第 9 卷第 1 期 2011 年 3 月
1672-6553/2011/09(1)/036-4
动 力 学 与 控 制 学 报
JOURNAL OF DYNAMICS AND CONTROL
Vol. 9 No. 1
Mar. 2011

积分事件空间中 Birkhoff 参数方程的场方法 *

张毅
(苏州科技学院土木工程学院,苏州 215011)

摘要 研究事件空间中 Birkhoff 系统的积分方法. 给出了事件空间中 Birkhoff 系统的参数方程. 选取事件空间中某一个 Birkhoff 变量作为其余变量的函数,建立了拟线性的基本偏微分方程组,并将场方法推广应用于积分事件空间中 Birkhoff 参数方程. 由于具体问题中可以选取事件空间中任意一个 Birkhoff 变量作为余下变量的函数,因此本文方法具有灵活性. 该方法的主要困难在于求基本偏微分方程组的完全解. 一旦求出完全解,就可以不用进一步积分而求得系统的运动. 文末,举例说明结果的应用.

关键词 分析力学, Birkhoff 参数方程, 事件空间, 积分方法

引 言

积分方法的研究是分析力学研究的一个重要方面[1]. 著名的 Hamilton-Jacobi 方法在推广应用于非保守和非完整系统时遇到了严重困难并有极其严格的限制[2]. 南斯拉夫学者 Vujanovic B 提出的场方法[3]对积分完整非保守系统动力学方程提供了一个重要工具;梅凤翔将场方法的基本思想推广应用于积分一阶非线性非完整非保守系统的动力学方程[4-6],Birkhoff 系统的动力学方程[7]等. 迄今为止,关于积分约束力学系统的场方法研究已取得了一系列重要成果[8-12]. 本文进一步将场方法推广应用于积分事件空间中 Birkhoff 系统的参数方程.

1 事件空间中 Birkhoff 系统的参数方程

研究事件空间中 Birkhoff 系统,其 Birkhoff 变量为 $a^{\mu}(\mu=1,\cdots,2n)$. 建立 $(2n+1)$ 维事件空间,此空间中点的坐标为 $a^{\mu}(\mu=1,\cdots,2n)$ 和时间 t. 引入记号

$$x_{\mu}=a^{\mu}(\mu=1,\cdots,2n),x_{2n+1}=t \tag{1}$$

那么,所有变量 $x_{\alpha}(\alpha=1,\cdots,2n+1)$ 可作为某参数 τ 的已知函数,令 $x_{\alpha}=x_{\alpha}(\tau)$ 是 C^2 类曲线,使得

$$\frac{dx_{\alpha}}{d\tau}=x'_{\alpha} \tag{2}$$

不同时为零,有

$$\dot{x}_{\alpha}=\frac{dx_{\alpha}}{d\tau}=\frac{x'_{\alpha}}{x'_{2n+1}} \tag{3}$$

假设系统在位形空间中的 Birkhoff 函数为 $B=B(t,a^{v})$,Birkhoff 函数组为 $R_{\mu}=R_{\mu}(t,a^{v})(\mu=1,\cdots,2n)$,则事件空间中的 Birkhoff 函数组为

$$B_{\mu}(x_{\alpha})=R_{\mu}(x_1,x_2,\cdots,x_{2n+1})(\mu=1,\cdots,2n),$$
$$B_{2n+1}(x_{\alpha})=-B(x_1,x_2,\cdots,x_{2n+1}) \tag{4}$$

事件空间中 Birkhoff 系统的参数方程可表为

$$\left(\frac{\partial B_{\beta}}{\partial x_{\alpha}}-\frac{\partial B_{\alpha}}{\partial x_{\beta}}\right)x'_{\beta}=0 \quad (\alpha=1,\cdots,2n+1) \tag{5}$$

容易证明[13],参数方程(5)不全独立,其最后一个方程可由其前面 $2n$ 个方程导出. 不失一般性,假设从方程(5)的前 $2n$ 个方程可以解出

$$x'_{\mu}=-\Omega^{\mu\nu}\left(\frac{\partial B_{2n+1}}{\partial x_{\nu}}-\frac{\partial B_{\nu}}{\partial x_{2n+1}}\right)x'_{2n+1} \quad (\mu,\nu=1,\cdots,2n) \tag{6}$$

其中

$$\Omega_{\mu\nu}=\frac{\partial B_{\nu}}{\partial x_{\mu}}-\frac{\partial B_{\mu}}{\partial x_{\nu}} \quad (\mu,\nu=1,\cdots,2n),$$
$$\det(\Omega_{\mu\nu})\neq 0,\Omega^{\mu\nu}\Omega_{\nu\tau}=\delta_{\mu\tau} \tag{7}$$

