第 9 卷第 1 期 2011 年 3 月
1672-6553/2011/09(1)/027-5
动力学与控制学报
JOURNAL OF DYNAMICS AND CONTROL
Vol. 9 No. 1
Mar. 2011

双向弯曲与扭转耦合 Bernoulli-Euler 薄壁梁稳定问题求解*

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摘要　基于薄壁杆件理论,研究了 Bernoulli－Euler 薄壁梁双向弯曲与扭转耦合的稳定问题. 运用能量变分原理,推导了薄壁梁双向弯曲与扭转耦合的振动微分方程,引入轴向荷载的影响,建立了求解稳定的特征方程,运用降阶法和变轴向荷载的扫描法求得了薄壁梁稳定临界力的解析解. 通过算例验证,证明了本文计算的稳定临界力收敛的可行性,并与有限元法结果吻合较好,验证了本文方法的正确性.

关键词　Bernoulli－Euler 薄壁梁,　弯扭耦合,　临界荷载,　解析解

引言

薄壁梁具有明显的空间结构特征,对于截面非对称结构或在非对称荷载作用下,存在着弯扭耦合作用,其受力分析比较复杂,使得薄壁梁的极限承载力大多由稳定条件控制[1,2]. 文献[3]研究了质点对轴向受力的 Euler－ Bernoulli 梁结构任意位置的横向撞击问题,文献[4]研究作旋转运动的柔性梁的线接触正碰撞问题,但均没有考虑梁的稳定问题. 文献[5~7]利用能量原理,研究了在偏心轴力和横向均布荷载作用下的单轴对称截面薄壁结构的稳定问题,文献[8]利用能量法和假想荷载法,研究了薄壁结构弯扭失稳问题. 但上述研究大部分局限于薄壁结构的单向弯曲与扭转的稳定分析,对于结构的双向弯曲与扭转耦合的稳定研究尚不多见. 本文运用能量变分原理,利用推导出 Bernoulli－Euler 薄壁梁双向弯曲与扭转耦合的振动微分方程[9],加入轴向荷载的影响,运用动力法求解了薄壁结构的稳定的临界荷载.

1　稳定微分方程的建立

1.1　基本假定及自由振动微分方程

薄壁杆件的弯曲和扭转理论,建立在 Vlasov 两个基本假设之上.

(1)杆件受力变形的过程中,其横截面的周边形状投影始终保持不变.

(2)杆件除受有圣维南扭转剪切变形外,其它剪切变形为零.

(3)杆件截面任意一点的弯曲剪切变形相等.

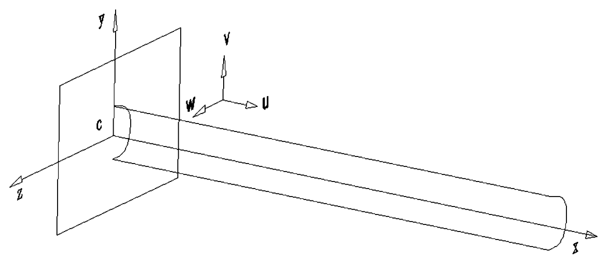

图 1　薄壁杆件位移模式

Fig. 1　Displacement mode of thin wall beam

得到薄壁杆件自由振动微分方程组[8]为

$$\begin{cases} EAu'' - \rho A\ddot{u} = 0 \\ EI_{yy}v_s^{(4)} - \rho I_{yy}\ddot{v}_s'' + \rho A\ddot{v}_s + \rho Az_s\ddot{\theta} = 0 \\ EI_{zz}w_s^{(4)} - \rho I_{zz}\ddot{w}_s'' + \rho A\ddot{w}_s + \rho Ay_s\ddot{\theta} = 0 \\ EI_w^s\theta^{(4)} - \rho I_w^s\ddot{\theta}'' = GI_k\theta'' + \rho(I_{yy} + I_{zz} + \\ \quad Ay_s^2 + Az_s^2)\ddot{\theta} + \rho Az_s\ddot{v}_s - \rho Ay_s\ddot{w}_s \end{cases} \tag{1}$$

