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矩形中厚板自由振动问题的辛本征展开定理*

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摘要　研究了矩形中厚板自由振动问题导出的一个Hamilton算子的本征值问题.在广义位移与内力构成的混合边界条件下,首先求解了相应算子的本征函数.接着,证明了本征函数系的完备性,这为使用分离变量法求解相应问题提供了可行性.最后,根据文中证明的展开定理获得了问题的一般解,并给出了具体的数值算例.

关键词　矩形中厚板,　Hamilton系统,　辛正交性,　本征展开,　一般解

引言

对于分离变量后可导向Strum - Liouville问题的偏微分方程,分离变量法是一种十分有效的求解方法,这依赖于自伴算子本征函数系的完备性.然而,对于某些非自伴问题,传统的分离变量法就显得无能为力.钟万勰教授[1]利用结构力学与最优控制相模拟的理论,与Hamilton算子相结合,建立了弹性力学求解新体系,理性求解了许多力学问题[2-14].新体系方法拓广了传统分离变量法的适用范围,导向了Hamilton算子的本征值问题.Hamilton算子是一类非自伴算子,在求解新体系中,辛本征函数系的完备性是首要解决的问题[15-17].

文献[18]建立了矩形中厚板自由振动问题的辛本征展开解法,但该文并没有考虑这一方法的可行性问题,即相应Hamilton算子本征函数系的完备性问题.利用Mathematica软件的帮助,结合Fourier分析方法,本文从矩形中厚板的自由振动问题中导出了一个完备的辛本征函数系,建立了相应的辛本征展开定理.基于展开定理,得到了问题的一般解,并给出了具体的数值算例.

1　预备知识

为叙述简洁,用Z表示非零整数集合.

定义　设$H: D(H)\subset X\times X\to X\times X$是稠定闭线性算子,如果

$$H=\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}$$

其中A是X中的稠定闭线性算子,B和C为自伴(对称)算子,则称H为无穷维Hamilton算子.

2　完备的辛本征展开

矩形中厚板的三个广义位移可用两个函数表示如下:

$$W=F-\frac{D}{C}\nabla^2F;\phi_x=\frac{\partial F}{\partial x}+\frac{\partial \Psi}{\partial y};\phi_y=\frac{\partial F}{\partial y}-\frac{\partial \Psi}{\partial x} \tag{1}$$

而矩形中厚板自由振动问题的基本方程为:

$$D\nabla^2\nabla^2F-\rho\omega^2(F-\frac{D}{C}\nabla^2F)=0 \tag{2}$$

$$\nabla^2\Psi-\frac{2C}{D(1-v)\Psi}=0 \tag{3}$$

其中$\nabla^2=\frac{\partial^2}{\partial x^2}$, $C=\frac{5}{6}Gh$为剪切刚度,$D=\frac{Eh^3}{12(1-v^2)}$为抗弯刚度,$G=\frac{E}{12(1+v)}$为材料的剪切模量.E,v,h,ρ,ω分别为材料的弹性模量、泊松比、板的厚度、板密度和板的固有频率.

板的内力可表示为:

$$M_x=-D[\frac{\partial^2F}{\partial x^2}+v\frac{\partial^2F}{\partial y^2}+(1+v)\frac{\partial^2\Psi}{\partial x\partial y}] \tag{4}$$

$$M_y=-D[\frac{\partial^2F}{\partial y^2}+v\frac{\partial^2F}{\partial x^2}-(1+v)\frac{\partial^2\Psi}{\partial x\partial y}] \tag{5}$$

$$M_{xy}=-D(1+v)[\frac{\partial^2\Psi}{\partial x\partial y}-\frac{1}{2}(\frac{\partial^2\Psi}{\partial y^2}-\frac{\partial^2\Psi}{\partial x^2})] \tag{6}$$

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$$Q_x = -D[\frac{\partial}{\partial x}\nabla^2 F + \frac{D}{C}\frac{\partial^2 \Psi}{\partial y}],$$
$$Q_y = -D[\frac{\partial}{\partial y}\nabla^2 F + \frac{D}{C}\frac{\partial^2 \Psi}{\partial x}] \tag{7}$$

在矩形区域 $\Omega = \{(x,y) \mid -\frac{a}{2} \leqslant x \leqslant \frac{a}{2}, 0 \leqslant y \leqslant b\}$ 内,考虑由广义位移与内力构成的混合边界条件:

$$x = \pm\frac{a}{2}\text{时}, W = 0,\quad \frac{\partial^2 \Psi}{\partial x} = 0,\quad \frac{\partial^2 F}{\partial x^2} = 0 \tag{8}$$

引入原变量 M, F, Ψ 的对偶变量[18]:

$$\theta = \frac{\partial \Psi}{\partial y}, \alpha = \frac{\partial F}{\partial y}, \beta = \frac{D}{\rho\omega}\frac{\partial^2 F}{\partial x^2}$$

则方程(1)至(7)可化为如下 Hamilton 系统:

$$\frac{\partial Z}{\partial y} = HZ \tag{9}$$

其中

$$H = \begin{pmatrix} 0 & 0 & 0 & \frac{\rho\omega^2}{D} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{D\partial^2}{\rho\omega^2 \partial x^2} - \frac{D}{C} & 1 & 0 & 0 & 0 & 0 \\ 1 & -\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2C}{D(1-v)} - \frac{\partial^2}{\partial x^2} & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

为 Hamilton 算子,$Z = (M, F, \Psi, \beta, \alpha, \theta)^T$ 为状态向量.

引入如下 Hilbert 空间:

$X = \{g \in L^2(-\frac{a}{2}, \frac{a}{2}) : (g, f^0) = 0, f^0$ 是 $L^2(-\frac{a}{2}, \frac{a}{2})$ 中的非零常函数$\}$,即 X 由函数空间 $L^2(-\frac{a}{2}, \frac{a}{2})$ 中与非零常函数 f^0 都正交的函数构成的 Hilbert 空间.

根据边界条件(8),Hamilton 算子(10)的定义域为:

$D(H) = \{[M(x), F(x), \Psi(x), \beta(x), \alpha(x), \theta(x)]^T \in W \mid M', F', \Psi'$ 绝对连续,且 $M'', F'', \Psi'' \in X$, $x = \pm\frac{a}{2}: F - \frac{D}{C}\nabla^2 F = 0, \frac{d\Psi}{dx} = 0, \frac{d^2 F}{dx^2} = 0\}$,其中 W 取作 Hilbert 空间 $X \times X \times X \times X \times X \times X$.

