界条件式(12),可得如下表达式:

$$\sum_{j=1}^{6}\sum_{n=-\infty}^{\infty}E_n^{ij}X_n^j=E^i\quad(i=1,2,3,4,5,6) \qquad (13)$$

其中

$$E_n^{11}=k_1^2(1-v)(\sigma_1-1)\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}H_{n-2}^{(1)}(k_1r_1)\cdot e^{i(n-2)\theta_1}-2\frac{1+v}{1-v}H_n^{(1)}(k_1r_1)e^{in\theta_1}+\frac{\overline{\eta_1\Omega'(\eta_1)}}{\eta_1\Omega'(\eta_1)}H_{n+2}^{(1)}(k_1r_1)e^{i(n+2)\theta_1}\right]$$

$$E_n^{12}=k_2^2(1-v)(\sigma_2-1)\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}K_{n-2}(k_2r_1)\cdot e^{i(n-2)\theta_1}+2\frac{1+v}{1-v}K_n(k_2r_1)e^{in\theta_1}+\frac{\overline{\eta_1\Omega'(\eta_1)}}{\eta_1\Omega'(\eta_1)}K_{n+2}(k_2r_1)e^{i(n+2)\theta_1}\right]$$

$$E_n^{13}=i\beta^2(1-v)\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}K_{n-2}(\beta r_1)e^{i(n-2)\theta_1}-\frac{\overline{\eta_1\Omega'(\eta_1)}}{\eta_1\Omega'(\eta_1)}K_{n+2}(\beta r_1)e^{i(n+2)\theta_1}\right]$$

$$E_n^{14}=k_1^2(1-v)(\sigma_1-1)\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}H_{n-2}^{(1)}(k_1r_2)\cdot e^{i(n-2)\theta_2}-2\frac{1+v}{1-v}H_n^{(1)}(k_1r_2)e^{in\theta_2}+\frac{\overline{\eta_1\Omega'(\eta_1)}}{\eta_1\Omega'(\eta_1)}H_{n+2}^{(1)}(k_1r_2)e^{i(n+2)\theta_2}\right]$$

$$E_n^{15}=k_2^2(1-v)(\sigma_2-1)\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}K_{n-2}(k_2r_2)\cdot e^{i(n-2)\theta_2}+2\frac{1+v}{1-v}K_n(k_2r_2)e^{in\theta_2}+\frac{\overline{\eta_1\Omega'(\eta_1)}}{\eta_1\Omega'(\eta_1)}K_{n+2}(k_2r_2)e^{i(n+2)\theta_2}\right]$$

$$E_n^{16}=i\beta^2(1-v)\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}K_{n-2}(\beta r_2)e^{i(n-2)\theta_2}-\frac{\overline{\eta_1\Omega'(\eta_1)}}{\eta_1\Omega'(\eta_1)}K_{n+2}(\beta r_2)e^{i(n+2)\theta_2}\right]$$

$$E_n^{21}=k_1^2(1-v)(\sigma_1-1)\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}H_{n-2}^{(1)}(k_1r_1)\cdot e^{i(n-2)\theta_1}-\frac{\overline{\eta_1\Omega'(\eta_1)}}{\eta_1\Omega'(\eta_1)}H_{n+2}^{(1)}(k_1r_1)e^{i(n+2)\theta_1}\right]$$

$$E_n^{22}=k_2^2(1-v)(\sigma_2-1)\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}K_{n-2}(k_2r_1)e^{i(n-2)\theta_1}-\frac{\overline{\eta_1\Omega'(\eta_1)}}{\eta_1\Omega'(\eta_1)}K_{n+2}(k_2r_1)e^{i(n+2)\theta_1}\right]$$

$$E_n^{23}=i\beta^2(1-v)\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}K_{n-2}(\beta r_1)e^{i(n-2)\theta_1}+\frac{\overline{\eta_1\Omega'(\eta_1)}}{\eta_1\Omega'(\eta_1)}K_{n+2}(\beta r_1)e^{i(n+2)\theta_1}\right]$$

$$E_n^{24}=k_1^2(1-v)(\sigma_1-1)\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}H_{n-2}^{(1)}(k_1r_2)\cdot e^{i(n-2)\theta_2}-\frac{\overline{\eta_1\Omega'(\eta_1)}}{\eta_1\Omega'(\eta_1)}H_{n+2}^{(1)}(k_1r_2)e^{i(n+2)\theta_2}\right]$$

