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局部流型分析的哈密顿系统小参数正则变换*

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摘要 表达二维不可压缩流动的流速分量与流函数关系的微分方程组是典型的具有一个自由度的哈密顿系统. 将流函数用 Taylor 级数展开,应用非线性系统动力学方法对流型及其分岔进行了分析. 对退化临界点,基于流动平面的小参数正则变换,导出了流函数的正形表达式和简化的微分方程,并对简化系统的一般特性进行了分析.

关键词 哈密顿系统, 正则变换, 临界点, 分岔

引言

流体内部流线的局部特性涉及到涡、驻点、分隔线、环流区等重要流动现象,借助流型分析正确识别这些流动特征的研究一直没有间断. 采用 Smith[1] 解决弹性薄板问题的双正交技术,将流函数表达为 Papkovich-Fadle 本征函数的级数解,Joseph 和 Sturges[2] 首先提出了分析腔洞流流型的正交本征函数法,Khuri[3] 等将这种方法推广到极坐标中,Gurcan[4] 等用这种方法对不同流场区域呈现的不同流动特点进行了讨论,Shankar[5] 用最小二乘法代替双正交条件,对同样的问题进行了研究,并与 Joseph 和 Sturges 的结果进行了比较. 上述方法均是在拉格朗日系统下对 Stokes 流问题进行的研究,涉及到双调和方程的求解. 从系统动力学观点而言,物理流动平面即为相平面,流线可视为微分方程 $\dot{\boldsymbol{x}}=\boldsymbol{v}$ 的轨线. 采用临界点和分岔理论,Dallmann[6] 研究了三维流动的涡结构,Chiang[7] 等研究了带坎槽道流流型,Brons 和 Hartnack[8] 以及 Gurcan[9] 将正则变换引入到流型分岔研究. 本文通过带小参数的正则变换,对描述流线的微分方程进行简化,获得了便于进行退化临界点附近流型分析的流函数的正则形式,进一步拓展了正则变换在流型分析中的应用.

1 退化临界点的小参数正则变换

对(x, y)平面内远离边界的二维不可压缩流动,流线可由下列流速场确定

$$\dot{x}=u=\frac{\partial\psi}{\partial y},\quad \dot{y}=v=-\frac{\partial\psi}{\partial x} \tag{1}$$

式(1)为典型的具有一个自由度的哈密顿系统[10],这里 ψ 为流函数. 对关于 y 轴对称的流动而言,流函数 ψ 可由下列关于原点(0,0)的 Taylor 级数展开

$$\psi=\sum_{i+j=1}^{\infty}a_{2i,j}x^{2i}y^{j} \tag{2}$$

式(2)代入式(1)可得流线方程的线性近似为

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}=\begin{pmatrix}a_{0,1}\\ 0\end{pmatrix}+\begin{pmatrix}0 & 2a_{0,2}\\ -2a_{2,0} & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} \tag{3}$$

当 $a_{0,1}=0$ 时,原点即为临界点,这时原点附近的局部流型可用标准的哈密顿系统理论确定,相应的 Jacobi 矩阵的行列式为

$$|J|=4a_{0,2}a_{2,0} \tag{4}$$

当$|J|>0$ 时,临界点为中心;当$|J|<0$ 时,临界点为鞍点;当$|J|=0$(不失一般性,假定 $a_{2,0}=0$,$a_{0,2}\neq0$)时,临界点为退化情形,这时高阶项对流型起决定作用. 从式(2)可以看出,随着阶次的增高,Taylor 级数的系数数量也随之增多,流型分析也变得更为困难. 而借助正则变换可使自由参数的数量大为减少.

