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Fokker - Planck 方程的非古典广义势对称及精确解

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摘要　分析了 Fokker-Planck 方程的非古典势对称,通过广义势系统而不是一般势系统求得了这些非古典势对称. 文中得到了这些方程的新的对称,同时也得到了伴随系统的新的对称,并用其求出了一些精确解. 这些解对进一步研究 Fokker-Planck 方程所描述的物理现象具有广泛的应用价值.

关键词　Fokker-Planck 方程，　非古典广义势对称，　广义势系统，　精确解

引 言

Fokker-Planck 方程主要是用于描述微粒或质点的位置与速度概率密度函数的演化规律,如布朗运动、湍流流动等. 该方程也被称为中观层次理论的关键方程,在物理学与相关工程领域具有重要的研究价值. ,所以寻找其精确解具有重要意义.

本文对一维 Fokker-Planck 方程进行了分类和研究. 我们主要采用了文献[1]中提出的广义势系统求得了这些方程的非古典势对称,同时也得到了其伴随系统的非古典对称,并利用这些对称求出了一些精确解.

现在以有两个自变量的偏微分方程为例,给出求偏微分方程非古典势对称的一般步骤:

假设 $u(x,t)$ 满足的偏微分方程可以写成守恒形式(散度形式)

$$D_x f(x,t,u,u_x,u_t) - D_t g(x,t,u,u_x,u_t) = 0 \quad (1)$$

引入势函数 v,得到原方程(1)的伴随系统

$$\begin{cases} \dfrac{\partial v}{\partial t} = f \\ \dfrac{\partial v}{\partial x} = g \end{cases} \quad (2)$$

不过,许多文献中经常考虑形如

$$u_t = D_x f(x,t,u,u_x) \quad (3)$$

的偏微分方程,则由文献[4]可得其伴随系统为

$$\begin{cases} v_x = u \\ v_t = f(x,t,u,u_x) \end{cases} \quad (4)$$

该系统是一般势系统. 而文献[1]中提出了广义势系统,即

$$\begin{cases} v_x = f_1(x)h_1(u) \\ v_t = f_2(x)h_2(u)u_x + f_3(x)h_3(u) \end{cases} \quad (5)$$

其中 $f_i(x)$ 和 $h_i(x)$ 满足一些条件($i=1,2,3$). 假设方程组(4)(或(5))具有如下形式的 Lie 对称群生成元

$$V = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \phi(x,t,u,v)\frac{\partial}{\partial u} + \eta(x,t,u,v)\frac{\partial}{\partial v} \quad (6)$$

为了寻找方程(3)的非古典势对称群生成元,就要附加下面的不变曲面条件

$$\begin{cases} \tau(x,t,u,v)u_t + \xi(x,t,u,v)u_x - \phi = 0 \\ \tau(x,t,u,v)v_t + \xi(x,t,u,v)v_x - \eta = 0 \end{cases} \quad (7)$$

然后利用式(4)(或(5))和式(7)求出偏导数 u_t, u_x, v_t, v_x,并将它们代入到对称群确定方程组中,得到确定方程组,进一步就可以求出 ξ,τ,ϕ,η. 只要 $\xi_v^2+\tau_v^2+\phi_v^2 \neq 0$,则称 V 就是(3)的一个非古典势对称.

1　Fokker-Planck 方程

$$u_t = [p(x)u_x + q(x)u]_x, \quad x,t \in R$$

下面根据 $p(x)$ 和 $q(x)$ 的不同分情况讨论:

1.1　$p(x) = a, q(x) = b \ (a,b \neq 0)$

则方程为

$$u_t = [au_x + bu]_x \quad (8)$$

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由文献[1]我们可得方程(8)的广义势系统

$$\begin{cases} v_x = e^{\frac{b}{a}x}u \\ v_t = ae^{\frac{b}{a}x}u_x + 1 \end{cases} \tag{9}$$

