

内共振关系对弹簧摆动力学行为的影响*

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摘要 基于数值方法,以弹簧摆为对象,讨论了不同的内共振关系对一类平方、立方非线性系统动力学行为的影响. 结果表明,对1:1内共振的情况,两个模态的振动均可能发生在偏离原来平衡位置的新的平衡位置附近,即出现平衡位置飘移的现象. 能量可以从低阶(摆动)模态传递到高阶(呼吸)模态,但不能从高阶(呼吸)模态传递到低阶(摆动)模态. 然而对1:3内共振的情况,这种能量在两个模态之间的传递却非常弱. 从仿真结果来看,对1:1和1:3内共振的情况,等幅的周期解是稳定的;但对1:2内共振的情况,出现的是调幅的周期运动即拍振,且拍频与初始条件有关.

关键词 弹簧摆, 内共振, 能量传递, 稳定性

引言

弹簧摆作为一个典型的两自由度系统,在许多关于非线性振动的专著中都有所论述,它既可以看成是单摆的推广也可以看成是单自由度弹簧-质量系统的推广. 由于我们对此两类单自由度系统的固有频率的概念非常熟悉,所以弹簧摆特别适合于说明非线性振动系统中的内共振以及由此引起的其他动力学现象^[1-4]. 事实上,弹簧摆的非线性动力学问题一直是人们的研究兴趣. 张耀辰等^[5]用多尺度法研究了弹簧摆的悬挂点作简谐运动时,1:2内共振平方非线性模型的主共振和主参数共振响应. 李银山等^[6]用 Melnikov 方法研究了平方、立方非线性模型弹簧摆的混沌运动. 石玉仁等^[7]直接对原微分方程(非线性项不作泰勒展开)进行数值计算,研究了受迫阻尼弹簧摆由倍周期分岔产生的混沌运动. 萧寒等^[8]用多尺度法研究了在单频激励下,1:2内共振平方非线性模型弹簧摆主共振响应的鞍结分岔控制问题. 郑建龙等^[9]用谐波平衡法和数值方法研究了平方非线性模型弹簧摆的内共振现象. 在文献[10]中, Lee 等利用谐波平衡法研究了1:2内共振平方非线性模型弹簧摆在简谐激励下主共振响应的全局吸引域. 在文献[11, 12]中,他们利用多尺度法研究了1:2内共振平方、立方非

线性模型弹簧摆在简谐激励下主共振响应的二阶近似和三阶近似解的混沌运动. van der Weele 等^[13]以摆动模态和呼吸模态的频率比以及系统能量与最小能量的比为控制参数讨论了1:2内共振弹簧的混沌运动. Zaki 等^[14]利用数值方法讨论了简谐激励下弹簧摆非主共振响应的分岔和通向混沌的道路. Eissa 等^[15]用多尺度法讨论了五阶非线性模型弹簧摆在两个模态都受到简谐激励时主共振、亚谐共振、超谐共振、组合共振响应的四阶近似解. Aria Alasty 等^[16]用多尺度法研究了1:2内共振弹簧摆在简谐激励下主共振响应的混沌运动. Amer 等^[17]用多尺度法研究了1:2内共振弹簧摆支点沿圆形轨道运动时简谐激励主共振响应的混沌运动. Gitterman 等^[18]用多尺度法讨论了悬挂点作简谐运动时四阶非线性模型弹簧摆的四阶近似解.

就弹簧摆的微幅振动而言,非线性项保留到四阶乃至五阶显然是不必要的. 但若只保留平方非线性项而忽略立方非线性项,则只需考虑1:2而无法考虑1:1和1:3内共振关系. 因此,本文对运动微分方程中的平方非线性项和立方非线性项同时保留,重点考虑不同形式的可能内共振关系对系统动力学行为的影响.

