应用非线性 Galerkin 方法求解微梁的动态响应*

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摘要　分析在电场力驱动下微共振器的非线性动力学特性.取前3阶模态,利用非线性 Galerkin 方法得到单自由度的降阶模型.用多尺度法计算降阶模型的动态响应,并得出了稳态响应的幅频特性曲线,与利用传统 Galerkin 方法直接取1阶模态所得的结果比较.以数值积分法求解3自由度模型得到的微共振器动力学响应为参考标准,验证了非线性 Galerkin 方法与传统 Galerkin 方法相比具有较高的精度.

关键词　MEMS,　微梁,　惯性流形,　非线性 Galerkin 方法

引言

微梁共振器从上世纪80年代起已被用于机械微传感器,其后在无线电通讯领域得到了广泛应用,致使高频与高性能微共振器得到快速发展.MEMS 共振器在上世纪90年代被提议向大结构共振器发展.对于 MEMS 装置有很多驱动方式,由于高效与结构的简单使得电场力驱动更优于其它的驱动方式[1].电场力驱动方式是通过在结构上加电压而产生电场以驱动结构运动.所加电压分为直流电压、交流电压或两者的组合.电压的频率和幅值可以精确地控制,从而实现机械元件的运动控制.在微梁共振器中,微梁在直流电偏置的基础上,再利用简谐交流电压驱动微梁振动.需要注意的是,这样的装置应避免所加电压超过一定范围使得驱动过程中出现失稳现象[2,3](pull - in instability),最终导致结构破坏.该现象是由于结构自身的恢复力不足以阻止外激励的作用而产生的失稳.与此同时,这一现象也可以用于设计射频(RF)微开关,改善通常情况下高压驱动的局面.有许多文献[2 - 4]已经研究过直流静电驱动失稳现象,提出了一些处理方法来预测失稳阈值以指导设计人员如何来避免或利用它.但微梁结构表现出来的非线性特性与所加电压交流部分有更大的关系[1,4-6].事实上,因交流电压而导致失稳的极限电压值会低于直流电压驱动的极限值[7].

文献[8]探讨了交流与直流混合电压驱动时系统的失稳现象与单用直流电压驱动时的区别,并指出利用 pull - in 现象可方便调节交流电压的频率或振幅以达到低压瞬间驱动 RF 微开关.文献[9]则分析了微共振器分别在超谐波与亚谐波激励下的动力学特性,指出在亚谐波的激励下微梁所受到的阻尼对动力学响应影响很小.基于这点可以设计出高效率的射频开关.

迄今为止,国内外学者对电场力激励下的微梁动力学特性已经做了大量的研究.在建立微梁的动力学模型时,大都采用有限元方法,但数量众多的有限单元不仅给求解系统的动态响应带来困难,而且也不便于定性分析.另一种方法是建立微梁的连续模型(偏微分方程及其相应的边界条件),利用线性部分的模态将其离散化,再采用传统 Galerkin 截断方法将系统降维.

非线性 Galerkin 方法是一种降维的方法,其理论基础是处理无穷维动力系统的理论 - 近似惯性流形理论[10,11].与之在同样的基础下产生的另一种相同性质的方法是 IU(Incremental Unknown)方法[12].传统 Galerkin 方法的局限在于通过将解向平直子空间投影进行简化而显粗糙,它完全忽略了高频部分对系统响应的影响.非线性 Galerkin 方法通过将解向近似惯性流形投影以将高频分量整合到低频分量之中而更精确地描述系统的响应.另一方面,在同样的精度下,非线性 Galerkin 方法比传统 Galerkin 方法更能节省数值计算量,给数值计算带来很大的方便.

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Matthies 和 Meyer[13]介绍了一种近似惯性流形投影方法，并将其用于研究风涡轮的疲劳问题，实现了误差估计与控制. Sinha 等[14]从理论和应用两方面介绍了投影于相空间的系统与二阶动力系统的降维方法.

张新华和徐健学[15]将非线性 Galerkin 方法引入到梁的非线性动力学长期性态分析之中，处理了一个简支梁的问题并与传统 Galerkin 方法的处理结果进行了比较，验证了非线性 Galerkin 方法在动力学系统长期形态分析中的优越性.

本文以电场力激励下的共振器微梁为具体研究对象，建立用偏微分方程及其相应的边界条件表示的非线性动力系统. 在此基础上，分别采用两种 Galerkin 方法将系统降维成单自由度系统. 其中传统 Galerkin 方法仅截取第一阶模态，完全忽略高阶模态的影响，而非线性 Galerkin 方法考虑第二、第三阶模态对第一阶模态的贡献. 对两种方法得到的单自由度系统，采用多尺度法分别求解其在主共振情况下的动态响应. 最后，以直接求解三个自由度系统得到的响应作为标准，评价两种方法的优劣.