正则变换[11] 在分析力学中有着广泛的应用,其重要作用在于可以简化哈密顿函数的结构,从而有利于正则方程的分析求解. 变换时需要选择合适的生成函数以保持系统的对称性. 为了实现新、旧

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坐标的变换,生成函数必须同时含有新、旧坐标. 以旧坐标(x,y)和新坐标(ξ,η)的两两不同组合,有四种不同的生成函数. 若取生成函数为$g(y,\xi)$,则得到如下的正则变换

$$x=\frac{\partial g}{\partial y},\quad \eta=\frac{\partial g}{\partial \xi} \tag{5}$$

为了考察线性退化情况下的流型分岔,引入如下的小参数

$$\varepsilon_1=a_{0,1},\quad \varepsilon_2=a_{2,0} \tag{6}$$

为了简化系统(1)的三次项即ψ的四次项,具体选取如下包含小参数的生成函数

$$\begin{aligned}g={}&y\xi+s_{3,0,0,0}y^3+s_{0,3,0,0}\xi^3+s_{0,0,3,0}\varepsilon_1^3+\\&s_{0,0,0,3}\varepsilon_2^3+s_{2,1,0,0}y^2\xi+s_{2,0,1,0}y^2\varepsilon_1+s_{2,0,0,1}y^2\varepsilon_2+\\&s_{1,0,2,0}y\varepsilon_1^2+s_{0,1,2,0}\xi\varepsilon_1^2+s_{0,0,2,1}\varepsilon_1^2\varepsilon_2+\\&s_{1,1,1,0}y\xi\varepsilon_1+s_{1,1,0,1}y\xi\varepsilon_2+s_{1,0,1,1}y\varepsilon_1\varepsilon_2+\\&s_{0,1,1,1}\xi\varepsilon_1\varepsilon_2+s_{4,0,0,0}y^4+s_{0,0,4,0}\varepsilon_1^4+s_{2,0,2,0}y^2\varepsilon_1^2+\\&s_{3,1,0,0}y^3\xi+s_{3,0,1,0}y^3\varepsilon_1+s_{3,0,0,1}y^3\varepsilon_2+\\&s_{1,3,0,0}y\xi^3+s_{0,3,1,0}\xi^3\varepsilon_1+s_{0,3,0,1}\xi^3\varepsilon_2+\\&s_{1,0,3,0}y\varepsilon_1^3+s_{0,1,3,0}\xi\varepsilon_1^3+s_{0,0,3,1}\varepsilon_1^3\varepsilon_2+s_{1,0,0,3}y\varepsilon_2^3+\\&s_{0,1,0,3}\xi\varepsilon_2^3+s_{0,0,1,3}\varepsilon_1\varepsilon_2^3+s_{2,1,1,0}t^2\xi\varepsilon_1+\\&s_{2,1,0,1}y^2\xi\varepsilon_2+s_{2,0,1,1}y^2\varepsilon_1\varepsilon_2+s_{1,1,2,0}y\xi\varepsilon_1^2+\\&s_{1,0,2,1}y\varepsilon_1^2\varepsilon_2+s_{0,1,2,1}\xi\varepsilon_1^2\varepsilon_2+s_{1,1,1,1}y\xi\varepsilon_1\varepsilon_2\end{aligned} \tag{7}$$

则式(5)变为

$$\begin{aligned}x={}&\xi+3s_{3,0,0,0}y^2+2s_{2,1,0,0}y\xi+2s_{2,0,1,0}y\varepsilon_1+\\&2s_{2,0,0,1}y\varepsilon_2+s_{1,0,2,0}\varepsilon_1^2+s_{1,1,1,0}\xi\varepsilon_1+s_{1,1,0,1}\xi\varepsilon_2+\\&s_{1,0,1,1}\varepsilon_1\varepsilon_2+4s_{4,0,0,0}y^3+2s_{2,0,2,0}y\varepsilon_1^2+\\&3s_{3,1,0,0}y^2\xi+3s_{3,0,1,0}y^2\varepsilon_1+3s_{3,0,0,1}y^2\varepsilon_2+\\&s_{1,3,0,0}\xi^3+s_{1,0,3,0}\varepsilon_1^3+s_{1,0,0,3}\varepsilon_2^3+2s_{2,1,1,0}y\xi\varepsilon_1+\\&2s_{2,1,0,1}y\xi\varepsilon_2+2s_{2,0,1,1}y\varepsilon_1\varepsilon_2+s_{1,1,2,0}\xi\varepsilon_1^2+\\&s_{1,0,2,1}\varepsilon_1^2\varepsilon_2+s_{1,1,1,1}\xi\varepsilon_1\varepsilon_2\end{aligned} \tag{8a}$$