2 积分 Birkhoff 参数方程的场方法

根据场方法的基本思想[3],令 x_{α} 中的任意一个,例如 x_1,作为变量 $x_k(k=2,\cdots,2n)$ 和 x_{2n+1} 的函数,即令

$$x_1=u(x_k,x_{2n+1}) \tag{8}$$

将式(8)两端对参数 τ 求导数,并利用方程(6),得到

2010-09-12 收到第 1 稿,2010-11-24 收到修改稿.
* 国家自然科学基金资助项目(10972151)

$$\frac{\partial u}{\partial x_{2n+1}}-\frac{\partial u}{\partial x_k}\Omega^{k\nu}\left(\frac{\partial B_{2n+1}}{\partial x_\nu}-\frac{\partial B_\nu}{\partial x_{2n+1}}\right)+\Omega^{1\nu}\left(\frac{\partial B_{2n+1}}{\partial x_\nu}-\frac{\partial B_\nu}{\partial x_{2n+1}}\right)=0 \tag{9}$$

我们称拟线性偏微分方程(9)为基本偏微分方程. 假设方程(9)的完全解表为形式

$$x_1=u(x_k,x_{2n+1},C_\sigma)(k=2,\cdots,2n;\sigma=1,\cdots,2n) \tag{10}$$

则将式(10)代入方程(9)使方程成为恒等式. 令运动的初始条件为

$$x_\alpha|_{\tau=0}=x_{\alpha0}\quad(\alpha=1,\cdots,2n+1) \tag{11}$$

将式(11)代入式(10),可将一个常数,例如 C_1,用 $x_{\alpha0}$和其余常数 C_j 表出,这样,式(10)可写成

$$x_1=u(x_k,x_{2n+1},x_{\alpha0},C_j)\quad(k,j=2,\cdots,2n) \tag{12}$$

利用文献[3]给出的方法,容易证明,初值问题(6)和(11)的解可由式(12)和$(2n+1)$个代数方程

$$\frac{\partial u}{\partial C_j}=0\quad(j=2,\cdots,2n) \tag{13}$$

来确定. 实际上,假设函数行列式

$$\det\left(\frac{\partial^2 u}{\partial C_j\partial x_k}\right) \tag{14}$$

在 C_j,x_k 的相关域上处处不为零. 将(13)式对参数 τ 求导数,有

$$\frac{\partial^2 u}{\partial C_j\partial x_k}x'_k+\frac{\partial^2 u}{\partial C_j\partial x_{2n+1}}x'_{2n+1}=0 \tag{15}$$

再将基本偏微分方程(9)对 C_j 求偏导数,得

$$\frac{\partial^2 u}{\partial x_{2n+1}\partial C_j}-\frac{\partial^2 u}{\partial x_k\partial C_j}\Omega^{k\nu}\left(\frac{\partial B_{2n+1}}{\partial x_\nu}-\frac{\partial B_\nu}{\partial x_{2n+1}}\right)-\frac{\partial u}{\partial x_k}\frac{\partial}{\partial u}\left[\Omega^{k\nu}\left(\frac{\partial B_{2n+1}}{\partial x_\nu}-\frac{\partial B_\nu}{\partial x_{2n+1}}\right)\right]\frac{\partial u}{\partial C_j}+\frac{\partial}{\partial u}\left[\Omega^{1\nu}\left(\frac{\partial B_{2n+1}}{\partial x_\nu}-\frac{\partial B_\nu}{\partial x_{2n+1}}\right)\right]\frac{\partial u}{\partial C_j}=0 \tag{16}$$

比较式(15)和式(16),并利用式(13),得

$$x'_k=-\Omega^{k\nu}\left(\frac{\partial B_{2n+1}}{\partial x_\nu}-\frac{\partial B_\nu}{\partial x_{2n+1}}\right)x'_{2n+1}\quad(k=2,\cdots,2n;v=1,\cdots,2n) \tag{17}$$

式(17)就是方程(6)的后面$(2n-1)$个方程. 将式(12)对求 τ 导数,有

$$x'_1=\frac{\partial u}{\partial x_k}x'_k+\frac{\partial u}{\partial x_{2n+1}}x'_{2n+1} \tag{18}$$

上式中可由方程(9)消去$\frac{\partial u}{\partial x_{2n+1}}$,并利用方程(17),有

$$x'_1=\frac{\partial u}{\partial x_k}x'_k+\frac{\partial u}{\partial x_k}\Omega^{k\nu}\left(\frac{\partial B_{2n+1}}{\partial x_\nu}-\frac{\partial B_\nu}{\partial x_{2n+1}}\right)x'_{2n+1}-\Omega^{1\nu}\left(\frac{\partial B_{2n+1}}{\partial x_\nu}-\frac{\partial B_\nu}{\partial x_{2n+1}}\right)x'_{2n+1}=-\Omega^{1\nu}\left(\frac{\partial B_{2n+1}}{\partial x_\nu}-\frac{\partial B_\nu}{\partial x_{2n+1}}\right)x'_{2n+1} \tag{19}$$

式(19)是方程(6)的第一个方程.