1.2　边界条件

各种边界条件[8]如下:

(a)自由边界:

$$\partial^2 v/\partial x^2 = \partial^3 v/\partial x^3 = 0,$$

$$\partial^2 w/\partial x^2 = \partial^3 w/\partial x^3 = 0$$

$$\partial^2\theta/\partial x^2 = -EI_w^s\partial^3\theta/\partial x^3 + GI_k\partial\theta/\partial x = 0 \tag{2}$$

(b)固定边界:

$$v = \partial v/\partial x = 0, w = \partial w/\partial x = 0, \theta = \partial\theta/\partial x = 0 \tag{3}$$

2010-09-14 收到第 1 稿,2010-10-18 收到修改稿.

*国家自然科学基金资助项目(50978058);全国优秀博士学位论文作者专项资金资助项目(200954);广东省自然科学基金资助项目(9151063101000050)

(c)简支边界：

$$v=\partial^2 v/\partial x^2=0,$$

$$w=\partial^2 w/\partial x^2=0,$$

$$\theta=\partial^2\theta/\partial x^2=0 \tag{4}$$

1.3 稳定条件引入

由轴向荷载 P 所产生的弯矩分别是

$$M_x=Pi_0^2\theta'-Pw'y_s+Pv'z_s \tag{5}$$

$$M_\eta=M_z=P(v_s+z_s\theta) \tag{6}$$

$$M_\xi=M_y=P(w_s-y_s\theta) \tag{7}$$

其中：i_0 为极回转半径.

将式(5)和式(6)均微分两次和式(7)微分一次后代入式(1)，就得到了加入轴向荷载的 Bernoulli－Euler 薄壁杆件双向弯曲与扭转耦合的稳定微分方程：

$$\begin{cases} EI_{yy}v_s^{(4)}-\rho I_{yy}\ddot{v}_s''+\rho A v_s+\rho Az_s\ddot{\theta}+Pv_s''+Pz_s\theta''=0 \\ EI_{zz}w_s^{(4)}-\rho I_{zz}\ddot{w}_s''+\rho A\ddot{w}_s-\rho Ay_s\ddot{\theta}+Pw_s''-Py_s\theta''=0 \\ EI_w^s\theta^{(4)}-\rho I_w^s\ddot{\theta}''-GI_k\theta''+\rho(I_{yy}+I_{zz}+Ay_s^2+ \\ \quad Az_s^2)\ddot{\theta}+\rho Az_s\ddot{v}_s-\rho Ay_s\ddot{w}_s+Pi_0^2\theta''-Py_sw_s''+ \\ \quad Py_sw''s+Pz_sv_s'' \end{cases} \tag{8}$$

2 稳定微分方程的求解

2.1 特征方程的建立

假设 Bernoulli－Euler 薄壁杆件各个方向的自由振动均随时间按正弦变化的，其圆频率为 ω，分离变量并引入过渡变量，则双向弯扭耦合的方程组化为：

$$\begin{cases} (-\lambda_a+\lambda_bD^4+D^2+\lambda_md^2)\hat{v} \\ \quad (\lambda_mD^2-\lambda_a)z_s\hat{\theta}=0 \\ (-\lambda_c+\lambda_bD^4+D^2+\lambda_nD^2)\hat{w}+ \\ \quad (-\lambda_mD^2+\lambda_a)y_s\hat{\theta}=0 \\ (\lambda_cD^2-\lambda_d)z_s\hat{v}+(-\lambda_cD^2+\lambda_d)y_s\hat{w}+ \\ \quad (-\lambda_eD^2+\lambda_bD^4+D^2-\lambda_g+\lambda i_0^2D^2)\hat{\theta}=0 \end{cases} \tag{9}$$