其对应的一阶延拓无穷小算子为

$$V^{(1)} = V + \phi_x^{(1)} \frac{\partial}{\partial u_x} + \phi_t^{(1)} \frac{\partial}{\partial u_t} + \eta_x^{(1)} \frac{\partial}{\partial v_x} + \eta_t^{(1)} \frac{\partial}{\partial v_t} \tag{10}$$

把 $V^{(1)}$ 作用到系统(9)中,得到

$$\begin{cases} \eta_x^{(1)} - \frac{b}{a}e^{\frac{b}{a}x}\xi u - e^{\frac{b}{a}x}\phi \\ \eta_t^{(1)} - be^{\frac{b}{a}x}\xi u_x - ae^{\frac{b}{a}x}\phi_x^{(1)} = 0 \end{cases} \tag{11}$$

将系统(9)和不变曲面条件(7)联立,可得

$$v_x = e^{\frac{b}{a}x}u, v_t = \frac{\eta}{\tau} - \frac{\xi}{\tau}e^{\frac{b}{a}x}u,$$

$$u_x = \frac{\eta}{a\tau}e^{\frac{b}{a}x} - \frac{\xi}{a\tau}u - \frac{1}{a}e^{-\frac{b}{a}x},$$

$$u_t = \frac{\phi}{\tau} - \frac{\xi\eta}{a\tau^2}e^{\frac{b}{a}x} + \frac{\xi^2}{a\tau^2} + \frac{\xi}{a\tau}e^{-\frac{b}{a}x}$$

为了减少运算量,可设 $\xi = \xi(x,t,v)$, $\tau = \tau(t,v)$, $\eta = \eta(x,t,v)$. 将 v_x, v_t, u_x, u_t 代入到(11)里. 由(11)的第一式可知

$$\phi(x,t,u,v) = \eta_x e^{\frac{b}{a}x} + \eta_v u - \xi_x u - \xi_v e^{\frac{b}{a}x}u^2 - \frac{\tau_v\eta}{\tau}u + \frac{\tau_v\xi}{\tau}e^{\frac{b}{a}x}u^2 - \frac{b}{a}\xi u$$

把 ϕ 的表达式代入到(11)的第二式里,得到了下面的非线性确定方程组

$$\eta_t + \frac{\eta_v\eta}{\tau} - \frac{\tau_v\eta}{\tau} - \frac{\tau_v\eta^2}{\tau^2} - \frac{b\xi\eta}{a\tau} + \frac{b\xi}{a} + b\eta_x - a\eta_{xx} - \frac{\eta_v\eta}{\tau} + \eta_v + \frac{\xi_x\eta}{\tau} - \xi_x + \frac{\tau_v\eta^2}{\tau^2} - \frac{\tau_v\eta}{\tau} + \frac{b\xi\eta}{a\tau} - \frac{b\xi}{a} - \frac{\eta_v\eta}{\tau} + \eta_v + \frac{\xi_x\eta}{\tau} - \xi_x + \frac{\tau_v\eta^2}{\tau^2} - \frac{\tau_v\eta}{\tau} + \frac{b\xi\eta}{a\tau} - \frac{b\xi}{a} + \frac{\xi_x\eta}{\tau} - \xi_x = 0 \tag{12a}$$