$$E_n^{25}=k_2^2(1-v)(\sigma_2-1)\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}K_{n-2}(k_2r_2)\cdot e^{i(n-2)\theta_2}-\frac{\overline{\eta_1\Omega'(\eta_1)}}{\eta_1\Omega'(\eta_1)}K_{n+2}(k_2r_2)e^{i(n+2)\theta_2}\right]$$

$$E_n^{26}=i\beta^2(1-v)\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}K_{n-2}(\beta r_2)e^{i(n-2)\theta_2}+\frac{\overline{\eta_1\Omega'(\eta_1)}}{\eta_1\Omega'(\eta_1)}K_{n+2}(\beta r_2)e^{i(n+2)\theta_2}\right]$$

$$E_n^{31}=k_1\sigma_1\left[\frac{\eta_1\Omega'(\eta_1)}{|\Omega'(\eta_1)|}H_{n-1}^{(1)}(k_1r_1)e^{i(n-1)\theta_1}-\frac{\overline{\eta_1\Omega'(\eta_1)}}{|\Omega'(\eta_1)|}H_{n+1}^{(1)}(k_1r_1)e^{i(n+1)\theta_1}\right]$$

$$E_n^{32}=-k_2\sigma_2\left[\frac{\eta_1\Omega'(\eta_1)}{|\Omega'(\eta_1)|}K_{n-1}(k_2r_1)\mathrm{e}^{i(n-1)\theta_1}+\overline{\frac{\eta_1\Omega'(\eta_1)}{|\Omega'(\eta_1)|}}K_{n+1}(k_2r_1)\mathrm{e}^{i(n+1)\theta_1}\right]$$

$$E_n^{33}=-i\beta\left[\frac{\eta_1\Omega'(\eta_1)}{|\Omega'(\eta_1)|}K_{n-1}(\beta r_1)\mathrm{e}^{i(n-1)\theta_1}-\overline{\frac{\eta_1\Omega'(\eta_1)}{|\Omega'(\eta_1)|}}K_{n+1}(\beta r_1)\mathrm{e}^{i(n+1)\theta_1}\right]$$

$$E_n^{34}=k_1\sigma_1\left[\frac{\eta_1\Omega'(\eta_1)}{|\Omega'(\eta_1)|}H_{n-1}^{(1)}(k_1r_2)\mathrm{e}^{i(n-1)\theta_2}-\overline{\frac{\eta_1\Omega'(\eta_1)}{|\Omega'(\eta_1)|}}H_{n+1}^{(1)}(k_1r_2)\mathrm{e}^{i(n+1)\theta_2}\right]$$

$$E_n^{35}=-k_2\sigma_2\left[\frac{\eta_1\Omega'(\eta_1)}{|\Omega'(\eta_1)|}K_{n-1}(k_2r_2)\mathrm{e}^{i(n-1)\theta_2}+\overline{\frac{\eta_1\Omega'(\eta_1)}{|\Omega'(\eta_1)|}}K_{n+1}(k_2r_2)\mathrm{e}^{i(n+1)\theta_2}\right]$$

$$E_n^{36}=-i\beta\left[\frac{\eta_1\Omega'(\eta_1)}{|\Omega'(\eta_1)|}K_{n-1}(\beta r_2)\mathrm{e}^{i(n-1)\theta_2}-\overline{\frac{\eta_1\Omega'(\eta_1)}{|\Omega'(\eta_1)|}}K_{n+1}(\beta r_2)\mathrm{e}^{i(n+1)\theta_2}\right]$$

$$E_n^{41}=k_1^2(1-v)(\sigma_1-1)\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}H_{n-2}^{(1)}(k_1r_2)\cdot\mathrm{e}^{i(n-2)\theta_2}-2\frac{1+v}{1-v}H_n^{(1)}(k_1r_2)\mathrm{e}^{in\theta_2}+\overline{\frac{\eta_2\Omega'(\eta_2)}{\eta_2\Omega'(\eta_2)}}H_{n+2}^{(1)}(k_1r_2)\mathrm{e}^{i(n+2)\theta_2}\right]$$

$$E_n^{42}=k_2^2(1-v)(\sigma_2-1)\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}K_{n-2}(k_2r_2)\cdot\mathrm{e}^{i(n-2)\theta_2}+2\frac{1+v}{1-v}K_n(k_2r_2)\mathrm{e}^{in\theta_2}+\overline{\frac{\eta_2\Omega'(\eta_2)}{\eta_2\Omega'(\eta_2)}}K_{n+2}(k_2r_2)\mathrm{e}^{i(n+2)\theta_2}\right]$$