借助于 Mathematica 软件的帮助,可得算子(10)的所有本征值为:

$$\mu_{\pm m}^{(1)} = \pm\sqrt{\frac{2m^2\pi^2 CD - a^2 D\rho\omega^2 - a^2\omega\sqrt{4\rho DC^2 + D^2\rho^2\omega^2}}{2a^2 CD}} \tag{11}$$

或

$$\mu_{\pm m}^{(2)} = \pm\sqrt{\frac{2m^2\pi^2 CD - a^2 D\rho\omega^2 + a^2\omega\sqrt{4\rho DC^2 + D^2\rho^2\omega^2}}{2a^2 CD}} \tag{12}$$

或

$$\mu_{\pm m}^{(3)} = \pm\sqrt{\frac{D(1-v)m^2\pi^2 + 2a^2 C}{a^2 D(1-v)}} \tag{13}$$

$\mu_{\pm m}^{(3)} (m \in Z)$ 对应的本征函数为:

$$X_{\pm(2m-1)}^{(1)} = [Q_{11}\cos\alpha_{\pm(2m-1)}x, T_{11}\cos\alpha_{\pm(2m-1)}x, 0, T_{12}\cos\alpha_{\pm(2m-1)}x, Q_{12}\cos\alpha_{\pm(2m-1)}x, 0]^T,$$
$$X_{\pm 2m}^{(2)} = [Q_{11}\sin\alpha_{\pm 2m}x, T_{11}\sin\alpha_{\pm 2m}x, 0, T_{12}\sin\alpha_{\pm 2m}x, Q_{12}\sin\alpha_{\pm 2m}x, 0]^T \tag{14}$$

其中 $Q_{11}, T_{11}, T_{12}, Q_{12}$ 见附录中的(A1)式.

$\mu_{\pm m}^{(2)} (m \in Z)$ 对应的本征函数为:

$$X_{\pm(2m-1)}^{(2)} = [Q_{21}\cos\alpha_{\pm(2m-1)}x, T_{21}\cos\alpha_{\pm(2m-1)}x, 0, T_{22}\cos\alpha_{\pm(2m-1)}x, Q_{22}\cos\alpha_{\pm(2m-1)}x, 0]^T,$$
$$X_{\pm 2m}^{(2)} = [Q_{21}\sin\alpha_{\pm 2m}x, T_{21}\sin\alpha_{\pm 2m}x, 0, T_{22}\sin\alpha_{\pm 2m}x, Q_{22}\sin\alpha_{\pm 2m}x, 0]^T \tag{15}$$

其中 $Q_{21}, T_{21}, T_{22}, Q_{22}$ 见附录中的(A2)式.

$\mu_{\pm m}^{(3)} (m \in Z)$ 对应的本征函数为:

$$X_{\pm(2m-1)}^{(3)} = [0, 0, \sin\alpha_{\pm(2m-1)}x, 0, 0, \mu_{\pm(2m-1)}^{(3)}\sin\alpha_{\pm(2m-1)}x]^T$$
$$X_{\pm 2m}^{(2)} = [0, 0, \cos\alpha_{\pm 2m}x, 0, 0, \mu_{\pm 2m}^{(3)}\cos\alpha_{\pm 2m}x]^T \tag{16}$$

注 2.1 需要说明的是:本文得到的本征值(11)-(13)式及相应的本征函数(14)-(16)式与文献[18]中的结果不完全相同.

下面的引理在文中主要结果的证明中起关键作用.

引理 2.1　算子(20)的本征函数 $\{X_m^{(1)}\}_{-\infty}^{\infty}$, $\{X_m^{(2)}\}_{-\infty}^{\infty}$ 和 $\{X_m^{(3)}\}_{-\infty}^{\infty}$ $(m \in Z)$ 间成立如下辛正交关系:

$$< X_m^{(1)}, X_m^{(1)} > = -2aD\mu_m^{(1)}\rho\omega^2(4C^2 + D\rho\omega^2 - \sqrt{D\rho}\omega\sqrt{4C^2 + D\rho\omega^2}) \tag{17}$$
$$< X_m^{(2)}, X_m^{(2)} > = -2aD\mu_m^{(2)}\rho\omega^2(4C^2 + D\rho\omega^2 + \sqrt{D\rho}\omega\sqrt{4C^2 + D\rho\omega^2}) \tag{18}$$
$$< X_m^{(3)}, X_m^{(3)} > = (-1)^{m+1} a\mu_m^{(3)} \tag{19}$$

其中辛内积定义为

$< v_1, v_2 > = \int_{-\frac{a}{2}}^{\frac{a}{2}} v_1^T J v_2 \mathrm{d}x$，对于任意的 $v_1, v_2 \in W$，且 $J = \begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}$.

引理 2.2 下列正交函数集在 Hilbert 空间 X 中按标准的内积是完备的，进而对于任意的 $\phi \in X$，相应的 Fourier 级数在中收敛于 ϕ.

(i) $\varphi_{2m-1}(x) = \cos(\frac{(2m-1)\pi x}{a})$，$\varphi_{2m}(x) = \sin(\frac{2m\pi x}{a})$，$m \geqslant 1, m \in Z$

(ii) $\varphi_{2m-1}(x) = \cos(\frac{(2m-1)\pi x}{a})$，$\varphi_{2m}(x) = \sin(\frac{2m\pi x}{a})$，$m \geqslant 1, m \in Z$

下面的定理是本文的主要结果，它对文中考虑的中厚板自由振动问题施行 Hamilton 体系下的分离变量法提供了理论保障.

定理 2.1 算子(20)的本征函数系(14)，(15)和(16)在空间 W 中 Cauchy 主值意义下完备. 换言之，对于任意的 $g(x) \in W$，都存在常数 $\{g_{\pm m}^{(1)}\}_{m=1}^{+\infty}$，$\{g_{\pm m}^{(2)}\}_{m=1}^{\infty}$ 和 $\{g_{\pm m}^{(3)}\}_{m=1}^{+\infty}$，使得

$$g(x) = \sum_{m=1}^{\infty} [g_{2m-1}^{(1)} X_{2m-1}^{(1)} + g_{2m-1}^{(2)} X_{2m-1}^{(2)} + g_{2m-1}^{(3)} X_{2m-1}^{(3)} + g_{-(2m-1)}^{(1)} X_{-(2m-1)}^{(1)} + g_{-(2m-1)}^{(2)} X_{-(2m-1)}^{(2)} + g_{-(2m-1)}^{(3)} X_{-(2m-1)}^{(3)} + g_{2m}^{(1)} X_{2m}^{(1)} + g_{2m}^{(2)} X_{2m}^{(2)} + g_{2m}^{(3)} X_{2m}^{(3)} + g_{-2m}^{(1)} X_{-2m}^{(1)} + g_{-2m}^{(2)} X_{-2m}^{(2)} + g_{-2m}^{(3)} X_{-2m}^{(3)}]$$