1 微梁振动微分方程及其离散化

1.1 电场激励共振器微梁的建模

考虑图1所示电场激励下共振器微梁，利用达朗伯原理可得系统的运动微分方程如下：

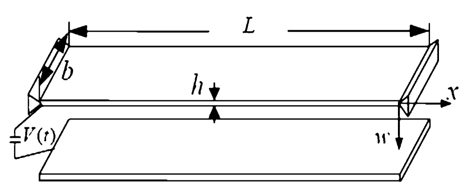

图1 电场激励微梁示意图

Fig. 1 A schematic of microbeam driven by electric force

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}+c\frac{\partial w}{\partial t}-[N+\frac{EA}{2L}\int_0^L(\frac{\partial w}{\partial x})^2\mathrm{d}x]\frac{\partial^2 w}{\partial x^2}=f_{jd}(x,t) \quad (1)$$

式中，$f_{jd}(x,t)=\frac{\varepsilon_0\varepsilon bV^2(t)}{2(d-w)^2}$，$V(t)=V_{dc}+V_{ac}\cos(\Omega t)$

各参数物理意义如下：

E 为材料的弹性模量；

$I=\frac{1}{12}bh^3$ 为绕横截面中性轴的惯性矩；

ρ 为材料的密度；

$A=bh$ 为横截面的面积；

c 为单位长度的粘性阻尼系数；

N 为初始轴力.

梁的两端固支，边界条件为

$$w(0,t)=w(L,t)=0,\frac{\partial w(0,t)}{\partial x}=\frac{\partial w(L,t)}{\partial x}=0 \quad (2)$$

令：$\bar{w}=\frac{w}{d}$，$\bar{x}=\frac{x}{l}$，$\bar{t}=\frac{t}{T}$，$T=\sqrt{\frac{\rho bhl^4}{EI}}=\sqrt{\frac{12\rho l^4}{Eh^2}}$，$\bar{\Omega}=\Omega T$，代入(1)式可得无量纲化方程：

$$\ddot{w}+\bar{c}\dot{w}+w_{xxxx}-[\bar{N}+\alpha_1\Gamma(w,w)]w_{xx}=\frac{\alpha_2V^2(t)}{(1-w)^2} \quad (3)$$

相应的边界条件为

$$w(0,t)=w(1,t)=0,\frac{\partial w(0,t)}{\partial x}=\frac{\partial w(1,t)}{\partial x}=0$$

其中

$$\begin{cases}\Gamma(f_1(x,t),f_2(x,t))=\int_0^1\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial x}\mathrm{d}x,\\ \bar{c}=\frac{Tc}{\rho A},\bar{N}=\frac{Nl^2}{EI},\alpha_1=6(\frac{d}{h})^2,\alpha_2=\frac{6\varepsilon\varepsilon_0 l^4}{Eh^3d^3}\end{cases}$$

1.2 振型计算

考虑无阻尼线性梁的振型 $\phi_i(x)$，即满足如下微分方程：

$$\phi_i^{iv}=N\phi_i''+\omega_i^2\phi_i \quad (4)$$

及相应的边界条件和归一化条件的解. 边界条件：

$$\phi_i(0)=\phi_i(1)=0,\phi_i'(0)=\phi_i'(1)=0 \quad (5)$$

正交性与归一化条件：

$$\int_0^1\phi_i\phi_j\mathrm{d}x=\delta_{ij} \quad (6)$$

设方程(4)的解为：

$$\phi_i=a_{i1}\cos\lambda_{i1}x+a_{i2}\sin\lambda_{i1}x+a_{i3}ch\lambda_{i2}x+a_{i4}sh\lambda_{i2}x \quad (7)$$

式中

$$\lambda_{i1}^2=\sqrt{(\frac{N}{2})^2+\omega_i^2}-\frac{N}{2},\lambda_{i2}^2=\sqrt{(\frac{N}{2})^2+\omega_i^2}+\frac{N}{2}$$

代入边界条件得

$$\begin{bmatrix}1&0&1&0\\0&\lambda_{i1}&0&\lambda_{i2}\\\cos\lambda_{i1}&\sin\lambda_{i1}&\mathrm{ch}\lambda_{i2}&\mathrm{sh}\lambda_{i2}\\-\lambda_{i1}\sin\lambda_{i1}&\lambda_{i1}\cos\lambda_{i1}&\lambda_{i2}\mathrm{sh}\lambda_{i2}&\lambda_{i2}\mathrm{ch}\lambda_{i2}\end{bmatrix}\begin{Bmatrix}a_{i1}\\a_{i2}\\a_{i3}\\a_{i4}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix}$$

方程具有非零解的条件是其系数行列式为零，即

$$N\sin\lambda_{i1}\mathrm{sh}\lambda_{i2}-2\omega_i\cos\lambda_{i1}\mathrm{ch}\lambda_{i2}+2\omega_i=0 \tag{8}$$

采用文[8]中提到的一组微梁参数(见表1),并设无量纲轴向力 $N=8.7$,则由式(8)可求得微梁的前三阶频率为:

$$\omega_1=24.6,\omega_2=64.8,\omega_3=124.4$$

表 1　微梁参数

Table 1　Parameters of the microbeam

L(μm)	h(μm)	b(μm)	d(μm)	E(Gpa)	ρ(kg/m^3)
510	1.5	100	1.18	149	2100

振型函数为

$$\phi_i=a_i\tilde{\phi}_i=a_i(-\cos\lambda_{i1}x+\mathrm{ch}\lambda_{i2}x)-a_i\frac{\lambda_{i1}\sin\lambda_{i1}+\lambda_{i2}\mathrm{sh}\lambda_{i2}}{\lambda_{i1}(\cos\lambda_{i1}-\mathrm{ch}\lambda_{i2})}(\sin\lambda_{i1}x-\frac{\lambda_{i1}}{\lambda_{i2}}\mathrm{sh}\lambda_{i2}x) \tag{9}$$

式中 $a_i=(\int_0^1\tilde{\phi}_i^2\mathrm{d}x)^{-1/2}$.