$$-\frac{\eta_v\xi}{\tau}e^{\frac{b}{a}x} - \xi_t e^{\frac{b}{a}x} - \frac{\xi_v\eta}{\tau}e^{\frac{b}{a}x} + \frac{\tau_t\xi}{\tau}e^{\frac{b}{a}x} + \frac{2\tau_v\eta\xi}{\tau^2}e^{\frac{b}{a}x} + \frac{b\xi^2}{a\tau}e^{\frac{b}{a}x} - a\eta_{vx}e^{\frac{b}{a}x} + \frac{\eta_v\xi}{\tau}e^{\frac{b}{a}x} + \frac{b\xi^2}{a\tau}e^{\frac{b}{a}x} + a\xi_{xx}e^{\frac{b}{a}x} - \frac{\xi_x\xi}{\tau}e^{\frac{b}{a}x} + \frac{2\xi_v\eta}{\tau}e^{\frac{b}{a}x} - 2\xi_v e^{\frac{b}{a}x} + \frac{a\tau_v\eta_x}{\tau}e^{\frac{b}{a}x} - \frac{\tau_v\eta\xi}{\tau}e^{\frac{b}{a}x} + b\xi_x e^{\frac{b}{a}x} - \frac{b\xi^2}{a\tau}e^{\frac{b}{a}x} - \frac{2\tau_v\xi\eta}{\tau^2}e^{\frac{b}{a}x} + \frac{2\tau_v\xi}{\tau}e^{\frac{b}{a}x} + \frac{\eta_v\xi}{\tau}e^{\frac{b}{a}x} - \frac{\xi_x\xi}{\tau}e^{\frac{b}{a}x} + \frac{2\xi_v\eta}{\tau}e^{\frac{b}{a}x} - 2\xi_v e^{\frac{b}{a}x} - \frac{\tau_v\eta\xi}{\tau^2}e^{\frac{b}{a}x} - \frac{2\tau_v\xi\eta}{\tau^2}e^{\frac{b}{a}x} + \frac{2\tau_v\xi}{\tau}e^{\frac{b}{a}x} - \frac{b\xi^2}{a\tau}e^{\frac{b}{a}x} - a\eta_{xv}e^{\frac{b}{a}x} - \frac{\xi_x\xi}{\tau}e^{\frac{b}{a}x} + \frac{\xi_x\eta}{\tau}e^{\frac{b}{a}x} - \xi_v e^{\frac{b}{a}x} + \frac{a\tau_v\eta_x}{\tau}e^{\frac{b}{a}x} - \frac{\tau_v\eta\xi}{\tau^2}e^{\frac{b}{a}x} + \frac{\tau_v\xi}{\tau}e^{\frac{b}{a}x} = 0 \tag{12b}$$

$$\frac{\xi_v\xi}{\tau}e^{\frac{2b}{a}x} - \frac{\tau_v\xi^2}{\tau^2}e^{\frac{2b}{a}x} + a\xi_{vx}e^{\frac{2b}{a}x} + b\xi_v e^{\frac{2b}{a}x} - \frac{2\xi_v\xi}{\tau}e^{\frac{2b}{a}x} - \frac{a\tau_v\xi_x}{\tau}e^{\frac{2b}{a}x} - \frac{b\tau_v\xi}{\tau}e^{\frac{2b}{a}x} + \frac{2\tau_v\xi^2}{\tau^2}e^{\frac{2b}{a}x} - \frac{2\xi_v\xi}{\tau}e^{\frac{2b}{a}x} + \frac{2\tau_v\xi^2}{\tau^2}e^{\frac{2b}{a}x} - a\eta_{vv}e^{\frac{2b}{a}x} + a\ \xi_{vx}e^{\frac{2b}{a}x} + a\ \frac{\tau_{vv}\eta}{\tau}e^{\frac{2b}{a}x} - a\ \frac{\tau_v^2\eta}{\tau^2}e^{\frac{2b}{a}x} + a\ \frac{\tau_v\eta_v}{\tau}e^{\frac{2b}{a}x} + b\xi_v e^{\frac{2b}{a}x} - \frac{\xi_v\xi}{\tau}e^{\frac{2b}{a}x} + a\ \frac{\tau_v\eta_v}{\tau}e^{\frac{2b}{a}x} - \frac{a\tau_v\xi_x}{\tau}e^{\frac{2b}{a}x} - a\ \frac{\tau_v^2\eta}{\tau^2}e^{\frac{2b}{a}x} - \frac{b\tau_v\xi}{\tau}e^{\frac{2b}{a}x} + \frac{\tau_v\xi^2}{\tau^2}e^{\frac{2b}{a}x} = 0 \tag{12c}$$

$$a\xi_{vv}e^{\frac{3b}{a}x} - \frac{a\tau_{vv}\xi}{\tau}e^{\frac{3b}{a}x} + \frac{2a\tau_v^2\xi}{\tau^2}e^{\frac{3b}{a}x} - \frac{2a\tau_v\xi_v}{\tau}e^{\frac{3b}{a}x} = 0 \tag{12d}$$