证明：对于任意的 $g(x) \in W$，将其写为矩阵形式

$$g(x) = [g_1(x), g_2(x), g_3(x), g_4(x), g_5(x), g_6(x)]^T$$

对于任意的正整数 m，令 $\alpha_i = \frac{i\pi}{a}$，$i \in Z$. 下面的符号

$Z_j^{(i)}$ $(i = 1,2,3; j = \pm(2m-1), \pm 2m, m \in Z)$ 见附录中的(A3)，(A4)和(A5)式. 根据引理 2.1，我们取

$$g_{2m-1}^{(1)} = \frac{1}{< X_{2m-1}^{(1)}, X_{-(2m-1)}^{(1)} >} < g(x), X_{-(2m-1)}^{(1)} > = \frac{1}{Z_{2m-1}^{(1)}} < g(x), X_{2m-1}^{(1)} > = \frac{1}{Z_{2m-1}^{(1)}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (-Q_{11}g_4 - T_{11}g_5 + T_{12}g_1 + Q_{12}g_2) \cos\alpha_{-(2m-1)}\xi \mathrm{d}\xi$$

$$g_{-(2m-1)}^{(1)} = \frac{1}{< X_{-(2m-1)}^{(1)}, X_{2m-1}^{(1)} >} < g(x), X_{2m-1}^{(1)} > = \frac{1}{Z_{2m-1}^{(1)}} < g(x), X_{2m-1}^{(1)} > = \frac{1}{Z_{-(2m-1)}^{(1)}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (-Q_{11}g_4 - T_{11}g_5 + T_{12}g_1 + Q_{12}g_2) \cos\alpha_{2m-1}\xi \mathrm{d}\xi$$

$$g_{2m}^{(1)} = \frac{1}{< X_{2m}^{(1)}, X_{-2m}^{(1)} >} < g(x), X_{-2m}^{(1)} > = \frac{1}{Z_{2m}^{(1)}} < g(x), X_{-2m}^{(1)} > = \frac{1}{Z_{2m}^{(1)}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (-Q_{11}g_4 - T_{11}g_5 + T_{12}g_1 + Q_{12}g_2) \sin\alpha_{-(2m-1)}\xi \mathrm{d}\xi$$

$$g_{-2m}^{(1)} = \frac{1}{< X_{-2m}^{(1)}, X_{2m}^{(1)} >} < g(x), X_{2m)}^{(1)} > = \frac{1}{Z_{-2m}^{(1)}} < g(x), X_{2m}^{(1)} > = \frac{1}{Z_{-2m}^{(1)}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (-Q_{11}g_4 - T_{11}g_5 + T_{12}g_1 + Q_{12}g_2) \sin\alpha_{2m)}\xi \mathrm{d}\xi$$

$$g_{2m-1}^{(2)} = \frac{1}{< X_{2m-1}^{(2)}, X_{-(2m-1)}^{(2)} >} < g(x), X_{-(2m-1)}^{(2)} > = \frac{1}{Z_{2m-1}^{(2)}} < g(x), X_{2m-1}^{(2)} > = \frac{1}{Z_{2m-1}^{(2)}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (-Q_{11}g_4 - T_{11}g_5 + T_{12}g_1 + Q_{12}g_2) \cos\alpha_{-(2m-1)}\xi \mathrm{d}\xi$$

$$g_{-(2m-1)}^{(2)} = \frac{1}{< X_{-(2m-1)}^{(2)}, X_{2m-1}^{(2)} >} < g(x), X_{2m-1}^{(2)} > = \frac{1}{Z_{-(2m-1)}^{(2)}} < g(x), X_{2m-1}^{(2)} > = \frac{1}{Z_{-(2m-1)}^{(2)}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (-Q_{21}g_4 - T_{21}g_5 + T_{22}g_1 + Q_{22}g_2) \cos\alpha_{2m-1}\xi \mathrm{d}\xi$$

$$g_{2m}^{(2)} = \frac{1}{< X_{2m}^{(2)}, X_{-2m}^{(2)} >} < g(x), X_{-2m}^{(2)} > = \frac{1}{Z_{2m)}^{(2)}} < g(x), X_{-2m}^{(2)} > = \frac{1}{Z_{2m}^{(2)}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (-Q_{21}g_4 - T_{21}g_5 + T_{22}g_1 + Q_{22}g_2) \sin\alpha_{-2m}\xi \mathrm{d}\xi$$

$$g_{-2m}^{(2)} = \frac{1}{< X_{-2m}^{(2)}, X_{2m}^{(2)} >} < g(x), X_{2m}^{(2)} > = \frac{1}{Z_{-2m}^{(2)}} < g(x), X_{2m}^{(2)} > = \frac{1}{Z_{-2m}^{(2)}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (-Q_{21}g_4 - T_{21}g_5 + T_{22}g_1 + Q_{22}g_2) \sin\alpha_{2m}\xi \mathrm{d}\xi$$

$$g_{2m-1}^{(3)} = \frac{1}{< X_{2m-1}^{(3)}, X_{-(2m-1)}^{(3)} >} < g(x), X_{-(2m-1)}^{(3)} > = \frac{1}{Z_{2m-1}^{(3)}} < g(x), X_{-(2m-1)}^{(3)} > = \frac{1}{Z_{2m-1}^{(3)}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (-g_6 + \mu g_3) \mathrm{c}\sin\alpha_{-(2m-1)}\xi \mathrm{d}\xi$$

$$g_{-(2m-1)}^{(3)} = \frac{1}{< X_{-(2m-1)}^{(3)}, X_{2m-1}^{(3)} >} < g(x), X_{2m-1}^{(3)} > = \frac{1}{Z_{-(2m-1)}^{(3)}} < g(x), X_{2m-1}^{(3)} > = \frac{1}{Z_{-(2m-1)}^{(3)}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (-g_6 + \mu g_3) \sin\alpha_{2m-1}\xi \mathrm{d}\xi$$

$$g_{2m}^{(3)} = \frac{1}{< X_{2m}^{(3)}, X_{-2m}^{(3)} >} < g(x), X_{-2m}^{(3)} > = \frac{1}{Z_{2m}^{(3)}} \times$$

$$< g(x), X_{-2m}^{(3)} > = \frac{1}{Z_{2m}^{(3)}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (-g_6 + \mu g_3) \cos\alpha_{-2m}\xi \mathrm{d}\xi$$

$$g_{-2m}^{(3)} = \frac{1}{< X_{-2m}^{(3)}, X_{2m}^{(3)} >} < g(x), X_{2m}^{(3)} > = \frac{1}{Z_{-2m}^{(3)}} \times$$

$$< g(x), X_{2m}^{(2)} > = \frac{1}{Z_{-2m}^{(3)}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (-g_6 + \mu g_3) \cos\alpha_{2m}\xi \mathrm{d}\xi$$