1 数学模型

为方便,分别用广义坐标 x_1 和 x_2 代表小球沿

弹簧轴向的位移和绕悬挂点摆动的角位移,此时弹簧摆的运动微分方程为^[1]

$$\begin{aligned} \ddot{x}_1 + \frac{k}{m}x_1 + g(1 - \cos x_2) - (l + x_1)\dot{x}_2^2 &= 0, \\ \ddot{x}_2 + \frac{g}{l+x_1}\sin x_2 + \frac{2}{l+x_1}\dot{x}_1\dot{x}_2 &= 0 \end{aligned} \quad (1)$$

其中 $l = l_0 + mg/k$, m , k 和 l_0 分别代表小球的质量、弹簧的刚度和原长.

将 $\cos x_2$, $\sin x_2$ 和 $1/(l + x_1)$ 分别作台劳展开,并保留到三次项,方程(1)化简为

$$\begin{aligned} \ddot{x}_1 + \frac{k}{m}x_1 + \frac{g}{2}x_2^2 - (l + x_1)\dot{x}_2^2 &= 0, \\ \ddot{x}_2 + \frac{g}{l}(x_2 - \frac{x_1x_2}{l} + \frac{x_1^2x_2}{l^2} - \frac{x_2^3}{6}) + \\ \frac{2}{l}\dot{x}_1\dot{x}_2(1 - \frac{x_1}{l}) &= 0 \end{aligned} \quad (2)$$

令 $\bar{x}_1 = x_1/l$, $\bar{t} = t/T_1$, 其中 $T_1 = \sqrt{m/k}$, 为方便 \bar{x}_1 仍记为 x_1 , \bar{x}_1 和 x_2 关于 \bar{t} 的一阶和二阶导数仍记为 \dot{x}_1 , \dot{x}_2 , \ddot{x}_1 和 \ddot{x}_2 , 则方程(2)可无量纲化为

$$\begin{aligned} \ddot{x}_1 + x_1 + \frac{g}{2l}T_1^2x_2^2 - (1 + x_1)\dot{x}_2^2 &= 0, \\ \ddot{x}_2 + \frac{g}{l}T_1^2(x_2 - x_1x_2 + x_1^2x_2 - \frac{x_2^3}{6}) + \\ 2\dot{x}_1\dot{x}_2(1 - x_1) &= 0 \end{aligned} \quad (3)$$

方程(3)可以改写成

$$\begin{aligned} \ddot{x}_1 + \omega_1^2x_1 &= (1 + x_1)\dot{x}_2^2 - \frac{g}{2l}T_1^2x_2^2, \\ \ddot{x}_2 + \omega_2^2x_2 &= \frac{g}{l}T_1^2(x_1x_2 - x_1^2x_2 + \frac{x_2^3}{6}) - \\ 2\dot{x}_1\dot{x}_2(1 - x_1), \end{aligned} \quad (4)$$

其中 $\omega_1^2 = 1$, $\omega_2^2 = \frac{g}{l}T_1^2$, 当它们相互可有理通约或接近可有理通约时,就可能产生内共振.

显然,控制微分方程(4)中耦合项都是以非线性项的形式出现的,且各非线性项的系数无论是与 ω_1^2 还是 ω_2^2 相比,均不是小量. 期待利用适于弱非线性系统的各种近似解析方法会遇到很大麻烦,因此这里采用数值分析的方法.

2 仿真分析

由于没考虑阻尼的作用,所以系统受到初始激励后一定会偏离原来的平衡位置而产生周期运动,即原平衡位置一定是不稳定的. 我们关心的是在不

同的内共振关系下,不同的初始扰动会引发系统产生什么样的运动. 下面的仿真分析结果是利用 MATLAB 中的 ODE 指令得到的. 由于初始的位移扰动比速度扰动更容易实现和量化,所以下数值分析的结果都是针对初始的位移扰动给出的.