从而得到了原方程(8)的一个非古典势对称

$$V_1 = te^v\frac{\partial}{\partial t} + tv\frac{\partial}{\partial u} + \frac{a}{b}te^v e^{\frac{b}{a}x}\frac{\partial}{\partial v} \tag{13}$$

和其伴随系统(9)的一个非古典对称

$$V_2 = \frac{\partial}{\partial t} + (1 - \frac{u}{t})\frac{\partial}{\partial u} + (\frac{a}{b}e^{\frac{b}{a}x} - \frac{v}{t} + 2 - \frac{1}{t})\frac{\partial}{\partial v}$$

对于 V_1:首先可得特征方程

$$\frac{dx}{0} = \frac{dt}{te^v} = \frac{dv}{\frac{a}{b}te^v e^{\frac{b}{a}x}},$$

进而

$$\begin{cases} x = \zeta_1 \\ v = \frac{a}{b}e^{\frac{b}{a}x}t + f(\zeta_1) \end{cases}$$

其中 $f(\zeta_1)$ 满足常微分方程

$$f''(\zeta_1) - \frac{b}{a}f'(\zeta_1) - \frac{1}{a}e^{\frac{b}{a}x} = 0 \tag{14}$$

则得到原方程(8)的一个解

$$u_1(x,t) = t + \frac{ac}{b} + \frac{x}{b} - \frac{a}{b^2}(c \neq 0).$$

对于 V_2: 步骤同理情况 V_1,得到了原方程(8)的另一个解

$$u_2(x,t)=e^{-\frac{b}{a}x}\frac{f'(x)}{t}+\frac{t}{2},$$

其中 $f(x)$ 满足常微分方程

$$ate^{-\frac{b}{a}x}f'''(x)-bte^{-\frac{b}{a}x}f''(x)+$$
$$e^{-\frac{b}{a}x}f'(x)-\frac{t^2}{2}=0$$

1.2 $p(x)=\frac{1}{x}$, $q(x)=b\ (b\neq 0)$

则方程为

$$u_t=[\frac{1}{x}u_x+bu]_x \tag{15}$$

根据文献[1],其广义势系统为

$$\begin{cases} v_x=e^{\frac{b}{2}x^2}x \\ v_t=\frac{1}{x}e^{\frac{b}{2}x^2}u_x+1 \end{cases} \tag{16}$$

把 $V^{(1)}$ 作用到系统(16)中,从而得到

$$\begin{cases} \eta_x^{(1)}-bxe^{\frac{bx^2}{2}}\xi u-e^{\frac{b}{2}x^2}\phi=0 \\ \eta_t^{(1)}+\frac{1}{x^2}e^{\frac{bx^2}{2}}\xi u_x-be^{\frac{bx^2}{2}}\xi u_x-\frac{1}{x}e^{\frac{bx^2}{2}}\phi_x^{(1)}=0 \end{cases} \tag{17}$$

将系统(16)与不变曲面条件(7)联立,可得 v_x,v_t,u_x,u_t. 为了减少运算量,可设 $\xi=\xi(x,t,v)$,$\tau=\tau(t,v)$,$\eta=\eta(x,t,v)$. 将 v_x,v_t,u_x,u_t 代入到(17)里. 由(17)的第一式可得 $\phi(x,t,u,v)$. 把 ϕ 的表达式代入到(17)的第二式里,则可得

$$\eta_t+\frac{\eta_v\eta}{\tau}-\frac{\tau_t\eta}{\tau}-\frac{\tau_v\eta^2}{\tau^2}+\frac{\xi\eta}{x\tau}-\frac{\xi}{x}-\frac{bx\xi\eta}{\tau}+bx\xi+$$
$$b\eta_x-\frac{\eta_{xx}}{x}-\frac{\eta_v\eta}{\tau}+\eta_v+\frac{\xi_x\eta}{\tau}-\xi_x+\frac{\tau_v\eta^2}{\tau^2}-\frac{\tau_v\eta}{\tau}+$$
$$\frac{bx\xi\eta}{\tau}-bx\xi-\frac{\eta_v\eta}{\tau}+\eta_v+\frac{\xi_x\eta}{\tau}-\xi_x+\frac{\tau_v\eta^2}{\tau^2}-$$
$$\frac{\tau_v\eta}{\tau}+\frac{bx\xi\eta}{\tau}-bx\xi+\frac{\xi_x\eta}{\tau}-\xi_x=0 \tag{18a}$$