$$\begin{aligned}\eta={}&y+3s_{0,3,0,0}\xi^2+s_{2,1,0,0}y^2+s_{0,1,2,0}\varepsilon_1^2+\\&s_{1,1,1,0}y\varepsilon_1+s_{1,1,0,1}y\varepsilon_2+s_{0,1,1,1}\varepsilon_1\varepsilon_2+s_{3,1,0,0}y^3+\\&3s_{1,3,0,0}y\xi^2+3s_{0,3,1,0}\xi^2\varepsilon_1+3s_{0,3,0,1}\xi^2\varepsilon_2+\\&s_{0,1,3,0}\varepsilon_1^3+s_{0,1,0,3}\varepsilon_2^3+s_{2,1,1,0}y^2\varepsilon_1+s_{2,1,0,1}y^2\varepsilon_2+\\&s_{1,1,2,0}y\varepsilon_1^2+s_{0,1,2,1}\varepsilon_1^2\varepsilon_2+s_{1,1,1,1}y\varepsilon_1\varepsilon_2\end{aligned} \tag{8b}$$

方程(8)的解可表示为下列Taylor级数的形式

$$\begin{aligned}&x=\xi+\sum_{i+j+k+l=2}^{\infty}\alpha_{i,j,k,l}\xi^i\eta^j\varepsilon_1^k\varepsilon_2^l,\\&y=\eta+\sum_{i+j+k+l=2}^{\infty}\beta_{i,j,k,l}\xi^i\eta^j\varepsilon_1^k\varepsilon_2^l\end{aligned} \tag{9}$$

将式(9)代入式(8),合并$\xi,\eta,\varepsilon_1,\varepsilon_2$的同阶次项,可得关于各系数$\alpha_{i,j,k,l},\beta_{i,j,k,l}$的线性方程组,取$s_{i,j,k,l}$为线性关系的所有状态变量和小参数的二、三阶次项系数的解为

$$\begin{aligned}&\beta_{2,0,0,0}=-3s_{0,3,0,0},\beta_{0,2,0,0}=-s_{2,1,0,0},\\&\beta_{0,0,2,0}=-s_{0,1,2,0},\beta_{0,1,1,0}=-s_{1,1,1,0},\\&\beta_{0,1,0,1}=-s_{1,1,0,1},\beta_{0,0,1,1}=-s_{0,1,1,1},\\&\beta_{0,0,0,2}=\beta_{1,1,0,1}=\beta_{1,0,1,0}=\beta_{1,0,0,1}=0\\&\beta_{0,3,0,0}=-s_{3,1,0,0},\beta_{0,0,3,0}=-s_{0,1,3,0},\\&\beta_{0,0,0,3}=-s_{0,1,0,3},\beta_{2,1,0,0}=-3s_{1,3,0,0},\\&\beta_{2,0,1,0}=-3s_{0,3,1,0},\beta_{2,0,0,1}=-s_{0,3,0,1},\\&\beta_{0,2,1,0}=-s_{2,1,1,0},\beta_{0,2,0,1}=-s_{2,1,0,1},\\&\beta_{0,1,2,0}=-s_{1,1,2,0},\beta_{0,0,2,1}=-s_{0,1,2,1},\\&\beta_{0,1,1,1}=-s_{1,1,1,1},\beta_{3,0,0,0}=\beta_{1,2,0,0}=\beta_{1,0,2,0}=\\&\quad\beta_{1,0,0,2}=\beta_{0,1,0,2}=\beta_{0,0,1,2}=\beta_{1,1,1,0}=\\&\quad\beta_{1,1,0,1}=\beta_{1,0,1,1}=0\\&\alpha_{0,2,0,0}=3s_{3,0,0,0},\alpha_{0,0,2,0}=s_{1,0,2,0},\\&\alpha_{1,1,0,0}=2s_{2,1,0,0},\alpha_{1,0,1,0}=s_{1,1,1,0},\\&\alpha_{1,0,0,1}=s_{1,1,0,1},\alpha_{0,1,1,0}=2s_{2,0,1,0},\\&\alpha_{0,1,0,1}=2s_{2,0,0,1},\alpha_{0,0,1,1}=s_{1,0,1,1},\\&\alpha_{2,0,0,0}=\alpha_{0,0,0,2}=0\\&\alpha_{3,0,0,0}=s_{1,3,0,0},\alpha_{0,3,0,0}=4s_{4,0,0,0},\\&\alpha_{0,0,3,0}=s_{1,0,3,0},\alpha_{0,0,0,3}=s_{1,0,0,3},\\&\alpha_{1,2,0,0}=3s_{3,1,0,0},\alpha_{0,2,1,0}=3s_{3,0,1,0},\\&\alpha_{0,2,0,1}=3s_{3,0,0,1},\alpha_{1,0,2,0}=s_{1,1,2,0},\\&\alpha_{0,1,2,0}=2s_{2,0,2,0},\alpha_{0,0,2,1}=s_{1,0,2,1},\\&\alpha_{1,1,1,0}=2s_{2,1,1,0},\alpha_{1,1,0,1}=2s_{2,1,0,1},\\&\alpha_{0,1,1,1}=2s_{2,0,1,1},\alpha_{1,0,1,1}=s_{1,1,1,1},\\&\alpha_{2,1,0,0}=\alpha_{2,0,1,0}=\alpha_{2,0,0,1}=\alpha_{0,1,0,2}=\\&\quad\alpha_{0,0,1,2}=\alpha_{1,0,0,2}=0\end{aligned}$$