2.1 1:1 内共振的情况

图 1-6 对应着六种不同的初始扰动,其中前四种对应着两个模态都被激发,后两种对应着只激发其中的一个模态. 从图 1 和 2 可以看出,两个模

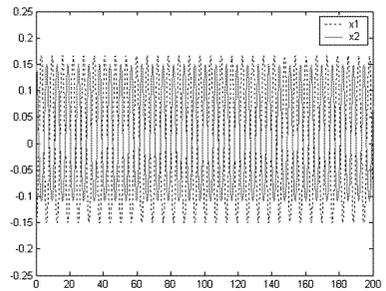


图 1 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (-0.15, 0.15, 0, 0)$ 时的时间历程

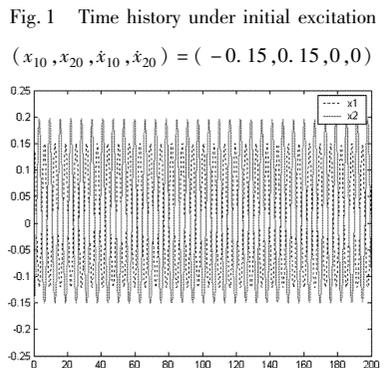


图 2 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0.15, -0.15, 0, 0)$ 时的时间历程

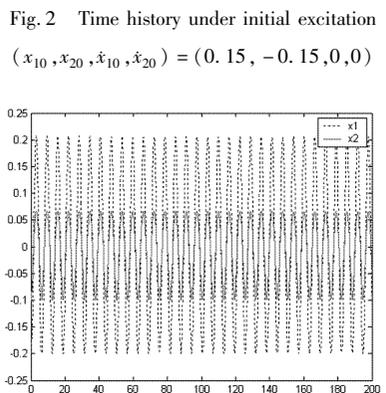


图 3 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (-0.2, -0.1, 0, 0)$ 时的时间历程

Fig. 3 Time history under initial excitation $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (-0.2, -0.1, 0, 0)$

态的振动都偏离了原来的平衡位置,并且都偏向于广义坐标正方向的一侧.虽然扰动的绝对值是一样的,但两个模态振动的峰峰值却相差很多.从图3和4可以看出,呼吸模态的振动基本上仍是在原平衡位置附近,只是摆动模态显著地偏向于广义坐标负方向的一侧.从图5和6可以看出,只激发呼吸模态,不会发生能量的传递从而诱发摆动模态的振动;但只激发摆动模态,却会由于能量的传递而诱发呼吸模态的振动,但此时能量的传递较弱.

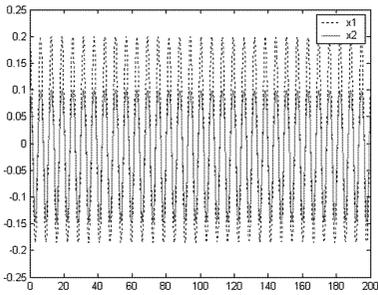


图4 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0.2, 0.1, 0, 0)$ 时的时间历程

Fig. 4 Time history under initial excitation

$$(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0.2, 0.1, 0, 0)$$

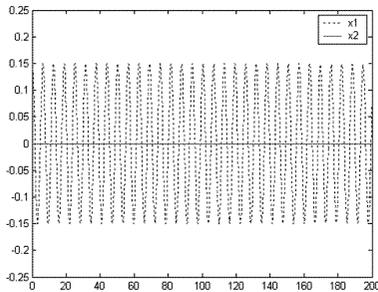


图5 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0.15, 0, 0, 0)$ 时的时间历程

Fig. 5 Time history under initial excitation

$$(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0.15, 0, 0, 0)$$

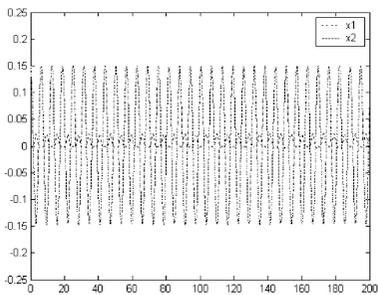


图6 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0.15, 0, 0)$ 时的时间历程

Fig. 6 Time history under initial excitation

$$(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0.15, 0, 0)$$

2.2 1:2 内共振的情况

图7-11是1:2内共振情况下,只激发摆动模态时对应的的时间历程,显见这种能量的传递是非常

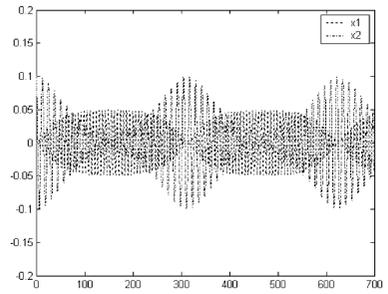


图7 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0.1, 0, 0)$ 时的时间历程

Fig. 7 Time history under initial excitation $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0.1, 0, 0)$