$$-\frac{\eta_v\xi}{\tau}e^{\frac{b}{2}x^2}-\xi_te^{\frac{b}{2}x^2}-\frac{\xi_v\eta}{\tau}e^{\frac{b}{2}x^2}+\frac{\tau_t\xi}{\tau}e^{\frac{b}{2}x^2}+$$
$$\frac{2\tau_v\eta\xi}{\tau^2}e^{\frac{b}{2}x^2}-\frac{\xi^2}{x\tau}e^{\frac{b}{2}x^2}+\frac{bx\xi^2}{\tau}e^{\frac{b}{2}x^2}-\frac{\eta_{vx}}{x}e^{\frac{b}{2}x^2}+$$
$$\frac{\eta_v\xi}{\tau}e^{\frac{b}{2}x^2}+\frac{\xi_{xx}}{x}e^{\frac{b}{2}x^2}-\frac{\xi_x\xi}{\tau}e^{\frac{b}{2}x^2}+\frac{2\xi_v\eta}{\tau}e^{\frac{b}{2}x^2}-$$
$$2\xi_ve^{\frac{b}{2}x^2}+\frac{\tau_v\eta_x}{x\tau}e^{\frac{b}{2}x^2}-\frac{\tau_v\eta\xi}{\tau^2}e^{\frac{b}{2}x^2}-\frac{2\tau_v\eta\xi}{\tau^2}e^{\frac{b}{2}x^2}+$$
$$\frac{2\tau_v\xi}{\tau}e^{\frac{b}{2}x^2}+\frac{b\xi}{x}e^{\frac{b}{2}x^2}+b\xi_xe^{\frac{b}{2}x^2}-\frac{bx\xi^2}{\tau}e^{\frac{b}{2}x^2}+$$
$$\frac{\eta_v\xi}{\tau}e^{\frac{b}{2}x^2}-\frac{\xi_x\xi}{\tau}e^{\frac{b}{2}x^2}+\frac{2\xi_v\eta}{\tau}e^{\frac{b}{2}x^2}-2\xi_ve^{\frac{b}{2}x^2}-$$
$$\frac{\tau_v\eta\xi}{\tau^2}e^{\frac{b}{2}x^2}-\frac{2\tau_v\eta\xi}{\tau^2}e^{\frac{b}{2}x^2}+\frac{2\tau_v\xi}{\tau}e^{\frac{b}{2}x^2}-\frac{bx\xi^2}{\tau}e^{\frac{b}{2}x^2}-$$
$$\frac{\eta_{vx}}{x}e^{\frac{b}{2}x^2}-\frac{\xi_x\xi}{\tau}e^{\frac{b}{2}x^2}+\frac{\xi_v\eta}{\tau}e^{\frac{b}{2}x^2}-\xi_ve^{\frac{b}{2}x^2}+$$
$$\frac{\tau_v\eta_x}{x\tau}e^{\frac{b}{2}x^2}-\frac{\tau_v\eta\xi}{\tau^2}e^{\frac{b}{2}x^2}+\frac{\tau_v\xi}{\tau}e^{\frac{b}{2}x^2}=0 \tag{18b}$$