$$E_n^{43}=i\beta^2(1-v)\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}K_{n-2}(\beta r_2)\mathrm{e}^{i(n-2)\theta_2}-\overline{\frac{\eta_2\Omega'(\eta_2)}{\eta_2\Omega'(\eta_2)}}K_{n+2}(\beta r_2)\mathrm{e}^{i(n+2)\theta_2}\right]$$

$$E_n^{44}=k_1^2(1-v)(\sigma_1-1)\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}H_{n-2}^{(1)}(k_1r_1)\cdot\mathrm{e}^{i(n-2)\theta_1}-2\frac{1+v}{1-v}H_n^{(1)}(k_1r_1)\mathrm{e}^{in\theta_1}+\overline{\frac{\eta_2\Omega'(\eta_2)}{\eta_2\Omega'(\eta_2)}}H_{n+2}^{(1)}(k_1r_1)\mathrm{e}^{i(n+2)\theta_1}\right]$$

$$E_n^{45}=k_2^2(1-v)(\sigma_2-1)\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}K_{n-2}(k_2r_1)\cdot\mathrm{e}^{i(n-2)\theta_1}+2\frac{1+v}{1-v}K_n(k_2r_1)\mathrm{e}^{in\theta_1}+\overline{\frac{\eta_2\Omega'(\eta_2)}{\eta_2\Omega'(\eta_2)}}K_{n+2}(k_2r_1)\mathrm{e}^{i(n+2)\theta_1}\right]$$

$$E_n^{46}=i\beta^2(1-v)\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}K_{n-2}(\beta r_1)\mathrm{e}^{i(n-2)\theta_1}-\overline{\frac{\eta_2\Omega'(\eta_2)}{\eta_2\Omega'(\eta_2)}}K_{n+2}(\beta r_1)\mathrm{e}^{i(n+2)\theta_1}\right]$$

$$E_n^{51}=k_1^2(1-v)(\sigma_1-1)\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}H_{n-2}^{(1)}(k_1r_2)\cdot\mathrm{e}^{i(n-2)\theta_2}-\overline{\frac{\eta_2\Omega'(\eta_2)}{\eta_2\Omega'(\eta_2)}}H_{n+2}^{(1)}(k_1r_2)\mathrm{e}^{i(n+2)\theta_2}\right]$$

$$E_n^{52}=k_2^2(1-v)(\sigma_2-1)\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}K_{n-2}(k_2r_2)\cdot\mathrm{e}^{i(n-2)\theta_2}-\overline{\frac{\eta_2\Omega'(\eta_2)}{\eta_2\Omega'(\eta_2)}}K_{n+2}(k_2r_2)\mathrm{e}^{i(n+2)\theta_2}\right]$$

$$E_n^{53}=i\beta^2(1-v)\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}K_{n-2}(\beta r_2)\mathrm{e}^{i(n-2)\theta_2}+\overline{\frac{\eta_2\Omega'(\eta_2)}{\eta_2\Omega'(\eta_2)}}K_{n+2}(\beta r_2)\mathrm{e}^{i(n+2)\theta_2}\right]$$

$$E_n^{54}=k_1^2(1-v)(\sigma_1-1)\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}H_{n-2}^{(1)}(k_1r_1)\cdot\mathrm{e}^{i(n-2)\theta_1}-\overline{\frac{\eta_2\Omega'(\eta_2)}{\eta_2\Omega'(\eta_2)}}H_{n+2}^{(1)}(k_1r_1)\mathrm{e}^{i(n+2)\theta_1}\right]$$

$$E_n^{55}=k_2^2(1-v)(\sigma_2-1)\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}K_{n-2}(k_2r_1)\cdot\mathrm{e}^{i(n-2)\theta_1}-\overline{\frac{\eta_2\Omega'(\eta_2)}{\eta_2\Omega'(\eta_2)}}K_{n+2}(k_2r_1)\mathrm{e}^{i(n+2)\theta_1}\right]$$

$$E_n^{56}=i\beta^2(1-v)\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}K_{n-2}(\beta r_1)\mathrm{e}^{i(n-2)\theta_1}+\overline{\frac{\eta_2\Omega'(\eta_2)}{\eta_2\Omega'(\eta_2)}}K_{n+2}(\beta r_1)\mathrm{e}^{i(n+2)\theta_1}\right]$$