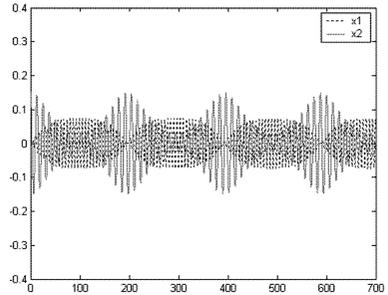


图8 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0.15, 0, 0)$ 时的时间历程

Fig. 8 Time history under initial excitation

$$(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0.15, 0, 0)$$

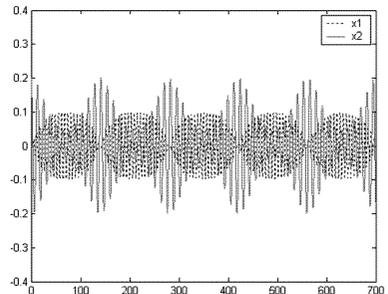


图9 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0.2, 0, 0)$ 时的时间历程

Fig. 9 Time history under initial excitation $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0.2, 0, 0)$

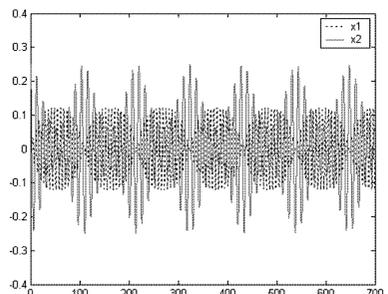


图10 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0.25, 0, 0)$ 时的时间历程

Fig. 10 Time history under initial excitation

$$(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0.25, 0, 0)$$

剧烈的.由于能量不断在两个模态之间传递,所以两个模态的振动都是调幅的,即俗称的拍振.但拍的频率却与扰动的强度有关,扰动越强,拍频越高.图12说明,在1:2内共振情况下,只激发呼吸

模态,仍不会发生能量的传递从而诱发摆动模态的振动.图13-16说明,两个模态都受到同样的激发,振动能量仍是在两个模态之间不断地交换.

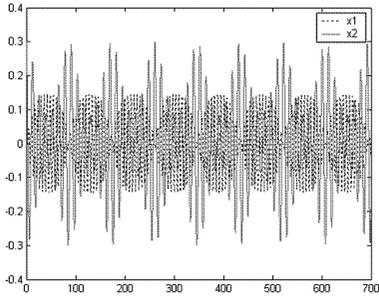


图11 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0, 3, 0, 0)$ 时的时间历程

Fig. 11 Time history under initial excitation $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0, 3, 0, 0)$

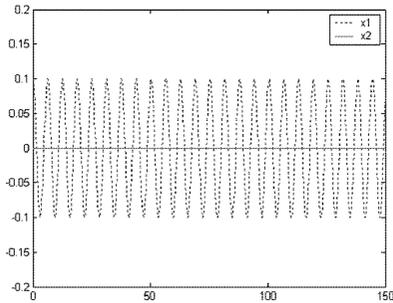


图12 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 1, 0, 0, 0)$ 时的时间历程

Fig. 12 Time history under initial excitation $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 1, 0, 0, 0)$