经计算可得如下等式:

$$g_{2m-1}^{(1)} X_{2m-1}^{(1)} + g_{2m-1}^{(2)} X_{2m-1}^{(2)} + g_{2m-1}^{(3)} X_{2m-1}^{(3)} = \begin{bmatrix} \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{11}^{(1)} g_4 + C_{12}^{(1)} g_5 + C_{13}^{(1)} g_1 + C_{14}^{(1)} g_2) \cos\alpha_{2m-1}\xi \mathrm{d}\xi \right) \cos\alpha_{2m-1} x \\ \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{21}^{(1)} g_4 + C_{22}^{(1)} g_5 + C_{23}^{(1)} g_1 + C_{24}^{(1)} g_2) \cos\alpha_{2m-1}\xi \mathrm{d}\xi \right) \cos\alpha_{2m-1} x \\ \int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{31}^{(1)} g_6 + C_{32}^{(1)} g_3) \sin\alpha_{2m-1}\xi \mathrm{d}\xi) \sin\alpha_{2m-1} x \\ \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{41}^{(1)} g_4 + C_{42}^{(1)} g_5 + C_{43}^{(1)} g_1 + C_{44}^{(1)} g_2) \cos\alpha_{2m-1}\xi \mathrm{d}\xi \right) \cos\alpha_{2m-1} x \\ \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{51}^{(1)} g_4 + C_{52}^{(1)} g_5 + C_{53}^{(1)} g_1 + C_{54}^{(1)} g_2) \cos\alpha_{2m-1}\xi \mathrm{d}\xi \right) \cos\alpha_{2m-1} x \\ \int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{61}^{(1)} g_6 + C_{62}^{(1)} g_3) \sin\alpha_{2m-1}\xi \mathrm{d}\xi) \sin\alpha_{2m-1} x \end{bmatrix}$$

其中 $C_{ij}^{(1)}(i=1,2,3,4,5,6;j=1,2,3,4)$ 见附录中的(A6)式;

$$g_{-(2m-1)}^{(1)} X_{-(2m-1)}^{(1)} + g_{-(2m-1)}^{(2)} X_{-(2m-1)}^{(2)} + g_{-(2m-1)}^{(3)} X_{-(2m-1)}^{(3)} = \begin{bmatrix} \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{11}^{(2)} g_4 + C_{12}^{(2)} g_5 + C_{13}^{(2)} g_1 + C_{14}^{(2)} g_2) \cos\alpha_{-(2m-1)}\xi \mathrm{d}\xi \right) \cos\alpha_{-(2m-1)} x \\ \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{21}^{(2)} g_4 + C_{22}^{(2)} g_5 + C_{23}^{(2)} g_1 + C_{24}^{(2)} g_2) \cos\alpha_{-(2m-1)}\xi \mathrm{d}\xi \right) \cos\alpha_{-(2m-1)} x \\ \int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{31}^{(2)} g_6 + C_{32}^{(2)} g_3) \sin\alpha_{-(2m-1)}\xi \mathrm{d}\xi) \sin\alpha_{-(2m-1)} x \\ \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{41}^{(2)} g_4 + C_{42}^{(2)} g_5 + C_{43}^{(2)} g_1 + C_{44}^{(2)} g_2) \cos\alpha_{-(2m-1)}\xi \mathrm{d}\xi \right) \cos\alpha_{-(2m-1)} x \\ \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{51}^{(2)} g_4 + C_{52}^{(2)} g_5 + C_{53}^{(2)} g_1 + C_{54}^{(2)} g_2) \cos\alpha_{-(2m-1)}\xi \mathrm{d}\xi \right) \cos\alpha_{-(2m-1)} x \\ \int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{61}^{(2)} g_6 + C_{62}^{(2)} g_3) \sin\alpha_{-(2m-1)}\xi \mathrm{d}\xi) \sin\alpha_{-(2m-1)} x \end{bmatrix}$$

其中 $C_{ij}^{(2)}(i=1,2,3,4,5,6;j=1,2,3,4)$ 见附录中的(A7)式;

$$g_{2m}^{(1)} X_{2m}^{(1)} + g_{2m}^{(2)} X_{2m}^{(2)} + g_{2m}^{(3)} X_{2m}^{(3)} = \begin{bmatrix} \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{11}^{(3)} g_4 + C_{12}^{(3)} g_5 + C_{13}^{(3)} g_1 + C_{14}^{(3)} g_2) \sin\alpha_{2m}\xi \mathrm{d}\xi \right) \sin\alpha_{2m} x \\ \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{21}^{(3)} g_4 + C_{22}^{(3)} g_5 + C_{23}^{(3)} g_1 + C_{24}^{(3)} g_2) \sin\alpha_{2m}\xi \mathrm{d}\xi \right) \sin\alpha_{2m} x \\ \int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{31}^{(3)} g_6 + C_{32}^{(3)} g_3) \cos\alpha_{2m}\xi \mathrm{d}\xi) \cos\alpha_{2m} x \\ \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{41}^{(3)} g_4 + C_{42}^{(3)} g_5 + C_{43}^{(3)} g_1 + C_{44}^{(3)} g_2) \sin\alpha_{2m}\xi \mathrm{d}\xi \right) \sin\alpha_{2m} x \\ \left(\int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{51}^{(3)} g_4 + C_{52}^{(3)} g_5 + C_{53}^{(3)} g_1 + C_{54}^{(3)} g_2) \sin\alpha_{2m}\xi \mathrm{d}\xi \right) \sin\alpha_{2m} x \\ \int_{-\frac{a}{2}}^{\frac{a}{2}} (C_{61}^{(3)} g_6 + C_{62}^{(3)} g_3) \cos\alpha_{2m}\xi \mathrm{d}\xi) \cos\alpha_{2m} x \end{bmatrix}$$

其中 $C_{ij}^{(3)}(i=1,2,3,4,5,6;j=1,2,3,4)$ 见附录中的($A8$)式；

$$g_{-2m}^{(1)}X_{-2m}^{(1)}+g_{-2m}^{(2)}X_{-2m}^{(2)}+g_{-2m}^{(3)}X_{-2m}^{(3)}=\begin{bmatrix}(\int_{-\frac{a}{2}}^{\frac{a}{2}}(C_{11}^{(4)}g_4+C_{12}^{(4)}g_5+C_{13}^{(4)}g_1+C_{14}^{(4)}g_2)\sin\alpha_{-2m}\xi d\xi)\sin\alpha_{-2m}x\\(\int_{-\frac{a}{2}}^{\frac{a}{2}}(C_{21}^{(4)}g_4+C_{22}^{(4)}g_5+C_{23}^{(4)}g_1+C_{24}^{(4)}g_2)\sin\alpha_{-2m}\xi d\xi)\sin\alpha_{-2m}x\\\int_{-\frac{a}{2}}^{\frac{a}{2}}(C_{31}^{(4)}g_6+C_{32}^{(4)}g_3)\cos\alpha_{-2m}\xi d\xi)\cos\alpha_{-2m}x\\(\int_{-\frac{a}{2}}^{\frac{a}{2}}(C_{41}^{(4)}g_4+C_{42}^{(4)}g_5+C_{43}^{(4)}g_1+C_{44}^{(4)}g_2)\sin\alpha_{-2m}\xi d\xi)\sin\alpha_{-2m}x\\(\int_{-\frac{a}{2}}^{\frac{a}{2}}(C_{51}^{(4)}g_4+C_{52}^{(4)}g_5+C_{53}^{(4)}g_1+C_{54}^{(4)}g_2)\sin\alpha_{-2m}\xi d\xi)\sin\alpha_{-2m}x\\\int_{-\frac{a}{2}}^{\frac{a}{2}}(C_{61}^{(4)}g_6+C_{62}^{(4)}g_3)\cos\alpha_{-2m}\xi d\xi)\cos\alpha_{-2m}x\end{bmatrix}$$