$$E_n^{61}=k_1\sigma_1\left[\frac{\eta_2\Omega'(\eta_2)}{|\Omega'(\eta_2)|}H_{n-1}^{(1)}(k_1r_2)\mathrm{e}^{i(n-1)\theta_2}-\overline{\frac{\eta_2\Omega'(\eta_2)}{|\Omega'(\eta_2)|}}H_{n+1}^{(1)}(k_1r_2)\mathrm{e}^{i(n+1)\theta_2}\right]$$

$$E_n^{62}=-k_2\sigma_2\left[\frac{\eta_2\Omega'(\eta_2)}{|\Omega'(\eta_2)|}K_{n-1}(k_2r_2)\mathrm{e}^{i(n-1)\theta_2}+\overline{\frac{\eta_2\Omega'(\eta_2)}{|\Omega'(\eta_2)|}}K_{n+1}(k_2r_2)\mathrm{e}^{i(n+1)\theta_2}\right]$$

$$E_n^{63}=-i\beta\left[\frac{\eta_2\Omega'(\eta_2)}{|\Omega'(\eta_2)|}K_{n-1}(\beta r_2)\mathrm{e}^{i(n-1)\theta_2}-\overline{\frac{\eta_2\Omega'(\eta_2)}{|\Omega'(\eta_2)|}}K_{n+1}(\beta r_2)\mathrm{e}^{i(n+1)\theta_2}\right]$$

$$E_n^{64}=k_1\sigma_1\left[\frac{\eta_2\Omega'(\eta_2)}{|\Omega'(\eta_2)|}H_{n-1}^{(1)}(k_1r_1)\mathrm{e}^{i(n-1)\theta_1}-\right.$$

$$\frac{\overline{\eta_2\Omega'(\eta_2)}}{|\Omega'(\eta_2)|}H_{n+1}^{(1)}(k_1r_1)e^{i(n+1)\theta_1}\Big]$$

$$E_n^{65}=-k_2\sigma_2\left[\frac{\eta_2\Omega'(\eta_2)}{|\Omega'(\eta_2)|}K_{n-1}(k_2r_1)e^{i(n-1)\theta_1}+\frac{\overline{\eta_2\Omega'(\eta_2)}}{|\Omega'(\eta_2)|}K_{n+1}(k_2r_1)e^{i(n+1)\theta_1}\right]$$

$$E_n^{66}=-i\beta\left[\frac{\eta_2\Omega'(\eta_2)}{|\Omega'(\eta_2)|}K_{n-1}(\beta r_1)e^{i(n-1)\theta_1}-\frac{\overline{\eta_2\Omega'(\eta_2)}}{|\Omega'(\eta_2)|}K_{n+1}(\beta r_1)e^{i(n+1)\theta_1}\right]$$

$$E_1=2k_{21}(\sigma_1-1)\left\{(1-v)+(1-v)\mathrm{Re}\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}\right]\right\}\cdot\exp\left\{\frac{ik_1}{2}\left[\Omega(\eta_1)+\overline{\Omega(\eta_1)}\right]\right\}$$

$$E_2=2ik_1^2(\sigma_1-1)(1-v)\mathrm{Im}\left[\frac{\eta_1\Omega'(\eta_1)}{\overline{\eta_1\Omega'(\eta_1)}}\right]\cdot\exp\left\{\frac{ik_1}{2}\left[\Omega(\eta_1)+\overline{\Omega(\eta_1)}\right]\right\}$$

$$E_3=-2ik_1^2\sigma_1\mathrm{Re}\left[\frac{\eta_1\Omega'(\eta_1)}{|\Omega'(\eta_1)|}\right]\cdot\exp\left\{\frac{ik_1}{2}\left[\Omega(\eta_1)+\overline{\Omega(\eta_1)}\right]\right\}$$

$$E_4=2k_1^2(\sigma_1-1)\left\{(1-v)+(1-v)\mathrm{Re}\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}\right]\right\}\cdot\exp\left\{\frac{ik_1}{2}\left[\Omega(\eta_2)+\overline{\Omega(\eta_2)}\right]\right\}$$

$$E_5=2ik_1^2(\sigma_1-1)(1-v)\mathrm{Im}\left[\frac{\eta_2\Omega'(\eta_2)}{\overline{\eta_2\Omega'(\eta_2)}}\right]\cdot\exp\left\{\frac{ik_1}{2}\left[\Omega(\eta_2)+\overline{\Omega(\eta_2)}\right]\right\}$$