对各状态变量和小参数保留到三次项,式(9)变为

$$\begin{aligned}x={}&\xi+3s_{3,0,0,0}\eta^2+s_{1,0,2,0}\varepsilon_1^2+2s_{2,1,0,0}\xi\eta+\\&s_{1,1,1,0}\xi\varepsilon_1+s_{1,1,0,1}\xi\varepsilon_2+2s_{2,0,1,0}\eta\varepsilon_1+\\&2s_{2,0,0,1}\eta\varepsilon_2+s_{1,0,1,1}\varepsilon_1\varepsilon_2+s_{1,3,0,0}\xi^3+\\&4s_{4,0,0,0}\eta^3+s_{1,0,3,0}\varepsilon_1^3+s_{1,0,0,3}\varepsilon_2^3+3s_{3,1,0,0}\xi\eta^2+\\&3s_{3,0,1,0}\eta^2\varepsilon_1+3s_{3,0,0,1}\eta^2\varepsilon_2+s_{1,1,2,0}\xi\varepsilon_1^2+\\&2s_{2,0,2,0}\eta\varepsilon_1^2+s_{1,0,2,1}\varepsilon_1^2\varepsilon_2+2s_{2,1,1,0}\xi\eta\varepsilon_1+\\&2s_{2,1,0,1}\xi\eta\varepsilon_2+2s_{2,0,1,1}\eta\varepsilon_1\varepsilon_2+s_{1,1,1,1}\xi\varepsilon_1\varepsilon_2\end{aligned} \tag{10a}$$

$$\begin{aligned}y={}&\eta-3s_{0,3,0,0}\xi^2-s_{2,1,0,0}\eta^2-s_{0,1,2,0}\varepsilon_1^2+\\&s_{0,1,1,1}\varepsilon_1\varepsilon_2-s_{1,1,1,0}\eta\varepsilon_1-s_{1,1,0,1}\eta\varepsilon_2-s_{3,1,0,0}\eta^3-\\&s_{0,1,3,0}\varepsilon_1^3-s_{0,1,0,3}\varepsilon_2^3-3s_{1,3,0,0}\xi^2\eta-\\&3s_{0,3,1,0}\xi^2\varepsilon_1-3s_{0,3,0,1}\xi^2\varepsilon_2-s_{2,1,1,0}\eta^2\varepsilon_1-\\&s_{2,1,0,1}\eta^2\varepsilon_2-s_{1,1,2,0}\eta\varepsilon_1^2-s_{0,1,2,1}\varepsilon_1^2\varepsilon_2-\end{aligned}$$

$s_{1,1,1,1}\eta\varepsilon_1\varepsilon_2$ (10b)