其中 $C_{ij}^{(4)}(i=1,2,3,4,5,6;j=1,2,3,4)$ 见附录中的($A9$)式. 我们有

$$\begin{aligned}&g_{2m-1}^{(1)}X_{2m-1}^{(1)}+g_{2m-1}^{(2)}X_{2m-a}^{(2)}+g_{2m-1}^{(3)}X_{2m-1}^{(3)}+g_{-(2m-1)}^{(1)}X_{-(2m-1)}^{(1)}+g_{-(2m-1)}^{(2)}X_{-(2m-1)}^{(2)}+g_{-(2m-1)}^{(3)}X_{-(2m-1)}^{(3)}+\\&\quad g_{2m}^{(1)}X_{2m}^{(1)}+g_{2m}^{(2)}X_{2m}^{(2)}+g_{-2m}^{(1)}X_{-2m}^{(1)}+g_{-2m}^{(2)}X_{-2m}^{(1)}+g_{-2m}^{(2)}X_{-2m}^{(2)}+g_{-2m}^{(3)}X_{-2m}^{(3)}=\\&\frac{2}{a}\begin{bmatrix}(\int_{-\frac{a}{2}}^{\frac{a}{2}}g_1\cos\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+\int_{-\frac{a}{2}}^{\frac{a}{2}}g_1\sin\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x\\(\int_{-\frac{a}{2}}^{\frac{a}{2}}g_2\cos\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+\int_{-\frac{a}{2}}^{\frac{a}{2}}g_2\sin\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x\\(\int_{-\frac{a}{2}}^{\frac{a}{2}}g_3\cos\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+\int_{-\frac{a}{2}}^{\frac{a}{2}}g_3\sin\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x\\(\int_{-\frac{a}{2}}^{\frac{a}{2}}g_4\cos\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+\int_{-\frac{a}{2}}^{\frac{a}{2}}g_4\sin\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x\\(\int_{-\frac{a}{2}}^{\frac{a}{2}}g_5\cos\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+\int_{-\frac{a}{2}}^{\frac{a}{2}}g_5\sin\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x\\(\int_{-\frac{a}{2}}^{\frac{a}{2}}g_6\cos\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+\int_{-\frac{a}{2}}^{\frac{a}{2}}g_6\sin\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x\end{bmatrix}\end{aligned}$$

注意到

$$\sum_{m=1}^{\infty}(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_3\cos\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_3\sin\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x(i=1,2,4,5)$$

和

$$\sum_{m=1}^{\infty}(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_3\cos\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_3\sin\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x(i=1,2,4,5)$$

分别是 $g_k(k=1,2,3,4,5,6)$ 按 X 中的正交函数系

$\{\cos(\pi x/a),\sin(2\pi x/a),\cos(3\pi x/a),\sin(4\pi x/a),\cdots\}$

和

$\{\sin(\pi x/a),\cos(2\pi x/a),\sin(3\pi x/a),\cos(4\pi x/a),\cdots\}$

展开的 Fourier 级数，应用引理 2.2，可得

$$\begin{aligned}g(x)&=\sum_{m=1}^{\infty}[g_{2m-1}^{(1)}X_{2m-1}^{(1)}+g_{2m-1}^{(2)}X_{2m-1}^{(2)}+g_{2m-1}^{(3)}X_{2m-1}^{(3)}+g_{-(2m-1)}^{(1)}X_{-(2m-1)}^{(1)}+g_{-(2m-1)}^{(2)}X_{-(2m-1)}^{(2)}+g_{-(2m-1)}^{(3)}X_{-(2m-1)}^{(3)}+\\&\quad g_{2m}^{(1)}X_{2m}^{(1)}+g_{2m}^{(2)}X_{2m}^{(2)}+g_{2m}^{(3)}X_{2m}^{(3)}+g_{-2m}^{(1)}X_{-2m}^{(1)}+g_{-2m}^{(2)}X_{-2m}^{(2)}+g_{-2m}^{(3)}X_{-2m}^{(3)}]=\\&\lim_{M\to+\infty}\sum_{m=1}^{M}[g_{2m-1}^{(1)}X_{2m-1}^{(1)}+g_{2m-1}^{(2)}X_{2m-1}^{(2)}+g_{2m-1}^{(3)}X_{2m-1}^{(3)}+g_{-(2m-1)}^{(1)}X_{-(2m-1)}^{(1)}+g_{-(2m-1)}^{(2)}X_{-(2m-1)}^{(2)}+\\&\quad g_{-(2m-1)}^{(3)}X_{-(2m-1)}^{(3)}+g_{2m}^{(1)}X_{2m}^{(1)}+g_{2m}^{(2)}X_{2m}^{(2)}+g_{2m}^{(3)}X_{2m}^{(3)}+g_{-2m}^{(1)}X_{-2m}^{(1)}+g_{-2m}^{(2)}X_{-2m}^{(2)}+\end{aligned}$$

$$g_{-2m}^{(3)}X_{-2m}^{(3)}]=\sum_{m=1}^{\infty}\begin{bmatrix}(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_1\cos\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_1\sin\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x\\(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_2\cos\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_2\sin\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x\\(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_3\sin\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_3\cos\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x\\(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_4\cos\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_4\sin\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x\\(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_5\cos\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_5\sin\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x\\(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_6\sin\alpha_{2m-1}\xi d\xi)\cos\alpha_{2m-1}x+(\frac{2}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}}g_6\cos\alpha_{2m}\xi d\xi)\sin\alpha_{2m}x\end{bmatrix}=$$

$$[g_1(x),g_2(x),g_3(x),g_4(x),g_5(x),g_6(x)]^T=g(x)$$

这就完成了定理的证明.

下面给出 Hamilton 方程(9)的一般解.

根据定理 2.1,本征函数系(14),(15)和(16)在空间 W 中 Cauchy 主值意义下完备. 根据叠加原理,方程(9)的解有如下形式:

$$Z(x,y)=\sum_{m=1}^{\infty}[T_{2m-1}^{(1)}X_{2m-1}^{(1)}+T_{2m-1}^{(2)}X_{2m-1}^{(2)}+T_{2m-1}^{(3)}X_{2m-1}^{(3)}+T_{-(2m-1)}^{(1)}X_{-(2m-1)}^{(1)}+T_{-(2m-1)}^{(2)}X_{-(2m-1)}^{(2)}+T_{-(2m-1)}^{(3)}X_{-(2m-1)}^{(3)}+T_{2m}^{(1)}X_{2m}^{(1)}+T_{2m}^{(1)}X_{2m}^{(1)}+T_{2m}^{(2)}X_{2m}^{(2)}+T_{2m}^{(3)}X_{2m}^{(3)}+T_{-2m}^{(1)}X_{-2m}^{(1)}+T_{-2m}^{(1)}X_{-2m}^{(1)}+T_{-2m}^{(2)}X_{-2m}^{(2)}+T_{-2m}^{(3)}X_{-2m}^{(3)}+T_{-2m}^{(3)}X_{-2m}^{(3)}] \tag{20}$$

将方程(20)代入方程(9)中,根据引理 2.1 的辛正交关系得:

$$\frac{\partial T_j^{(i)}}{\partial y}=\mu_j^{(i)}T_j^{(i)}$$

$$(i=1,2,3;j\pm(2m-1),\ \pm 2m,m\in Z)$$

因而

$$T_j^{(i)}(y)=A_j^{(i)}e^{\mu_j^{(i)}y} \tag{21}$$

其中 $A_j^{(i)}$ 为任意常数,它们由侧边的边界条件决定.