$$E_6=-2ik_1^2\sigma_1\mathrm{Re}\left[\frac{\eta_2\Omega'(\eta_2)}{|\Omega'(\eta_2)|}\right]\cdot\exp\left\{\frac{ik_1}{2}\left[\Omega(\eta_2)+\overline{\Omega(\eta_2)}\right]\right\}$$

$$\eta_1=\exp(ia\theta_1),\ r_2=\sqrt{a^2+d^2+2ad\sin\theta_1},$$

$$\eta_2=\exp(ia\theta_2),\ r_1=\sqrt{a^2+d^2-2ad\sin\theta_2},$$

$$\theta_2=\arccos\frac{a\cos\theta_1}{r_2},\theta_1=-\arccos\frac{a\cos\theta_2}{r_1}.$$

用 $\exp(-is\theta_j)$ 乘以式(13)的两端,并且在区间 $(-\pi,\pi)$ 上积分,可得无穷代数方程组如下:

$$\sum_{-\infty}^{\infty}E_{ns}X_n=E_s \tag{14}$$

其中 $E_{ns}=\frac{1}{2\pi}\int_{-\pi}^{\pi}E_n\exp(-is\theta_j)d\theta_j$,

$$E_s=\frac{1}{2\pi}\int_{-\pi}^{\pi}E\exp(-is\theta_j)d\theta_j.$$

3　动应力集中系数

由开孔动应力集中的定义:动应力集中系数是开孔周边上的环向动弯矩与入射波在入射方向上的弯矩幅值之比,即:

$$M_\theta^*=\frac{M_\theta}{M_0} \tag{15}$$

式中,M_θ^* 为无量纲弯矩,表示动应力集中系数;M_0 为入射弯矩的幅值,$M_0=-Dk_1^2(\sigma_1-1)W_0$,$M_\theta$ 为开孔周边自由时开孔周边上任意一点的环向动弯矩,它可以表示成

$$\begin{aligned}M_\theta&=D(1+v)\left[(\sigma_1-1)\nabla^2(W_1^{(i)}+W_1^{(s)})+(\sigma_2-1)\nabla^2W_2^{(s)}\right]\\&=-D(1+v)\left[(\sigma_1-1)k_1^2(W_1^{(i)}+W_1^{(s)})-(\sigma_2-1)k_2^2W_2^{(s)}\right]\\&=-D(1+v)(\sigma_1-1)k_1^2\left\{\sum_{n=-\infty}^{\infty}A_n^mH_n^{(1)}\cdot(k_1|\Omega(\eta)|)\left\{\frac{\Omega(\eta)}{|\Omega(\eta)|}\right\}^n-\frac{\sigma_2-1}{\sigma_1-1}\left(\frac{k_2}{k_1}\right)^2\cdot\right.\\&\left.\sum_{n=-\infty}^{\infty}B_n^mK_n(k_2|\Omega(\eta)|)\left\{\frac{\Omega(\eta)}{|\Omega(\eta)|}\right\}^n+W_0\exp\left[\frac{ik_1(\Omega(\eta)+\overline{\Omega(\eta)})}{2}\right]\right\}\end{aligned} \tag{16}$$

因此,对于含两个开孔的 Mindlin 板,第 m 个开孔周边动应力集中系数可表示为

$$\begin{aligned}M_\theta^*=&-(1+v)k_1^2\mathrm{Re}\left\{\frac{1}{W_0}\sum_{n=-\infty}^{\infty}A_n^mH_n^{(1)}\cdot(k_1|\Omega(\eta)|)\left\{\frac{\Omega(\eta_m)}{|\Omega(\eta_m)|}\right\}^n-\frac{\sigma_2-1}{\sigma_1-1}\left(\frac{k_2}{k_1}\right)^2\cdot\right.\\&\frac{1}{W_0}\sum_{n=-\infty}^{\infty}B_n^mK_n(k_2|\Omega(\eta_m)|)\left\{\frac{\Omega(\eta_m)}{|\Omega(\eta_m)|}\right\}^n+\\&\left.\exp\left[\frac{ik_1(\Omega(\eta_m)+\overline{\Omega(\eta_m)})}{2}-i\omega t\right]\right\}\end{aligned} \tag{17}$$

式(17)即是 Mindlin 型厚板含两个开孔第 m 个开孔周边动应力集中系数的一般表达式.