其中：$\hat{v},\hat{w},\hat{\theta}$ 为三个方向的振型函数；

$$\lambda_a=L^2A/I_{yy},$$

$$\lambda_b=E/L^2\rho\omega^2,$$

$$\lambda_c=L^2A/I_{zz},$$

$$\lambda_d=L_A^2/I_w^s,$$

$$\lambda_e=GI_k/\rho\omega^2I_w^s,$$

$$\lambda_g=L^2(I_{yy}+I_{zz}+Ay_s^2+Az_s^2)/I_w^s,$$

$$\lambda_w=EI_w^s/L^2GI_k,$$

$$\lambda_m=P/(\rho\omega^2I_{yy}),$$

$$\lambda_n=P/(\rho\omega^2I_{zz}),$$

$$\lambda_m=P/(\rho\omega^2I_w),$$

$$i_0=(I_{zz}+I_{yy})/A+y_s^2+z_s^2;$$

$$D\equiv Ld/dx. \tag{10}$$

把方程组(9)消除 $\hat{w},\hat{\theta}$ 两个变量，可得只有 $\hat{v}$ 的方程，并把 $\hat{v}$ 改写为关于 τ 的多项式方程 $f(\tau)$，有：

$$\begin{aligned} f(\tau)=&\tau^6\lambda_b^3+\tau^5\lambda_b^2(3-\lambda_e+\lambda_m+\lambda_n+i_0^2\lambda_o)+ \\ &\tau^4\lambda_b(3-\lambda_b(\lambda_a+\lambda_c+\lambda_g)+2\lambda_m+2\lambda_n+ \\ &\lambda_m\lambda_n-\lambda_e(2+\lambda_m+\lambda_n)+(-z_s^2\lambda_m-y_s^2\lambda_n+ \\ &i_0^2(2+\lambda_m+\lambda_n))\lambda_o+\tau^3(-(-1+\lambda_e)(1+ \\ &\lambda_m)(1+\lambda_n)+(-y_s^2(1+\lambda_m)\lambda_n-z_s^2\lambda_m(1+ \\ &\lambda_n)+i_0^2(1+\lambda_m)(1+\lambda_n))\lambda_o+\lambda_a\lambda_b(-2+ \\ &\lambda_e-\lambda_n-i_0^2\lambda_o+z_s^2\lambda_o)+\lambda_b(-\lambda_g(2+\lambda_m+ \\ &\lambda_n)\lambda+\lambda_d(z_s^2\lambda_m+y_s^2\lambda_n)+\lambda_c(-2+\lambda_e- \\ &\lambda_m-i_0^2\lambda_o+y_s^2\lambda_o)))+\tau^2(-\lambda_g+(z_s^2\lambda_d- \\ &\lambda_g)\lambda_m+(z_s^2\lambda_d\lambda_m+y_s^2\lambda_d(1+\lambda_m)-\lambda_g(1+ \\ &\lambda_m))\lambda_n+\lambda_a(\lambda_b(\lambda_c-z_s^2\lambda_d+\lambda_g)+(-1+ \\ &\lambda_e)(1+\lambda_n)-(-y_s^2\lambda_n+i_0^2(1+\lambda_n)- \\ &z_s^2(1+\lambda_n))\lambda_o)+\lambda_c(\lambda_b\lambda_g+(-1+ \\ &\lambda_e)(1+\lambda_m)-i_0^2\lambda_o+(i_0^2+z_s^2)\lambda_m\lambda_o+ \\ &y_s^2(-\lambda_b\lambda_d+(1+\lambda_m)\lambda_o)))- \\ &\tau(\lambda_c(z_s^2\lambda_d\lambda_m+y_s^2(1+\lambda_m)-\lambda_g(1+\lambda_m))+ \\ &\lambda_a(y_s^2\lambda_d\lambda_n+z_s^2\lambda_d(1+\lambda_n)-\lambda_g(1+\lambda_n)+ \\ &\lambda_c(-1+\lambda_e+(-i_0^2+y_s^2+z_s^2)\lambda_o)))+ \\ &\lambda_a\lambda_c((y_s^2+z_s^2)\lambda_d-\lambda_g) \end{aligned} \tag{11}$$