将方程(21)代入到(20)中,得到方程(9)的一般解 $Z(x,y)$ 为:

$$Z(x,y)=\sum_{m=1}^{\infty}[A_{2m-1}^{(1)}e^{\mu_{2m-1}^{(1)}y}X_{2m-1}^{(1)}+A_{2m-1}^{(2)}e^{\mu_{2m-1}^{(2)}y}X_{2m-1}^{(2)}+A_{2m-1}^{(3)}e^{\mu_{2m-1}^{(3)}y}X_{2m-1}^{(3)}+A_{-(2m-1)}^{(1)}e^{\mu_{-(2m-1)}^{(1)}y}X_{-(2m-1)}^{(1)}+A_{-(2m-1)}^{(2)}e^{\mu_{-(2m-1)}^{(2)}y}X_{-(2m-1)}^{(2)}+A_{-(2m-1)}^{(3)}e^{\mu_{-(2m-1)}^{(3)}y}X_{-(2m-1)}^{(3)}+A_{2m}^{(1)}e^{\mu_{2m}^{(1)}y}X_{2m}^{(1)}+A_{2m}^{(2)}e^{\mu_{2m}^{(2)}y}X_{2m}^{(2)}+A_{2m}^{(3)}e^{\mu_{2m}^{(3)}y}X_{2m}^{(3)}+A_{-2m}^{(1)}e^{\mu_{-2m}^{(1)}y}X_{-2m}^{(1)}+A_{-2m}^{(2)}e^{\mu_{-2m}^{(2)}y}X_{-2m}^{(2)}+A_{-2m}^{(3)}e^{\mu_{-2m}^{(3)}y}X_{-2m}^{(3)}] \tag{22}$$

3　数值算例

令 $a=10,b=1$. 考虑矩形区域 $\Omega=\{(x,y)\mid-5\leqslant x\leqslant 5,0\leqslant y\leqslant 1\}$ 和 y 侧边的如下边界条件:

$$F=0,\frac{\partial\Psi}{\partial y}=0,\frac{\partial F}{\partial y}=0,当\ y=0,\ -5\leqslant x\leqslant 5;$$

$$F=\cos x,\frac{\partial\Psi}{\partial y}=0,\frac{\partial F}{\partial y}=0,当\ y=1,\ -5\leqslant x\leqslant 5.$$

根据上面的条件,可确定一般解(22)中的任意常数 $A_j^{(i)}$. 取下列计算参数:$E=3.0\times10^6$Pa,$v=0.15$,$h=0.4$m,$\rho=2.4\times10^3$kg/m^3,$C=2.0\times10^5$Ns/m^3. 取(22)的前 5 项. 下表列出了板的固有频率 $\omega=5.6$ 时的计算结果.

表 1　计算结果

Table 1　Computed results

(x,y)	M	F	Ψ	β	α	θ
(-4.50,0.05)	-0.492887	-0.000617	0.00000	0.0047942	-0.024657	0.00000
(-2.00,0.25)	-0.796597	-0.024882	0.00000	-0.005735	-0.199636	0.00000
(-0.05,0.40)	1.88965	0.151085	0.00000	0.0229974	0.760087	0.00000
(0.20,0.05)	1.846620	0.0023084	0.00000	0.0006378	0.0923442	0.00000
(3.00,0.60)	-1.96627	-0.346875	0.00000	−0.087059	-1.178010	0.00000
(4.80,0.96)	-0.071415	-0.065228	0.00000	0.0470644	-0.099958	0.00000

4　结论

文中用 Hamilton 体系下的辛本征展开精确求解了中厚板自由振动问题,更重要的是,证明了相应本征函数系的完备性. 文中定理为中厚板自由振

动问题施行 Hamilton 体系下的分离变量法提供了理论保证,而且基于展开定理换得到了问题的一般解.

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附录:

$$Q_{11}=2C\rho\omega^2,$$

$$T_{11}=D\rho\omega^2-\sqrt{D}\sqrt{\rho}\omega\sqrt{4C^2+D\rho\omega^2},$$

$$T_{12}2CD\mu,$$

$$Q_{12}=\mu D\rho\omega^2-\mu\sqrt{D}\sqrt{\rho}\omega\sqrt{4C^2+D\rho\omega^2}. \quad (A1)$$

$$Q_{21}=2C\rho\omega^2, Q_{22}=2CD\mu,$$

$$T_{21}=\sqrt{\rho}\omega(D\sqrt{\rho}\omega+\sqrt{4C^2+D\rho\omega^2}),$$

$$T_{22}=\mu\sqrt{D}\sqrt{\rho}\omega(\sqrt{D}\sqrt{\rho}\omega+\sqrt{4C^2+D\rho\omega^2}) \quad (A2)$$

$$<X_m^{(1)},X_{-m}^{(1)}> = Z_m^{(1)} \tag{A3}$$

$$<X_m^{(2)},X_{-m}^{(2)}> = Z_m^{(2)} \tag{A4}$$

$$<X_m^{(3)},X_{-m}^{(3)}> = Z_m^{(3)} \tag{A5}$$

$$C_{11}^{(1)} = -\left(\frac{Q_{11}^2}{Z_{2m-1}^{(1)}}+\frac{Q_{21}^2}{Z_{2m-1}^{(2)}}\right),$$

$$C_{12}^{(1)} = -\left(\frac{Q_{11}T_{11}}{Z_{2m-1}^{(1)}}+\frac{Q_{21}T_{21}}{Z_{2m-1}^{(1)}}\right),$$

$$C_{13}^{(1)} = \left(\frac{Q_{11}T_{12}}{Z_{2m-1}^{(1)}}+\frac{Q_{21}T_{22}}{Z_{2m-1}^{(2)}}\right),$$

$$C_{14}^{(1)} = \left(\frac{Q_{11}T_{12}}{Z_{2m-1}^{(1)}}+\frac{Q_{21}T_{22}}{Z_{2m-1}^{(2)}}\right),$$

$$C_{21}^{(1)} = -\left(\frac{T_{11}Q_{11}}{Z_{2m-1}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{2m-1}^{(2)}}\right),$$

$$C_{22}^{(1)} = -\left(\frac{T_{11}^2}{Z_{2m-1}^{(1)}}+\frac{T_{21}^2}{Z_{2m-1}^{(2)}}\right),$$