将式(10)代入式(2)并保留到四次项得

$$\psi=\lambda_1+\lambda_2\xi+\lambda_3\eta+\lambda_4\xi^2+\bar{a}_{2,0}\eta^2+((4s_{2,0,1,0}+2a_{2,1}s_{1,0,1,1})\varepsilon_1\varepsilon_2+2a_{2,1}s_{1,0,2,0}\varepsilon_1^2+4s_{2,0,0,1}\varepsilon_2^2)\xi\eta+(a_{2,1}s_{2,0,1,0}\varepsilon_1+(6s_{3,0,0,0}+4a_{2,1}s_{2,0,0,1})\varepsilon_2)\xi\eta^2+(a_{2,1}\text{-}6a_{0,2}s_{0,3,0,0}+(a_{2,1}s_{1,1,1,0}+6a_{0,2}s_{1,1,1,0}s_{0,3,0,0}-3s_{1,3,0,0}-6a_{0,2}s_{0,3,1,0})\varepsilon_1+(a_{2,1}s_{1,1,0,1}+4s_{2,1,0,0}+6a_{0,2}s_{1,1,0,1}s_{0,3,0,0}-6a_{0,2}s_{0,3,0,1})\varepsilon_2)\xi^2\eta+(a_{0,3}-2a_{0,2}s_{2,1,0,0}+(2a_{0,2}s_{1,1,1,0}s_{2,1,0,0}-s_{3,1,0,0}-2a_{0,2}s_{2,1,1,0}-3a_{0,3}s_{1,1,1,0})\varepsilon_1+(2a_{0,2}s_{1,1,0,1}s_{2,1,0,0}-2a_{0,2}s_{2,1,0,1}-3a_{0,3}s_{1,1,0,1})\varepsilon_2)\eta^3+6a_{2,1}s_{3,0,0,0}\xi\eta^3+(6a_{0,2}s_{0,3,0,0}s_{2,1,0,0}-6a_{0,2}s_{1,3,0,0}+3a_{2,1}s_{2,1,0,0}-9a_{0,3}s_{0,3,0,0}+a_{2,2})\xi^2\eta^2+(a_{0,2}s_{2,1,0,0}^2-2a_{0,2}s_{3,1,0,0}-3a_{0,3}s_{2,1,0,0}+a_{0,4})\eta^4+\bar{a}_{4,0}\xi^4 \tag{11}$$

其中 $\lambda_1,\lambda_2,\lambda_3,\lambda_4$ 为由小参数 $\varepsilon_1,\varepsilon_2$ 的非线性组合所构成的新的小参数,具体表达式为

$$\lambda_1=a_{0,2}s_{0,1,2,0}^2\varepsilon_1^4+a_{0,2}s_{0,1,1,1}^2\varepsilon_1^2\varepsilon_2^2+2a_{0,2}s_{0,1,1,1}s_{0,1,2,0}\varepsilon_1^3\varepsilon_2-s_{0,1,2,0}\varepsilon_1^3-s_{0,1,1,1}\varepsilon_1^2\varepsilon_2-s_{0,1,3,0}\varepsilon_1^4-s_{0,1,0,3}\varepsilon_1\varepsilon_2^3-s_{0,1,2,1}\varepsilon_1^3\varepsilon_2 \tag{12}$$

$$\lambda_2=2s_{1,0,2,0}\varepsilon_1^2\varepsilon_2+2s_{1,0,1,1}\varepsilon_1\varepsilon_2^2 \tag{13}$$

$$\lambda_3=\varepsilon_1-s_{1,1,1,0}\varepsilon_1^2-s_{1,1,0,1}\varepsilon_1\varepsilon_2-s_{1,1,2,0}\varepsilon_1^3-s_{1,1,1,1}\varepsilon_1^2\varepsilon_2+2a_{0,2}s_{1,1,0,1}s_{0,1,1,1}\varepsilon_1\varepsilon_2^2+2a_{0,2}s_{1,1,0,1}s_{0,1,2,0}\varepsilon_1^2\varepsilon_2+2a_{0,2}s_{1,1,1,0}s_{0,1,1,1}\varepsilon_1^2\varepsilon_2+2a_{0,2}s_{1,1,1,0}s_{0,1,2,0}\varepsilon_1^3-2a_{0,2}s_{0,1,1,1}\varepsilon_1\varepsilon_2-2a_{0,2}s_{0,1,2,0}\varepsilon_1^2-2a_{0,2}s_{0,1,3,0}\varepsilon_1^3-2a_{0,2}s_{0,1,0,3}\varepsilon_2^3-2a_{0,2}s_{0,1,2,1}\varepsilon_1^2\varepsilon_2 \tag{14}$$