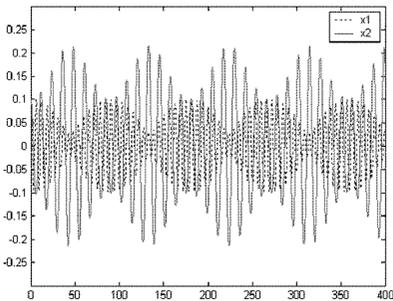


图13 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 1, 0, 1, 0, 0)$ 时的时间历程

Fig. 13 Time history under initial excitation

$$(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 1, 0, 1, 0, 0)$$

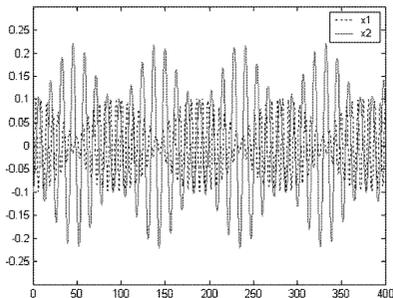


图14 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (-0.1, -0.1, 0, 0)$ 时的时间历程

Fig. 14 Time history under initial excitation

$$(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (-0.1, -0.1, 0, 0)$$

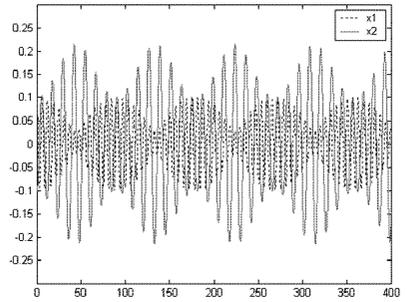


图15 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0.1, -0.1, 0, 0)$ 时的时间历程

Fig. 15 Time history under initial excitation

$$(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0.1, -0.1, 0, 0)$$

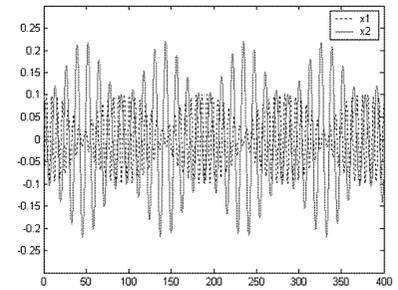


图16 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (-0.1, 0, 1, 0, 0)$ 时的时间历程

Fig. 16 Time history under initial excitation

$$(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (-0.1, 0, 1, 0, 0)$$

2.3 1:3 内共振的情况

图17-22是1:3内共振情况下,不同初始扰动对应的时间历程曲线.图17和18表明,只激发

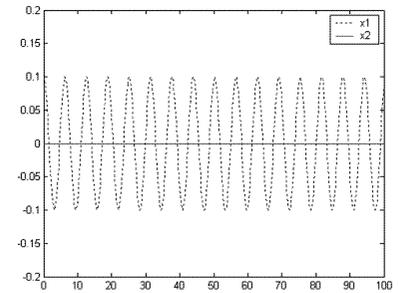


图17 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 1, 0, 0, 0)$ 时的时间历程

Fig. 17 Time history under initial excitation $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 1, 0, 0, 0)$

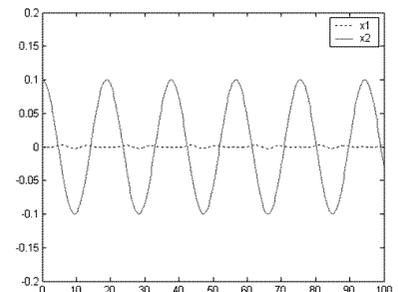


图18 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0, 1, 0, 0)$ 时的时间历程

Fig. 18 Time history under initial excitation $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0, 0, 1, 0, 0)$

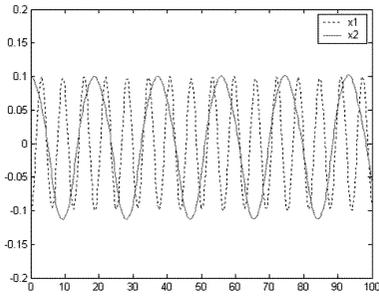


图19 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (-0.1, 0, 1, 0, 0)$ 时的时间历程