$$C_{23}^{(1)} = \frac{T_{11}T_{12}}{Z_{2m-1}^{(1)}}+\frac{T_{21}T_{22}}{Z_{2m-1}^{(2)}},C_{24}^{(1)} = \frac{T_{11}Q_{12}}{Z_{2m-1}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{2m-1}^{(2)}},$$

$$C_{31}^{(1)} = -\frac{\mu}{Z_{2m-1}^{(3)}},C_{32}^{(1)} = -\frac{1}{Z_{2m-1}^{(3)}},$$

$$C_{41}^{(1)} = -\left(\frac{T_{12}Q_{11}}{Z_{2m-1}^{(1)}}+\frac{Q_{22}Q_{21}}{Z_{2m-1}^{(2)}}\right),$$

$$C_{42}^{(1)} = -\left(\frac{T_{12}T_{11}}{Z_{2m-1}^{(1)}}+\frac{Q_{22}Q_{21}}{Z_{2m-1}^{(2)}}\right),$$

$$C_{43}^{(1)} = \frac{T_{12}^2}{Z_{2m-1}^{(1)}}+\frac{Q_{22}Q_{21}}{Z_{2m-1}^{(2)}},C_{44}^{(1)} = \frac{T_{12}Q_{12}}{Z_{2m-1}^{(1)}}+\frac{Q_{22}^2}{Z_{2m-1}^{(2)}},$$

$$C_{51}^{(1)} = -\left(\frac{Q_{12}Q_{11}}{Z_{2m-1}^{(1)}}+\frac{T_{22}Q_{21}}{Z_{2m-1}^{(2)}}\right),$$

$$C_{52}^{(1)} = -\left(\frac{Q_{12}T_{11}}{Z_{2m-1}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{2m-1}^{(2)}}\right),$$

$$C_{53}^{(1)} = \frac{T_{12}Q_{12}}{Z_{2m-1}^{(1)}}+\frac{T_{22}^2}{Z_{2m-1}^{(2)}},C_{54}^{(1)} = \frac{Q_{12}^2}{Z_{2m-1}^{(1)}}+\frac{Q_{22}T_{22}}{Z_{2m-1}^{(2)}},$$

$$C_{61}^{(1)} = -\frac{\mu^2}{Z_{2m-1}^{(3)}},C_{62}^{(1)} = -\frac{\mu}{Z_{2m-1}^{(3)}}. \tag{A6}$$

$$C_{11}^{(2)} = -\left(\frac{Q_{11}^2}{Z_{-(2m-1)}^{(1)}}+\frac{Q_{21}^2}{Z_{-(2m-1)}^{(2)}}\right),$$

$$C_{12}^{(2)} = -\left(\frac{Q_{11}T_{11}}{Z_{-(2m-1)}^{(1)}}+\frac{Q_{21}T_{21}}{Z_{-(2m-1)}^{(1)}}\right),$$

$$C_{13}^{(2)} = \left(\frac{Q_{11}T_{12}}{Z_{-(2m-1)}^{(1)}}+\frac{Q_{21}T_{22}}{Z_{-(2m-1)}^{(2)}}\right),$$

$$C_{14}^{(2)} = \left(\frac{T_{11}Q_{11}}{Z_{-(2m-1)}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{-(2m-1)}^{(2)}}\right),$$

$$C_{21}^{(2)} = -\left(\frac{T_{11}Q_{11}}{Z_{-(2m-1)}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{-(2m-1)}^{(2)}}\right),$$

$$C_{22}^{(2)} = -\left(\frac{T_{11}^2}{Z_{-(2m-1)}^{(1)}}+\frac{T_{21}^2}{Z_{-(2m-1)}^{(2)}}\right),$$

$$C_{23}^{(2)} = \frac{T_{11}T_{12}}{Z_{-(2m-1)}^{(1)}}+\frac{T_{21}T_{22}}{Z_{-(2m-1)}^{(2)}},$$

$$C_{24}^{(2)} = \frac{T_{11}Q_{12}}{Z_{-(2m-1)}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{-(2m-1)}^{(2)}},$$

$$C_{31}^{(2)} = -\frac{\mu}{Z_{-(2m-1)}^{(3)}},C_{32}^{(2)} = -\frac{1}{Z_{-(2m-1)}^{(3)}},$$

$$C_{41}^{(2)} = -\left(\frac{T_{12}Q_{11}}{Z_{-(2m-1)}^{(1)}}+\frac{Q_{22}Q_{21}}{Z_{-(2m-1)}^{(2)}}\right),$$

$$C_{42}^{(2)} = -\left(\frac{T_{12}T_{11}}{Z_{-(2m-1)}^{(1)}}+\frac{Q_{22}Q_{21}}{Z_{-(2m-1)}^{(2)}}\right),$$

$$C_{43}^{(2)} = \frac{T_{12}^2}{Z_{-(2m-1)}^{(1)}}+\frac{Q_{22}Q_{21}}{Z_{-(2m-1)}^{(2)}},$$

$$C_{44}^{(2)} = \frac{T_{12}Q_{12}}{Z_{-(2m-1)}^{(1)}}+\frac{Q_{22}^2}{Z_{-(2m-1)}^{(2)}},$$

$$C_{51}^{(2)} = -\left(\frac{Q_{12}Q_{11}}{Z_{-(2m-1)}^{(1)}}+\frac{T_{22}Q_{21}}{Z_{-(2m-1)}^{(2)}}\right),$$

$$C_{52}^{(2)} = -\left(\frac{Q_{12}T_{11}}{Z_{-(2m-1)}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{-(2m-1)}^{(2)}}\right),$$

$$C_{53}^{(2)} = \frac{T_{12}Q_{12}}{Z_{-(2m-1)}^{(1)}}+\frac{T_{22}^2}{Z_{-(2m-1)}^{(2)}},$$

$$C_{54}^{(2)} = \frac{Q_{12}^2}{Z_{-(2m-1)}^{(1)}}+\frac{Q_{22}T_{22}}{Z_{-(2m-1)}^{(2)}},$$

$$C_{61}^{(2)} = -\frac{\mu^2}{Z_{-(2m-1)}^{(3)}},C_{62}^{(2)} = -\frac{\mu}{Z_{-(2m-1)}^{(3)}}. \tag{A7}$$

$$C_{11}^{(3)} = -\left(\frac{Q_{11}^2}{Z_{2m}^{(1)}}+\frac{Q_{21}^2}{Z_{2m}^{(2)}}\right),$$

$$C_{12}^{(3)} = -\left(\frac{Q_{11}T_{11}}{Z_{2m}^{(1)}}+\frac{Q_{21}T_{21}}{Z_{2m}^{(1)}}\right),$$

$$C_{13}^{(3)} = \left(\frac{Q_{11}T_{12}}{Z_{2m}^{(1)}}+\frac{Q_{21}T_{22}}{Z_{2m}^{(2)}}\right),$$

$$C_{14}^{(3)} = \left(\frac{Q_{11}Q_{12}}{Z_{2m}^{(1)}}+\frac{Q_{21}Q_{22}}{Z_{2m}^{(2)}}\right),$$