$$\lambda_4=\varepsilon_2-3s_{3,0,0,0}\varepsilon_1-3s_{0,3,1,0}\varepsilon_1^2-3s_{0,3,0,1}\varepsilon_1\varepsilon_2+2s_{1,1,1,0}\varepsilon_1\varepsilon_2+2s_{1,1,0,1}\varepsilon_2^2+6a_{0,2}s_{0,1,1,1}s_{0,3,0,0}\varepsilon_1\varepsilon_2+6a_{0,2}s_{0,1,2,0}s_{0,3,0,0}\varepsilon_1^2-a_{2,1}s_{0,1,2,0}\varepsilon_1^2-a_{2,1}s_{0,1,1,1}\varepsilon_1\varepsilon_2 \tag{15}$$

另外

$$\bar{a}_{0,2}=a_{0,2}-(s_{2,1,0,0}+2a_{0,2}s_{1,1,1,0})\varepsilon_1+(a_{0,2}s_{1,1,1,0}^2+2a_{0,2}s_{0,1,2,0}s_{2,1,0,0}-s_{2,1,1,0}-2a_{0,2}s_{1,1,2,0}-3a_{0,3}s_{0,1,2,0})\varepsilon_1^2+(2a_{0,2}s_{1,1,1,0}s_{1,1,0,1}+2a_{0,2}s_{0,1,1,1}s_{2,1,0,0}-s_{2,1,0,1}-2a_{0,2}s_{1,1,1,1}-3a_{0,3}s_{0,1,1,1})\varepsilon_1\varepsilon_2+a_{0,2}s_{1,1,0,1}^2\varepsilon_2^2-2a_{0,2}s_{1,1,0,1}\varepsilon_2 \tag{16}$$

$$\bar{a}_{4,0}=(9a_{0,2}s_{0,3,0,0}^2-3a_{2,1}s_{0,3,0,0}+a_{4,0}) \tag{17}$$

由于各 $s_{i,j,k,l}$ 选择的自由性,令

$$s_{2,1,0,0}=\frac{a_{0,3}}{2a_{0,2}},s_{0,3,0,0}=\frac{a_{2,1}}{6a_{0,2}},s_{0,3,0,1}=\frac{a_{0,3}}{3a_{0,2}^2},$$

$$s_{1,3,0,0}=\frac{a_{2,2}}{6a_{0,2}}+\frac{a_{2,1}a_{0,3}}{12a_{0,2}^2},s_{2,1,1,0}=\frac{5a_{0,3}^2}{16a_{0,2}^3}-\frac{a_{0,4}}{4a_{0,2}^2},$$

$$s_{0,3,1,0}=-\frac{a_{2,2}}{12a_{0,2}^2}-\frac{a_{2,1}a_{0,3}}{24a_{0,2}^3},$$

$$s_{3,1,0,0}=\frac{a_{0,4}^2}{2a_{0,2}}-\frac{5a_{0,3}^2}{8a_{0,2}^2}$$

同时令生成函数式(7)中其余系数为零,则式(12)~(17)变为

$$\lambda_1=0 \tag{18}$$

$$\lambda_2=0 \tag{19}$$

$$\lambda_3=\varepsilon_1 \tag{20}$$

$$\lambda_4=\varepsilon_2+\left(\frac{a_{2,2}}{4a_{0,2}^2}+\frac{a_{2,1}a_{0,3}}{8a_{0,2}^3}\right)\varepsilon_1^2-\frac{a_{0,3}}{a_{0,2}^2}\varepsilon_1\varepsilon_2 \tag{21}$$