Fig. 19 Time history under initial excitation

$$(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (-0.1, 0, 1, 0, 0)$$

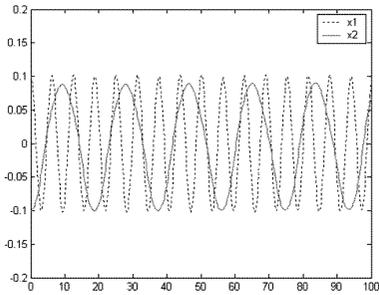


图20 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0.1, -0.1, 0, 0)$ 时的时间历程

Fig. 20 Time history under initial excitation

$$(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0.1, -0.1, 0, 0)$$

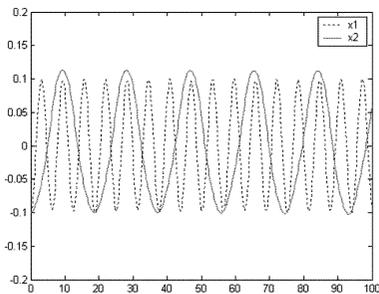


图21 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (-0.1, -0.1, 0, 0)$ 时的时间历程

Fig. 21 Time history under initial excitation

$$(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (-0.1, -0.1, 0, 0)$$

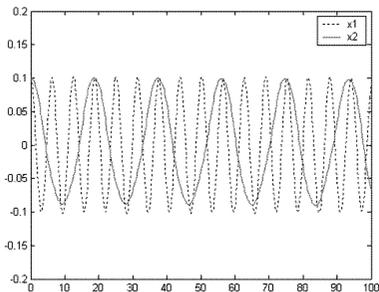


图22 初始扰动为 $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0.1, 0, 1, 0, 0)$ 时的时间历程

Fig. 22 Time history under initial excitation $(x_{10}, x_{20}, \dot{x}_{10}, \dot{x}_{20}) = (0.1, 0, 1, 0, 0)$

呼吸模态,仍不会发生能量的传递从而诱发摆动模态的振动.与前两种情况相反,只激发摆动模态时,能量的传递也是非常微弱的.与1:2内共振的情况

不同,两个模态都被激发时,能量传递的结果不是导致调幅运动,而是两个模态都保持等幅运动.与1:1内共振的情况相反,只有摆动模态的振动有平衡位置的偏离,但偏离很小.

3 讨论与结论

对弹簧摆的运动微分方程保留到三次非线性项,利用数值方法讨论了内共振关系对其动力学行为的影响,得到了如下结论.对1:1内共振的情况,两个模态的振动均可能发生在偏离原来平衡位置的新的平衡位置附近,即出现平衡位置飘移的现象.能量不能从高阶(呼吸)模态传递到低阶(摆动)模态,但可以从低阶(摆动)模态传递到高阶(呼吸)模态;但对1:3内共振的情况,这种能量在两个模态之间的传递却非常弱.对1:1和1:3内共振的情况,能量在两个模态之间的传递导致的是等幅的周期振动.但对1:2内共振的情况,出现的却是调幅的周期运动即拍振.初始条件对拍频有重要影响,初始激励越小,拍频越低;初始激励越大,拍频越高.

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INFLUENCE OF INTERNAL RESONANCE ON DYNAMICS OF SPRING PENDULUMS *

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Abstract On the assumption that both quadratic and cubic nonlinear terms are considered, the influence of internal resonance on the dynamics of spring-pendulum systems was investigated by numerical simulations. Based on the numerical results, it is observed that vibrations in both modes can take place away from its static equilibrium position for the case of 1:1 internal resonance. For any kind of internal resonance, it is found that transfer of energy from pendulum mode into spring mode is possible, whereas the reverse transfer of energy is impossible. Moreover, transfer of energy between the two modes is very weak for the case of 1:3 internal resonance. Periodic motions with constant amplitude are stable for both 1:1 and 1:3 internal resonance. However the periodic motions for the case of 1:2 internal resonance exhibit a phenomenon known as beats, and the beat frequency is associated with initial conditions.

Key words spring pendulums, internal resonance, transfer of energy, stability

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