$$C_{21}^{(3)} = -\left(\frac{T_{11}Q_{11}}{Z_{2m}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{2m}^{(2)}}\right),$$

$$C_{22}^{(3)} = -\left(\frac{T_{11}^2}{Z_{2m}^{(1)}}+\frac{T_{21}^2}{Z_{2m}^{(2)}}\right),$$

$$C_{23}^{(3)} = \frac{T_{11}T_{12}}{Z_{2m}^{(1)}}+\frac{T_{21}T_{22}}{Z_{2m}^{(2)}},C_{24}^{(3)} = \frac{T_{11}Q_{12}}{Z_{2m}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{2m}^{(2)}},$$

$$C_{31}^{(3)} = -\frac{\mu}{Z_{2m}^{(3)}},C_{32}^{(3)} = -\frac{1}{Z_{2m}^{(3)}},$$

$$C_{41}^{(3)} = -\left(\frac{T_{12}Q_{11}}{Z_{2m}^{(1)}}+\frac{Q_{22}Q_{21}}{Z_{2m}^{(2)}}\right),$$

$$C_{42}^{(3)}=-\left(\frac{T_{12}Q_{11}}{Z_{2m}^{(1)}}+\frac{Q_{22}Q_{21}}{Z_{2m}^{(2)}}\right),$$

$$C_{43}^{(3)}=-\left(\frac{T_{12}^2}{Z_{2m}^{(1)}}+\frac{Q_{22}Q_{21}}{Z_{2m}^{(2)}}\right),$$

$$C_{44}^{(3)}=-\left(\frac{T_{12}Q_{12}}{Z_{2m}^{(1)}}+\frac{Q_{22}^2}{Z_{2m}^{(2)}}\right),$$

$$C_{51}^{(3)}=\frac{Q_{12}Q_{11}}{Z_{2m}^{(1)}}+\frac{T_{22}Q_{21}}{Z_{2m}^{(2)}},C_{52}^{(3)}=\frac{Q_{12}T_{11}}{Z_{2m}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{2m}^{(2)}},$$

$$C_{53}^{(3)}=-\left(\frac{T_{12}Q_{12}}{Z_{2m}^{(1)}}+\frac{T_{22}^2}{Z_{2m}^{(2)}}\right),$$

$$C_{54}^{(3)}=-\left(\frac{Q_{12}^2}{Z_{2m}^{(1)}}+\frac{Q_{22}T_{22}}{Z_{2m}^{(2)}}\right),$$

$$C_{61}^{(3)}=-\frac{\mu^2}{Z_{2m}^{(3)}},C_{62}^{(3)}=-\frac{\mu}{Z_{2m}^{(3)}}. \qquad (A8)$$

$$C_{11}^{(4)}=\frac{Q_{11}^2}{Z_{-2m}^{(1)}}+\frac{Q_{21}^2}{Z_{-2m}^{(2)}},C_{12}^{(4)}=\frac{Q_{11}T_{11}}{Z_{-2m}^{(1)}}+\frac{Q_{21}T_{21}}{Z_{-2m}^{(1)}},$$

$$C_{13}^{(4)}=-\left(\frac{Q_{11}T_{12}}{Z_{-2m}^{(1)}}+\frac{Q_{21}T_{22}}{Z_{-2m}^{(2)}}\right),$$

$$C_{14}^{(4)}=-\left(\frac{Q_{11}T_{12}}{Z_{-2m}^{(1)}}+\frac{Q_{21}T_{22}}{Z_{-2m}^{(2)}}\right),$$

$$C_{21}^{(4)}=\frac{T_{11}Q_{11}}{Z_{-2m}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{-2m}^{(2)}},C_{22}^{(4)}=\frac{T_{11}^2}{Z_{-2m}^{(1)}}+\frac{T_{21}^2}{Z_{-2m}^{(2)}},$$

$$C_{23}^{(4)}=-\left(\frac{T_{11}T_{12}}{Z_{-2m}^{(1)}}+\frac{T_{21}T_{22}}{Z_{-2m}^{(2)}}\right),$$

$$C_{24}^{(4)}=-\left(\frac{T_{11}Q_{12}}{Z_{-2m}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{-2m}^{(2)}}\right),$$

$$C_{31}^{(4)}=-\frac{\mu}{Z_{-2m}^{(3)}},C_{32}^{(4)}=-\frac{1}{Z_{-2m}^{(3)}},$$

$$C_{41}^{(4)}=\frac{T_{12}Q_{11}}{Z_{-2m}^{(1)}}+\frac{Q_{22}Q_{21}}{Z_{-2m}^{(2)}},C_{42}^{(4)}=\frac{T_{12}T_{11}}{Z_{-2m}^{(1)}}+\frac{Q_{22}Q_{21}}{Z_{-2m}^{(2)}},$$

$$C_{43}^{(4)}=-\left(\frac{T_{12}^2}{Z_{-2m}^{(1)}}+\frac{Q_{22}Q_{21}}{Z_{-2m}^{(2)}}\right),$$

$$C_{44}^{(4)}=-\left(\frac{T_{12}Q_{12}}{Z_{-2m}^{(1)}}+\frac{Q_{22}^2}{Z_{-2m}^{(2)}}\right),$$

$$C_{51}^{(4)}=\frac{Q_{12}Q_{11}}{Z_{-2m}^{(1)}}+\frac{T_{22}Q_{21}}{Z_{-2m}^{(2)}},$$

$$C_{52}^{(4)}=\frac{Q_{12}T_{11}}{Z_{-2m}^{(1)}}+\frac{T_{21}Q_{22}}{Z_{-2m}^{(2)}},$$

$$C_{53}^{(4)}=-\left(\frac{T_{12}Q_{12}}{Z_{-2m}^{(1)}}+\frac{T_{22}^2}{Z_{-2m}^{(2)}}\right),$$

$$C_{54}^{(4)}=-\left(\frac{Q_{12}^2}{Z_{-2m}^{(1)}}+\frac{Q_{22}T_{22}}{Z_{-2m}^{(2)}}\right),$$

$$C_{61}^{(4)}=-\frac{\mu^2}{Z_{-2m}^{(3)}},C_{62}^{(4)}=-\frac{\mu}{Z_{-2m}^{(3)}}. \qquad (A9)$$

SYMPLECTIC EIGENFUNCTION EXPANSION THEOREM FOR FREE VIBRATION OF MODERATELY THICK RECTANGULAR PLATES *

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Abstract The eigenvalue problem of the Hamiltonian operator associated with the free vibration of a moderately thick rectangular plate was investigated. First, the eigenfunctions of the operator with the mixed boundary conditions of the generalized displacements and generalized internal forces were solved directly. Then, the completeness of the eigenfunctinons was proved, demonstrating the feasibility of using separation of variables to solve the problems. Finally, the general solution was obtained by using the proved expansion theorem, and a concrete numerical example was given.

Key words moderately thick rectangular plates, Hamiltonian system, symplectic orthogonality, eigenfunction expansion, general solution

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