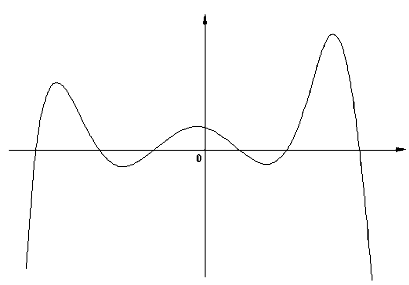

图 2 Bernoulli－Euler 薄壁杆件动力稳定特征函数形状

Fig. 2 Shape of the dynamic stabilization eigenfunction of the Bernoulli－Euler beam

$f(\tau)$的函数曲线如图 2 所示，方程虽然增加了轴向

力的影响,但并没有改变方程的物理性质,所以我们可得到方程 $f(\tau)$ 的 6 个实根 $\tau_1,\tau_2,\tau_3,-\tau_4,-\tau_5,-\tau_6$. 由此,特征方程的 12 个根为 $s_1,-s_1,s_2,-s_2,s_3,-s_3,is_4,-is_4,is_5,-is_5,is_6,-is_6$.

由特征方程求出的 12 个根可以得出 $\hat{v},\hat{w}$ 和 $\hat{\theta}$ 的振型:

$$\begin{aligned}\hat{v} = &A_c\mathrm{osh}(s_1x/L)+A_2\sinh(s_1x/L)+\\&A_3\cosh(s_2x/L)+A_4\sinh(s_2x/L)+\\&A_5\cosh(s_3x/L)+A_6\sinh(s_3x/L)+\\&A_7\cos(s_4x/L)+A_8\sin(s_4x/L)+\\&A_9\cos(s_5x/L)+A_{10}\sin(s_5x/L)+\\&A_{11}\cos(s_6x/L)+A_{12}\sin(s_6x/L)\end{aligned}\tag{12}$$

$$\begin{aligned}\hat{w} = &B_c\mathrm{osh}(s_1x/L)+B_2\sinh(s_1x/L)+\\&B_3\cosh(s_2x/L)+B_4\sinh(s_2x/L)+\\&B_5\cosh(s_3x/L)+B_6\sinh(s_3x/L)+\\&B_7\cos(s_4x/L)+B_8\sin(s_4x/L)+\\&B_9\cos(s_5x/L)+B_{10}\sin(s_5x/L)+\\&B_{11}\cos(s_6x/L)+B_{12}\sin(s_6x/L)\end{aligned}\tag{13}$$

$$\begin{aligned}\hat{\theta} = &C_c\mathrm{osh}(s_1x/L)+C_2\sinh(s_1x/L)+\\&C_3\cosh(s_2x/L)+C_4\sinh(s_2x/L)+\\&C_5\cosh(s_3x/L)+C_6\sinh(s_3x/L)+\\&C_7\cos(s_4x/L)+C_8\sin(s_4x/L)+\\&C_9\cos(s_5x/L)+C_{10}\sin(s_5x/L)+\\&C_{11}\cos(s_6x/L)+C_{12}\sin(s_6x/L)\end{aligned}\tag{14}$$