$$\bar{a}_{0,2}=a_{0,2}-\frac{a_{0,3}}{2a_{0,2}}\varepsilon_1+\left(\frac{a_{0,4}}{4a_{0,2}^2}-\frac{5a_{0,3}^2}{16a_{0,2}^3}\right)\varepsilon_1^2 \tag{22}$$

$$\bar{a}_{4,0}=a_{4,0}-\frac{a_{2,1}^2}{4a_{0,2}} \tag{23}$$

从而式(11)变为

$$\psi=\varepsilon_1\eta+\bar{a}_{0,2}\eta^2+\lambda_4\xi^2+\bar{a}_{4,0}\xi^4 \tag{24}$$

进一步,式(24)中的 η 项可通过坐标平移消除. 以 $\eta+\eta_0$ 代替 η,并令 $\eta_0=-\varepsilon_1/2\bar{a}_{0,2}$,$c_0'=\varepsilon_1\eta_0+\bar{a}_{0,2}\eta_0^2$,式(24)变为

$$\psi=c_0'+\bar{a}_{0,2}\eta^2+\lambda_4\xi^2+\bar{a}_{4,0}\xi^4 \tag{25}$$

为便于后面的分析,将式(25)两边同除以 $4\bar{a}_{4,0}$,并以 ψ 代替 $\psi/4\bar{a}_{4,0}$作比尺变换,得到最终的简化形式为

$$\psi=c_0+\frac{\sigma}{2}\eta^2+c_2\xi^2+\frac{1}{4}\xi^4 \tag{26}$$

其中 $c_0=c_0/4\bar{a}_{4,0}$,$c_2=\lambda_4/4\bar{a}_{4,0}$为由 $\varepsilon_1,\varepsilon_2$ 所确定的新的小参数,$\sigma=\bar{a}_{2,0}/2\bar{a}_{4,0}$. 由于等流函数线即为流线,式(26)中的 c_0 可进一步省略. 如果 $\bar{a}_{4,0}$也为小参数,则逼近六阶或更高阶的非线性退化,需计算更高阶次项. 返回到旧坐标(x,y),可得如下的定理.

定理: 设 $a_{0,1},a_{2,0}$和 $\bar{a}_{4,0},\bar{a}_{6,0},\cdots,\bar{a}_{2N-2,0}$为小参数,非退化条件为 $a_{0,2}\neq0,\bar{a}_{2N,0}\neq0$,则阶次为 $2N$ 的流函数的正则形式为

$$\psi_{2N}=\frac{\sigma}{2}y^2+h(x) \tag{27}$$

其中

$$h(x)=\sum_{i=1}^{N}c_{2i}x^{2i},\quad c_{2N}=\frac{1}{2N},\quad \sigma=\bar{a}_{2,0}/N\bar{a}_{2N,0}$$

$c_{2i}(i=1,2,\cdots,N\geqslant2)$为变换后的小参数,$\bar{a}_{2,0}$为

$a_{0,2}$加上小参数 $a_{0,1}, a_{2,0}$ 的非线性组合，$\bar{a}_{2k,0}$ 为 $a_{2k,0}$ 加上 $a_{2i,j}(2i+j<2k)$ 的非线性组合.

同时可得如下的推论.

推论： 如果小参数 $a_{0,1}=a_{2,0}=\bar{a}_{4,0}=\bar{a}_{6,0}=\cdots=\bar{a}_{2N-2,0}=0$，但 $a_{0,2}\neq 0$，$\bar{a}_{2N,0}\neq 0$，则发生 $2N$ 阶退化，这时的正则形式为

$$\psi_{2N}=\frac{\sigma}{2}y^2+\frac{1}{2N}x^{2N} \tag{28}$$

其中，$\sigma=a_{0,2}/N\bar{a}_{2N,0}$.

2 退化临界点附近的流型分析

根据非线性系统临界点和分岔理论[12]，可对式(27)和式(28)正则形式的流函数所揭示的流型进行定性分析.