以上我们将场方法推广到积分事件空间中 Birkhoff 系统的参数方程. 场方法的主要困难在于求基本偏微分方程(9)的完全解. 只要能够求出基本偏微分方程(9)的完全解,不用任何进一步积分,便可由式(12)和代数方程(13)求得事件空间中 Birkhoff 系统的解.

3　算例

设事件空间中四阶 Birkhoff 系统的 Birkhoff 函数组为

$$B_1=x_3,B_2=x_4,B_3=B_4=0,B_5=-x_2-\frac{1}{2}(x_3^2+x_4^2) \tag{20}$$

试用场方法求系统的解.

事件空间中 Birkhoff 系统的参数方程(5)给出

$$-x'_3=0,-x'_4-x'_5=0,x'_1-x_3x'_5=0,x'_2-x_4x'_5=0,x'_2+x_3x'_3+x_4x'_4=0 \tag{21}$$

由方程(21)的前 4 个方程,可解出

$$x'_1=x_3x'_5,x'_2=x_4x'_5,x'_3=0,x'_4=-x'_5 \tag{22}$$

令

$$x_1=u(x_2,x_3,x_4,x_5) \tag{23}$$

基本偏微分方程(9)给出为

$$\frac{\partial u}{\partial x_5}+\frac{\partial u}{\partial x_2}x_4-\frac{\partial u}{\partial x_4}-x_3=0 \tag{24}$$

令方程(24)的完全解有形式

$$u=f_1(x_5)+f_2(x_5)x_2+f_3(x_5)x_3+f_4(x_5)x_4 \tag{25}$$

将式(25)代入方程(24),并令自由项以及含 x_2, x_3,x_4 的项分别为零,得到为确定f_1,f_2,f_3,f_4 的微分方程

$$\frac{df_1}{dx_5}-f_4=0,\frac{df_2}{dx_5}=0,\frac{df_3}{dx_5}-1=0,\frac{df_4}{dx_5}+f_2=0 \tag{26}$$

积分之,可得

$$f_1=C_1-\frac{1}{2}C_2x_5^2+C_4x_5,f_2=C_2,$$

$$f_3 = C_3 + x_5, f_4 = C_4 - C_2 x_5 \tag{27}$$

其中 C_1, C_2, C_3, C_4 为常数. 将式(27)代入式(25),得

$$u = C_1 - \frac{1}{2}C_2 x_5^2 + C_4 x_5 + C_2 x_2 + (C_3 + x_5)x_3 + (C_4 - C_2 x_5)x_4 \tag{28}$$

设初始条件为

$$x_\alpha|_{\tau=0} = x_{\alpha 0} \quad (\alpha = 1, \cdots, 5) \tag{29}$$

将初始条件(29)代入式(28),解出

$$C_1 = x_{10} + \frac{1}{2}C_2 x_{50}^2 - C_4 x_{50} - C_2 x_{20} - (C_3 + x_{50})x_{30} - (C_4 - C_2 x_{50})x_{40} \tag{30}$$

将式(30)代入式(28),有

$$u = x_{10} + \frac{1}{2}C_2 x_{50}^2 - C_4 x_{50} - C_2 x_{20} - (C_3 + x_{50})x_{30} - (C_4 - C_2 x_{50})x_{40} - \frac{1}{2}C_2 x_5^2 + C_4 x_5 + C_2 x_2 + (C_3 + x_5)x_3 + (C_4 - C_2 x_5)x_4 \tag{31}$$

方程(13)给出

$$\frac{\partial u}{\partial C_2} = 0, \frac{\partial u}{\partial C_3} = 0, \frac{\partial u}{\partial C_4} = 0 \tag{32}$$

即

$$\frac{1}{2}x_{50}^2 - x_{20} + x_{50}x_{40} - \frac{1}{2}x_5^2 + x_2 - x_4 x_5 = 0, \quad -x_{30} + x_3 = 0, \quad -x_{50} - x_{40} + x_5 + x_4 = 0 \tag{33}$$

由此解得

$$x_2 = x_{20} + x_{40}(x_5 - x_{50}) - \frac{1}{2}(x_5 - x_{50})^2, \quad x_3 = x_{30}, x_4 = x_{40} + x_{50} - x_5 \tag{34}$$

将式(34)代入式(31),得

$$u = x_1 = x_{10} + x_{30}(x_5 - x_{50}) \tag{35}$$

式(34)和(35)即为所论事件空间中 Birkhoff 系统的参数方程的解.