其中,系数 $A_i,B_i,C_i(i=1,2,\cdots,12)$ 并非相互独立的,根据方程组(9),我们可建立如下的关系式:

$$A_l=C_lz_s(\lambda_c+\lambda_ds_1^2-1)/(\lambda_as_1^4+\lambda_bs_1^2+\lambda_c-1),$$
$$B_l=-C_ly_s(\lambda_g+\lambda_hs_1^2-1)/(\lambda_es_1^4+\lambda_fs_1^2+\lambda_g-1);$$
$$(l=1,2)$$

$$A_m=C_mz_s(\lambda_c+\lambda_ds_2^2-1)/(\lambda_as_2^4+\lambda_bs_2^2+\lambda_c-1),$$
$$B_m=-C_my_s(\lambda_g+\lambda_hs_2^2-1)/(\lambda_es_2^4+\lambda_fs_2^2+\lambda_g-1);$$
$$(m=3,4)$$

$$A_n=C_nz_s(\lambda_c+\lambda_ds_3^2-1)/(\lambda_as_3^4+\lambda_bs_3^2+\lambda_c-1),$$
$$B_n=-C_ny_s(\lambda_g+\lambda_hs_3^2-1)/(\lambda_es_3^4+\lambda_fs_3^2+\lambda_g-1);$$
$$(n=5,6)$$

$$A_p=C_pz_s(\lambda_c+\lambda_ds_4^2-1)/(\lambda_as_4^4+\lambda_bs_4^2+\lambda_c-1),$$
$$B_p=-C_py_s(\lambda_g+\lambda_hs_4^2-1)/(\lambda_es_4^4+\lambda_fs_4^2+\lambda_g-1);$$
$$(p=7,8)$$

$$A_q=C_qz_s(\lambda_c+\lambda_ds_5^2-1)/(\lambda_as_5^4+\lambda_bs_5^2+\lambda_c-1),$$
$$B_q=-C_qy_s(\lambda_g+\lambda_hs_5^2-1)/(\lambda_es_5^4+\lambda_fs_5^2+\lambda_g-1);$$
$$(m=9,10)$$

$$A_r=C_rz_s(\lambda_c+\lambda_ds_6^2-1)/(\lambda_as_6^4+\lambda_bs_6^2+\lambda_c-1),$$
$$B_r=-C_ry_s(\lambda_g+\lambda_hs_6^2-1)/(\lambda_es_6^4+\lambda_fs_6^2+\lambda_g-1);$$
$$(m=11,12)$$

引入边界条件,每个方向有 4 个边界条件,总共 12 个边界条件,就得到方程:

$$[D]_{12\times12}\{C\}_{12\times1}=0\tag{15}$$

$[D]_{12\times12}$ 由两端边界条件组合而成. 要使 $\{C\}_{12\times1}$ 不全为 0,只能令 $[D]_{12\times12}$ 的行列式为 0,从 0 开始扫描轴向荷载 P,当轴向荷载 P 增大到适当大小时,$[D]_{12\times12}$ 的行列式就为 0,这个 P 值就是我们所要求解的失稳临界荷载.

3　算例验证

为了验证本文方法的正确性,现选取一组不同跨高比、截面相同但无任何对称、两端固定的薄壁梁的本文计算方法与有限元软件 Ansys 的计算结果进行比较.

梁的各个尺寸和支撑条件如图 3 所示,图中尺寸单位为 mm. 材料特性 $E=2.10\times10^{10}\mathrm{N/mm^2}$,$G=8.10\times10^{9}\mathrm{N/mm^2}$,$\rho=0.0078\mathrm{g/mm^3}$,截面特性 $A=2772\mathrm{mm^2}$,$I_k=45790\mathrm{mm^4}$,$I_{yy}=1836310\mathrm{mm^4}$,$I_{zz}=23790000\mathrm{mm^4}$,$I_w^s=1.172\times10^{10}\mathrm{mm^6}$,$y_s=-35.7\mathrm{mm}$,$z_s=53.69\mathrm{mm}$.