由式(28)，临界点可能的分隔线由 $\psi_{2N}=0$ 确定，即

$$\frac{\sigma}{2}y^2+\frac{1}{2N}x^{2N}=0 \tag{29}$$

由此可以判断，当 $\sigma>0$ 时，不存在分隔线，临界点为退化的中心；当 $\sigma<$ 时，临界点为拓扑鞍点. 值得指出的是，这一结论是在相关参数取给定值的条件下得出的，相应的流型在结构上是不稳定的，参数及其组合的小的变化将会导致流型的改变而出现分岔. 为了进一步考察线性退化情况下的流型分岔，对包含小参数的正则形式的流函数式(27)所揭示的流型特征分析如下：

(1)由式(27)，可得关于流线的微分方程为

$$\dot{x}=\sigma y,\quad \dot{y}=-h'(x) \tag{30}$$

由此可见，原点始终为临界点之一，总的临界点个数为奇数，且最多为 $2N-1$. 所有的临界点均位于 x 坐标轴上，相应的 x 坐标可由 $h'(x)=0$ 求得.

(2)临界点相应的 Jacobi 矩阵的行列式为

$$|J|=\sigma h''(x) \tag{31}$$

由此可见，若 $|J|=\sigma h''(x)>0$，则临界点为中心；若 $|J|=\sigma h''(x)<0$，则临界点为鞍点.

(3)如果 $|J|=\sigma h''(x)=0$，即 $h''(x)=0$，则临界点退化，分岔发生. 退化的临界点 $(x_d,0)$ 类型可由下列 Taylor 展开式确定

$$\psi_{2N}=\frac{\sigma}{2}y^2+h(x_d)+\frac{1}{3!}(x-x_d)^3h'''(x_d)+\cdots+\frac{1}{n!}(x-x_d)^n h^{(n)}(x_d) \tag{32}$$

分隔线由 $\psi_{2N}=h(x_d)$ 确定，退化的临界点的类型可由从低阶到高阶首先出现的不为零的 $h(x)$ 的导数确定. 如果首先出现的不为零的导数项的阶次为奇数，则退化点为尖点[13]. 如果首先出现的不为零的导数项的阶次 $i(4\leqslant i\leqslant n)$ 为偶数，则退化点的类型取决于 $h^{(i)}(x_d)/\sigma$ 的符号，当 $h^{(i)}(x_d)/\sigma>0$ 时，临界点为中心；当 $h^{(i)}(x_d)/\sigma<0$ 时，临界点为拓扑鞍点.

(4)对于退化的临界点，除发生局部分岔外，也有可能存在连接不同临界点(鞍点)的异宿轨线，从而发生全局分岔. 此时流函数 ψ 在各临界点具有相同的值，各临界点 $(x_i,0)(i=1,2,\cdots,m)$ 在同一异宿环上. 即满足下列条件

$$h(x_1)=h(x_2)=\cdots=h(x_m),$$
$$h'(x_1)=h'(x_2)=\cdots=h'(x_m)=0 \tag{33}$$

3 结语

二维不可压缩流动的速度场构成标准的具有一个自由度的哈密顿系统，当该动力系统出现线性退化时，流型将表现为结构上的不稳定而出现分岔，形成新的流型结构. 本文将流函数展开为 Taylor 级数，Taylor 级数的系数视为分岔参数，通过系统的小参数正则变换，极大地较少了自由参数的数量，获得了统一的便于进行退化流型和分岔分析的简化动力系统. 从退化临界点附近流型特征的分析可见，动力系统正则形式的流型结构主要取决于 Jacobi 行列式，由此可方便地进行局部流型分析.

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SMALL PARAMETER CANONICAL TRANSFORMATIONS OF HAMILTONIAN SYSTEM FOR THE LOCAL STREAMLINE PATTERN ANALYSIS *

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Abstract The differential equations expressing the relations of velocity components and streamfunction for incompressible two-dimensional flow are typical one-degree-of-freedom Hamiltonian system. The streamfunction was expanded in Taylor series. The streamline patterns and their bifurcations were examined using methods from nonlinear system dynamics. Based on the small parameter canonical transformations in the physical flow plane, the normal form expressions of streamfunction and simplified differential equations were derived for the degenerate critical points. Some general characteristics of the simplified system were analyzed.

Key words Hamiltonian system, canonical transformation, critical point, bifurcation

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