4 结论

本文给出了积分事件空间中 Birkhoff 参数方程的一种新的积分方法,即场方法。本文方法的优越性在于:其一,基本偏微分方程(9)是拟线性的;其二,方法具有灵活性,对于具体问题可选取事件空间中 Birkhoff 变量中的任意一个变量作为余下 2n 个变量的函数。本文方法的主要困难在于求基本偏微分方程(9)的完全解。只要求出完全解,就可以不用进一步积分而求得系统的运动。

参考文献

1 梅凤翔,刘端,罗勇. 高等分析力学. 北京:北京理工大学出版社, 1991 (Mei Fengxiang, Liu Duan, Luo Yong. Advanced analytical mechanics. Beijing: Beijing Institute of Technology Press, 1991 (in Chinese))

2 Rumyantzev V V, Sumbatov A S. On the problem of a generalization of the Hamilton-Jacobi method for nonholonomic systems. *ZAMM*, 1978, 58: 477 ~ 481

3 Vujanović B. A field method and its application to the theory of vibrations. *Int J Non-Linear Mechanics*, 1984, 19: 383 ~ 396

4 Mei Fengxiang. A field method for solving the equations of motion of nonholonomic systems. *Acta Mechanica Sinica*, 1989, 5(3): 260 ~ 268

5 Mei Fengxiang. Parametric equations of nonholonomic nonconservative systems in the event space and their integration method. *Acta Mechnica Sinica*, 1990, 6(2): 160 ~ 168

6 Mei Fengxiang. A field method for integrating the equations of motion of nonholonomic controllable systems. *Applied Mathematics and Mechanics*, 1992, 13(2): 181 ~ 187

7 梅凤翔,史荣昌,张永发,吴惠彬. BIRKHOFF 系统动力学. 北京:北京理工大学出版社, 1996 (Mei Fengxiang, Shi Rongchang, Zhang Yongfa, Wu Huibin. Dynamics of Birkhoffian systems. Beijing: Beijing Institute of Technology Press, 1996 (in Chinese))

8 Luo Shaokai. The integration methods of Vacco dynamics equation of nonlinear nonholonomic systems. *Applied Mathematics and Mechanics*, 1995, 16(11): 1055 ~ 1964

9 罗绍凯,郭永新,陈向炜,傅景礼. 转动相对论 Birkhoff 系统动力学的场方法. 物理学报, 2001, 50(11): 2049 ~ 2052 (Luo Shaokai, Guo Yongxin, Chen Xiangwei and Fu Jingli. A field method for solving the equations of motion of a rotational realtivistic Birkhoffian system. *Acta Phys. Sinica*, 2001, 50(11): 2049 ~ 2052 (in Chinese))

10 Kovacic I. A field method in the study of weakly non-linear two-degree-of-freedom oscillatory systems. *Journal of Sound and Vibration*, 2004, 27(1): 464 ~ 468

11 Kovacic I. On the field method in non-holonomic mechanics. *Acta Mechanica Sinica*, 2005, 21(2): 192 ~ 196

12 葛伟宽. Whittaker 方程的场方法. 物理学报, 2006, 55(1): 10 ~ 12 (Ge Weikuan. A field method for solving Whittaker equations. *Acta Phys. Sinica*, 2006, 55(1): 10

~12(in Chinese))

13 张毅. 事件空间中 Birkhoff 系统的参数方程及其第一积分. 物理学报,2008,57(5):2649~2653 (Zhang Yi. Parametric equations and its first integrals for Birkhoffian systems in the event space. *Acta Phys. Sinica*, 2008,57(5):2649~2653(in Chinese))

A FIELD METHOD FOR INTEGRATING BIRKHOFF'S PARAMETRIC EQUATIONS IN EVENT SPACE *

Zhang Yi

(*College of Civil Engineering*, *Suzhou University of Science and Technology*, *Suzhou* 215011, *China*)

Abstract The integration method of Birkhoffian systems in event space was studied, and the parametric equations of the Birkhoffian systems in the event space were given. By choosing a Birkhoff's variable in the event space as the function of the rest of the variables, the basic system of quasi-linear partial differential equation was set up, and the idea of field method was generalized to the integration of Birkhoff's parametric equations in the event space. Since any Birkhoff's variable can be chosen as the function of the remaining variables in a specific problem, so the method has flexibility. The major difficulty of this method lies in how to find the complete solution of the basic partial differential equation. Once the complete solution was found, the motion of the systems can be obtained without doing further integration. At the end of the paper, an example was given to illustrate the application of the results.

Key words analytical mechanics, Birkhoff's parametric equation, event space, integration method

Received 12 September 2010, revised 24 November 2010.

* The project supported by the National Natural Science Foundation of China(10972151)