4　数值算例

上述分析可以用来计算含两个自由孔洞中厚板的动应力集中系数. 根据文献[6]中 Mindlin 板理论的公式,编制了 Mindlin 板中两个自由开孔周围动应力集中系数的计算程序. 取 $n=10$,Poisson 比 $v=0.3$,无量纲波数 $ka=0.1\sim3.0$.

动弯矩集中系数的分布如图 1 ~ 12 所示,图中计算的圆孔是上下部署的上圆孔,两圆孔中心的连线与 x 轴垂直,两圆孔中心的距离为 L. 其中图 1 ~ 12 的上半部分为 $t=0$ 时单圆孔动弯矩集中系数随周向角度($0\sim\pi$)的分布规律,下半部分为 $t=0$ 时双圆孔间的动弯矩集中系数随周向角度($-\pi\sim0$)的分布规律. 图 13 则给出了双圆孔间的动弯矩集中系数($-\pi\sim0$)随无量纲孔间距 L/a 变化的规律.

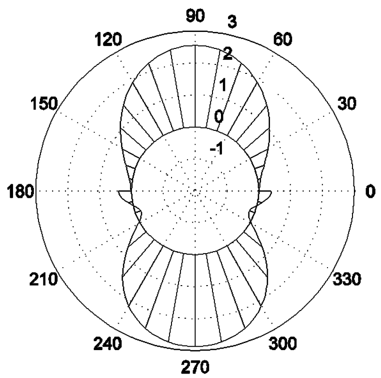

图 1 Mindlin 板两孔孔边动弯矩分布

Fig. 1 Dynamic moment factor

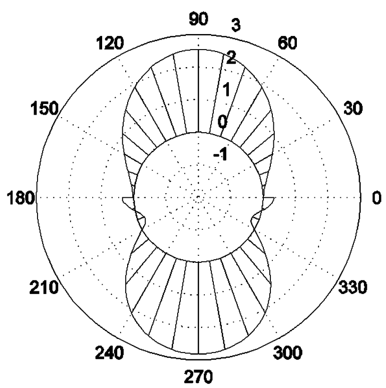

图 2 Mindlin 板两孔孔边动弯矩分布

Fig. 2 Dynamic moment factor

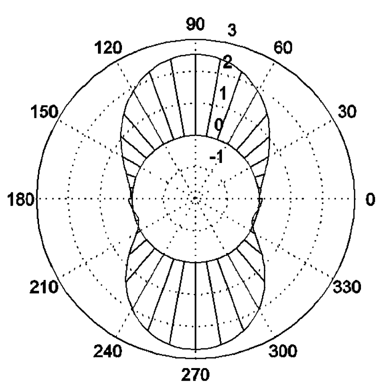

图 3 Mindlin 板两孔孔边动弯矩分布

Fig. 3 Dynamic moment factor

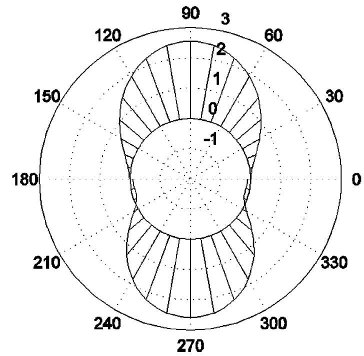

图 4 Mindlin 板两孔孔边动弯矩分布

Fig. 4 Dynamic moment factor

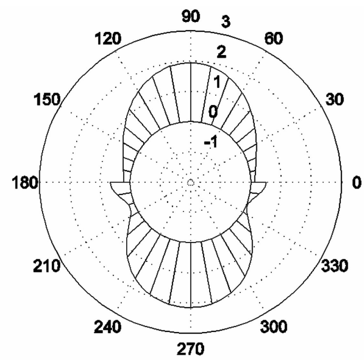

图 5 Mindlin 板两孔孔边动弯矩分布

Fig. 5 Dynamic moment factor