$$\frac{\xi_v\xi}{\tau}e^{bx^2}-\frac{\tau_v\xi^2}{\tau^2}e^{bx^2}+\frac{\xi_{vx}}{x}e^{bx^2}+b\xi_ve^{bx^2}-\frac{2\xi_v\xi}{\tau}e^{bx^2}-$$
$$\frac{\tau_v\xi_x}{x\tau}e^{bx^2}-\frac{b\tau_v\xi}{\tau}e^{bx^2}+\frac{2\tau_v\xi^2}{\tau^2}e^{bx^2}-\frac{2\xi_v\xi}{\tau}e^{bx^2}+$$
$$\frac{2\tau_v\xi^2}{\tau^2}e^{bx^2}-\frac{\eta_{vv}}{x}e^{bx^2}+\frac{\xi_{vx}}{x}e^{bx^2}+\frac{\tau_{vv}\eta}{x\tau}e^{bx^2}+$$
$$\frac{\tau_v\eta_v}{x\tau}e^{bx^2}-\frac{\tau_v^2\eta}{x\tau^2}e^{bx^2}+b\xi_ve^{bx^2}-\frac{\xi_v\xi}{\tau}e^{bx^2}+$$
$$\frac{\tau_v\eta_v}{x\tau}e^{bx^2}-\frac{\xi_x\tau_v}{x\tau}e^{bx^2}-\frac{\tau_v^2\eta}{x\tau^2}e^{bx^2}-$$
$$\frac{b\tau_v\xi}{\tau}e^{bx^2}+\frac{\tau_v\xi^2}{\tau^2}e^{bx^2}=0 \tag{18c}$$

$$\frac{\xi_{vv}}{x}e^{\frac{3}{2}bx^2}-\frac{\tau_{vv}\xi}{x\tau}e^{\frac{3}{2}bx^2}-\frac{2\tau_v\xi_v}{x\tau}e^{\frac{3}{2}bx^2}+\frac{2\tau_v^2\xi}{x\tau^2}e^{\frac{3}{2}bx^2}=0 \tag{18d}$$

从而得到方程(15)的一个非古典势对称

$$V_3=tv\frac{\partial}{\partial t}+btv\frac{\partial}{\partial v},$$

和其伴随系统(16)的一个非古典对称

$$V_4=tv\frac{\partial}{\partial t}-\frac{u}{t}\frac{\partial}{\partial u}+(-\frac{v}{t}+2-\frac{1}{t})\frac{\partial}{\partial v}$$

对于 V_3: 步骤同理情况 V_1,得到了原方程(15)的一个解

$$u_3(x,t)=\frac{c_1}{b}+e^{-\frac{b}{2}bx^2}c_2\ (c_1,c_2\neq 0).$$

对于 V_4: 则得到原方程(15)的又一个解

$$u_x(x,t)=e^{-\frac{b}{2}bx^2}\frac{f'(x)}{t},$$

其中 $f(x)$ 满足常微分方程

$$\frac{t}{x}f'''(x)-(bt+\frac{t}{x^2})f''(x)+f'(x)=0 \tag{19}$$

1.3 $p(x)=x^2$, $q(x)=b\ (b\neq 0)$

则方程为

$$u_t=[x^2u_x+bu]_x \tag{20}$$

同理 1.2,首先得其广义势系统为

$$\begin{cases} v_x = e^{-\frac{b}{x}}u \\ v_t = x^2 e^{-\frac{b}{x}}u_x + 1 \end{cases} \tag{21}$$

把 $V^{(1)}$ 作用到系统(21)中,从而得到

$$\begin{cases} \eta_x^{(1)} - \frac{b}{x^2}e^{-\frac{b}{x}}\xi u - e^{-\frac{b}{x}}\phi = 0 \\ \eta_t^{(1)} - 2xe^{-\frac{b}{x}}\xi u_x - be^{-\frac{b}{x}}\xi u_x - x^2 e^{-\frac{b}{x}}\phi_x^{(1)} = 0 \end{cases} \tag{22}$$