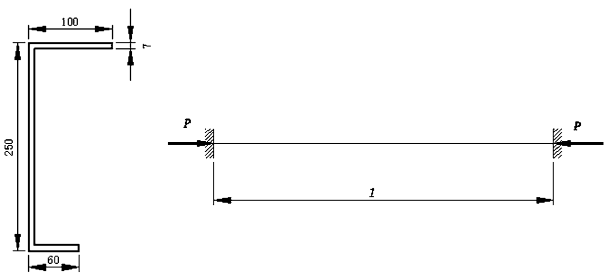

图 3　梁支承及截面图(尺寸单位:mm)

Fig. 3　Boundary conditions and cross - sectional shape of beam

3.1　临界荷载收敛性探讨

在推导求解屈曲临界荷载的过程中,对薄壁杆件定义一个接近于 0 的自振频率,使得求解特征方程的过程有意义,现在需要讨论的自振频率接近 0 的程度对屈曲临界荷载收敛有多大的影响. 从表 1 可看出当自振频率缩小到小数后两位时,稳定屈曲荷载已不再变化,即使自振频率取到整数位,相对误差也不超过 1%,证明了整个计算过程具有良好的收敛性.

表1 稳定屈曲荷载随自振频率的收敛关系

Table 1 Relation of Convergence of buckling load and frequency

Natural vibration frequencies	1	0.1	0.01	0.001	0.0001	0.00001
Buckling loads of stability	133997	134159	134161	134161	134161	134161
Relative error	0.1222%	0.0015%	0.0000%	0.0000%	0.0000%	0.0000%

3.2 临界荷载收敛性探讨

取自振频率为0.01,按照此截面计算跨高比由5、10、20、40、60的稳定屈曲荷载临界值,并与beam188和beam189的有限元计算值相互比较.表2反映了不同高跨比之间的计算结果,而理论计算的结果更接近于一次单元beam188的有限元计算结果,这是由于本文的动力稳定理论还仅限于运用Bernoulli - Euler薄壁杆件计算所造成的,其最大相对误差不超过10%,可供实际的工程计算参考.

表2 稳定屈曲荷载计算对比

Table 2 Contrast of the buckling load

Ratio of span to height	Theoretical calculating value	Beam188	Beam189	Maximum relative error
5	4212691	3954100	3902200	7.37%
10	1188601	1165000	1150900	3.17%
20	400001	386960	383330	4.17%
40	134161	124260	123070	8.27%
60	64051	58083	57935	9.55%

4 小结

(1)本文通过能量泛函变分原理建立了Bernoulli - Euler薄壁梁双向弯曲与扭转耦合的振动微分方程,加入了轴向荷载的影响,以动力法求解薄壁结构的稳定问题,运用降阶法和变轴向荷载的扫描法求得薄壁梁稳定临界力的解析解.

(2)通过算例验证,证明了稳定临界力收敛的可行性,经与有限元法结果比较吻合较好,验证了本文方法的正确性,可供研究设计参考.

(3)文本研究是基于Bernoulli - Euler薄壁梁理论求解结构的临界荷载.对于跨高比较小的情况,考虑剪切变形的Timoshenko薄壁梁对临界荷载的影响,还有待进一步研究.

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STABILIZATION SOLUTION IN BERNOULLI-EULER THIN-WALLED BEAM CONSIDERING COUPLED COMPOUND BENDING AND TWISTING *

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Abstract Based on the theory of thin-walled structures, the stability of coupled bending and torsional vibration of thin-walled beam was researched. According to the variation method and energy principles in mechanic, the oscillatory differential equation of Bernoulli-Euler thin-walled beam was deduced. The effect of axial force on the structure was considered, and then the characteristic equation of critical force was built. Finally, the analytic solution of differential equation was deduced by using method of order reducing and sweep method of axial force modifying. Calculation examples show that the results are convergent and agree well with the results of finite element method, which prove the effectiveness of this method.

Key words Bernoulli-Euler thin-walled beam, coupling of bending and torsional, critical force, analytic solution

Received 14 September 2010, revised 18 October 2010.

* The authors wish to acknowledge the financial support of National Natural Science Foundation of China (50978058), Foundation for the Author of National Excellent Doctoral Dissertation of P. R. China (200954) and the Natural Science Foundation of Guangdong Province (9151063101000050)