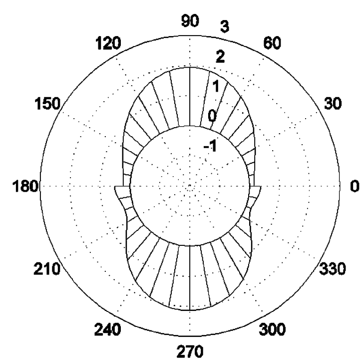

图 6 Mindlin 板两孔孔边动弯矩分布

Fig. 6 Dynamic moment factor

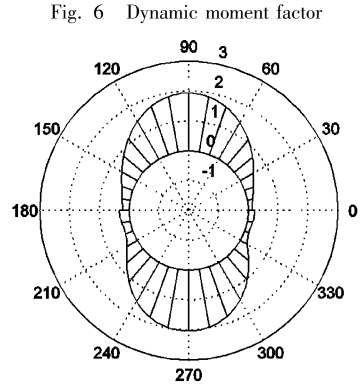

图 7 Mindlin 板两孔孔边动弯矩分布

Fig. 7 Dynamic moment factor

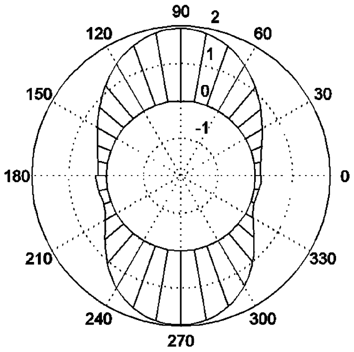

图 8　Mindlin 板两孔孔边动弯矩分布

Fig. 8　Dynamic moment factor

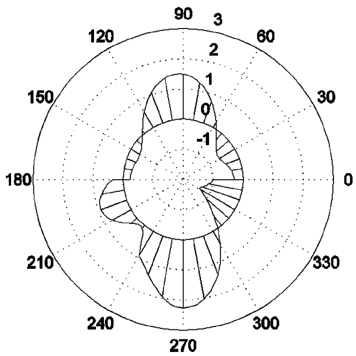

图 9　Mindlin 板两孔孔边动弯矩分布

Fig. 9　Dynamic moment factor

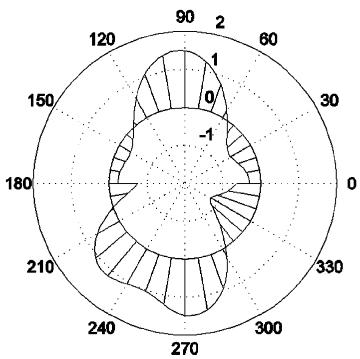

图 10　Mindlin 板两孔孔边动弯矩分布

Fig. 10　Dynamic moment factor

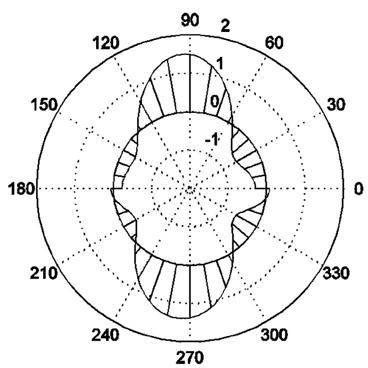

图 11　Mindlin 板两孔孔边动弯矩分布

Fig. 11　Dynamic moment factor

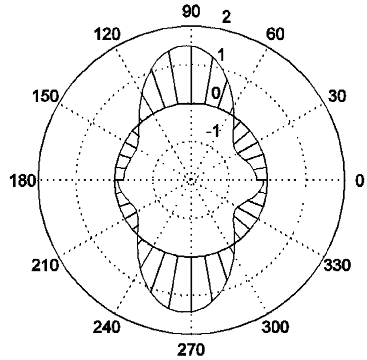

图 12　Mindlin 板两孔孔边动弯矩分布

Fig. 12　Dynamic moment factor

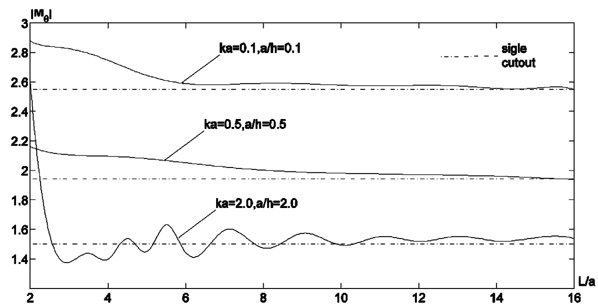

图 13　含双圆孔 Mindlin 板动弯矩集中系数随两孔间距变化规律

Fig. 13　Dynamic moment factors vs dimensionless wave number

5　结论

当孔间动应力集中系数与单孔动应力集中系数之商减去 1 的绝对值小于 2% 时,认为开孔互不影响.