将系统(21)与不变曲面条件(7)联立,可得 v_x, v_t, u_x, u_t. 为了减少运算量,可设 $\xi = \xi(x,t,v)$, $\tau = \tau(t,v)$, $\eta = \eta(x,t,v)$. 将 v_x, v_t, u_x, u_t 代入到(22)里. 由(22)的第一式可得 $\phi(x,t,u,v)$. 把 ϕ 的表达式代入到(22)的第二式里,则可得(21)的确定方程组,进一步就得到了方程(20)的一个非古典势对称

$$V_5 = tv\frac{\partial}{\partial t} + btv\frac{\partial}{\partial v}$$

和其伴随系统(21)的一个非古典对称

$$V_6 = \frac{\partial}{\partial t} - \frac{u}{t}\frac{\partial}{\partial u} + \left(-\frac{v}{t} + 2 - \frac{1}{t}\right)\frac{\partial}{\partial v}.$$

对于 V_5: 得到原方程(20)的一个解

$$u_5(x,t) = \frac{c}{b} + c_1 e^{\frac{b}{x}} \quad (c, c_1 \neq 0).$$

对于 V_6: 也得到了原方程(20)的又一个解

$$u_6(x,t) = e^{\frac{b}{x}}\frac{f'(x)}{t},$$

其中 $f(x)$ 满足常微分方程

$$x^2 t f'''(x) + (2x - b)tf''(x) + f'(x) = 0$$

1.4 $p(x) = e^x$, $q(x) = 1$

则方程为

$$u_t = [e^x u_x + u]_x \tag{23}$$

同理1.2和1.3,首先得其广义势系统为

$$\begin{cases} v_x = e^{-e^x}u \\ v_t = e^x e^{-e^x}u_x + 1 \end{cases} \tag{24}$$

类似1.2,1.3,得到了方程(23)的两个非古典势对称

$$V_7 = tv\frac{\partial}{\partial t} + btv\frac{\partial}{\partial v},$$

$$V_8 = v\frac{\partial}{\partial t} + ve^{e^{-x}}\frac{\partial}{\partial u} + (x - t)v\frac{\partial}{\partial v}$$

和其伴随系统(24)的一个非古典对称

$$V_9 = \frac{\partial}{\partial t} - \frac{u}{t}\frac{\partial}{\partial u} + \left(-\frac{v}{t} + 2 - \frac{1}{t}\right)\frac{\partial}{\partial v}.$$

对于 V_7: 得到原方程(23)的一个解

$$u_7(x,t) = c_1 + c_2 e^{e^{-x}} \quad (c_1, c_2 \neq 0).$$

对于 V_8: 也得到原方程(23)的又一个解

$$u_8(x,t) = e^{e^{-x}}t + e^{e^{-x}}f'(x),$$

其中 $f(x)$ 满足常微分方程

$$e^x f'''(x) - (1 - e^x)f''(x) - 1 = 0.$$

对于 V_9: 则得到原方程(23)的第三个解

$$u_9(x,t) = e^{e^{-x}}t\frac{f'(x)}{t},$$

其中 $f(x)$ 满足常微分方程

$$te^x f'''(x) + te^x f''(x) - tf''(x) + f'(x) = 0$$

2 结论

综上,我们利用广义势系统得到了方程(8)的一个非古典势对称 V_1,方程(15)的一个非古典势对称 V_3,方程(20)的一个非古典势对称 V_5,以及方程(23)的两个非古典势对称 V_7、V_8,同时也得到这些方程的伴随系统的非古典对称,并用上述对称求出了原方程的一些精确解. 方程存在广义势系统的充分条件是什么,以及什么条件下方程可以利用这些广义势系统

来求得原方程的解,这些都是有待于研究的问题.

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NONCLASSICAL GENERAL POTENTIAL SYMMETRIES AND EXACT SOLUTIONS OF FOKKER-PLANCK EQUATION

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Abstract This paper analyzed the nonclassical potential symmetries Fokker-Planck equation. And these nonclassical potential symmetries were determined by considering a generalized potential system rather than the natural potential system. The new symmetries of these given PDEs and auxiliary potential systems were obtained. Furthermore, the corresponding exact solutions can be obtained by using the above symmetries. The solutions have extensive application value for further researching the physical phenomena described by Fokker-Planck equation.

Key words Fokker-Planck equation, nonclassical general potential symmetries, general potential system, exact solutions

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