通过对计算结果分析可以看到:

(1) 在入射波频率较低的情况下,例如 $ka=0.1$ 时,开孔互不影响间距为 $L/a=5.5$;例如 $ka=0.5$ 时,开孔互不影响间距为 $L/a=9$. 在入射波频率较高的情况下,动弯矩集中系数随两孔间距的变化出现波动,开孔互不影响间距也将变大,例如 $ka=2.0$ 时,开孔互不影响间距为 $L/a=18$.

(2) 与单圆孔情况相比,由于开孔之间的相互影响,双圆孔间的动弯矩集中系数会发生比较复杂的变化. 动应力状态有时会缓解,但有时也会加剧. 因此,做动态强度设计时,不能全部套用静载强度设计标准或规范,应做全面的动态应力分析.

参　考　文　献

1　Pao Y H. Dynamical stress concentration in an elastic plate. *ASME Journal of Applied Mechanics*, 1962: 299 ~ 305

2　Reissner E. The effect of transverse shear deformation on the bending of elastic plates. *ASME Journal of Applied Mechanics*, *1945*, *12*(*2*): *69~77*

3 Mindlin R D. Influence of rotatory inertia and shear on flexural motions of isotropic elastic plates. *ASME Journal of Applied Mechanics*, 1951, 18: 31~38

4 鲍亦兴, 毛昭宙. 弹性波的衍射与动应力. 刘殿魁, 苏先樾译. 北京: 科学出版社, 1993: 230~231 (Pao Y H, Mao C C. Diffraction of elastic waves and dynamic stress concentrations. Liu D K, Su X Y, Trans. Beijing: Science Press, 1993: 230~231 (in Chinese))

5 Pao Y H. Diffractions of flexural waves by a cavity in an elastic plate. *AIAA Journal*, 1964, 2(11): 2004~2010

6 Hu C, Chen T, Han G, et al. Flexural wave propagation and localized vibration in narrow Mindlin's plate. *Journal of Sound and Vibration*, 2007, 306(3-5): 389~399

7 郭旺, 胡超. 悬臂板条结构动力学与振动控制分析. 动力学与控制学报, 2010, 08(1): 80~86 (Guo W, Hu C. Dynamics of catilever plates and its vibration control. *Journal of Dynamics and Control*, 2010, 08(1): 80~86 (in Chinese))

8 Liu D K, Hu C. Scattering of flexural wave and dynamic stress concentration in Mindlin thick plates. *Acta Mechanica Sinica*, 1996, 12(2): 169~185

9 胡超, 马兴瑞, 黄文虎. 含双圆孔平板弹性波散射与动应力分析. 力学学报, 1998, 30(5): 587~596 (Hu C, Ma X R, Hung W H. Dynamic stress concentrations in thin plates with two circular cutouts. *Acta Mechanica Sinica*, 1998, 30(5): 587~596 (in Chinese))

10 Muskhelishvili N I 著. 数学弹性力学的几个基本问题. 赵惠元译. 北京: 科学出版社, 1958. 236 (Muskhelishvili N I. Some basic problems in mathematical elasticity. Zhao H Y Trans. Beijing: Science Press, 1958: 236 (in Chinese))

11 刘殿魁, 盖秉政, 陶贵源. 论孔附近的动应力集中. 力学学报, 1981: 65~77 (Liu D K, Gai B Z, Tao G Y. On dynamic stress concentration in the neighbourhood of a cavity. *Acta Mechanica Sinica*, 1981: 65~77 (in Chinese))

ELASTIC WAVE SCATTERING AND DYNAMIC STRESS CONCENTRATIONS IN MINDLIN'S PLATE WITH TWO CUTOUTS *

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Abstract Based on Mindlin's theory of plates, elastic wave scattering and dynamic stress concentrations in plates with two circular cutouts were studied by using the wave function expansion method and local coordinate system. Numerical results of dynamic moment concentration factors around the cutouts were obtained by setting different parameters and the influence of the parameters on dynamic moment distributions were discussed. It is shown that unlike single-cutout situation, the complex changes take place on dynamic moment concentration factors at two-cutout situation due to the interaction between two openings. Sometimes the dynamic moment was increased, or sometimes the one was reduced. When the plate is thicker at the high-frequency, the opening spacing of the non-significant influence is larger. Therefore, in the dynamic design the comprehensive dynamic stress analysis would be done instead of directly applying the static standard and code for the structure design.

Key words Mindlin theory of plates, elastic wave scattering and dynamic stress concentrations, two circular cutouts, local coordinate system

Received 21 March 2013, revised 29 March 2013.

* The project supported by the National Natural Science Foundation of China (51378451, 51276129)

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