1.3　动力学方程离散化

令方程(3)解的形式为:

$$w(x,t)=\sum_{i=1}^{M}u_i(t)\phi_i(x)$$

将上式代入方程(3),两端分别乘以 $\phi_n(x)(1-w)^2$,考虑到式(4),得到离散化的动力学方程为:

$$\begin{aligned}\ddot{u}_n+c\dot{u}_n+\omega_n^2u_n=&2\sum_{i,j=1}^{M}u_i\ddot{u}_j\int_0^1\phi_i\phi_j\phi_n\mathrm{d}x-\sum_{i,j,k=1}^{M}u_iu_j\ddot{u}_k\int_0^1\phi_i\phi_j\phi_k\phi_n\mathrm{d}x+2\sum_{i,j=1}^{M}\omega_i^2u_iu_j\int_0^1\phi_i\phi_j\phi_n\mathrm{d}x-\\&\sum_{i,j,k=1}^{M}\omega_i^2u_iu_ju_k\int_0^1\phi_i\phi_j\phi_k\phi_n\mathrm{d}x+2c\sum_{i,j=1}^{M}u_i\dot{u}_j\int_0^1\phi_i\phi_j\phi_n\mathrm{d}x-\\&c\sum_{i,j,k=1}^{M}u_iu_j\dot{u}_k\int_0^1\phi_i\phi_j\phi_k\phi_n\mathrm{d}x+\alpha_1\sum_{i,j,k=1}^{M}u_iu_ju_k\Gamma(\phi_i,\phi_j)\times\\&\int_0^1\phi_k''\phi_n\mathrm{d}x-2\alpha_1\sum_{i,j,k,l=1}^{M}u_iu_ju_ku_l\Gamma(\phi_i,\phi_j)\int_0^1\phi_k\phi_l''\phi_n\mathrm{d}x+\\&\alpha_1\sum_{i,j,k,l,m=1}^{M}u_iu_ju_ku_lu_m\Gamma(\phi_i,\phi_j)\int_0^1\phi_k\phi_l\phi_m''\phi_n\mathrm{d}x+\\&\alpha_2[V_{dc}+V_{ac}\cos(\Omega t)]^2\int_0^1\phi_n\mathrm{d}x\end{aligned} \tag{10}$$

2　两种 Galerkin 方法降维

以下用传统 Galerkin 方法和非线性 Galerkin 方法分别将原系统降为 1 个自由度系统.

2.1　传统 Galerkin 方法降维

令 $n=1,M=1$ 得:

$$\begin{aligned}&(\ddot{u}_1+c\dot{u}_1+\omega_1^2u_1)(1-2u_1\int_0^1\phi_1^3\mathrm{d}x+u_1^2\int_0^1\phi_1^4\mathrm{d}x)=\\&\alpha_2[V_{dc}+V_{ac}\cos(\Omega t)]^2\int_0^1\phi_1\mathrm{d}x+\\&\alpha_1u_1^3\Gamma(\phi_1,\phi_1)\int_0^1\phi_1\phi_1''\mathrm{d}x-\alpha_1u_1^3\Gamma(\phi_1,\phi_1)\times\\&(2u_1\int_0^1\phi_1^2\phi_1''\mathrm{d}x-u_1^2\int_0^1\phi_1^3\phi_1''\mathrm{d}x)\end{aligned} \tag{11}$$

将位移 u_1 写成

$$u_1=u_{1s}+u_{1d} \tag{12}$$

u_{1s},u_{1d}分别为直流、交流电压产生的位移,则有

$$\begin{aligned}&\omega_1^2u_{1s}(1-2\beta_2u_{1s}+\beta_3u_{1s}^2)=\alpha_2\beta_1V_{dc}^2+\\&\alpha_1\beta_7u_{1s}^3(\beta_4-2\beta_5u_{1s}+\beta_bu_{1s}^2)\end{aligned} \tag{13}$$

式中

$$\beta_1=\int_0^1\phi_1\mathrm{d}x,\beta_2=\int_0^1\phi_1^3\mathrm{d}x,\beta_3=\int_0^1\phi_1^4\mathrm{d}x,$$

$$\beta_4=\int_0^1\phi_1\phi_1''\mathrm{d}x,\beta_5=\int_0^1\phi_1^2\phi_1''\mathrm{d}x,$$

$$\beta_6=\int_0^1\phi_1^3\phi_1''\mathrm{d}x,\beta_7=\int_0^1\phi_1'\phi_1'\mathrm{d}x$$

根据(11)、(12),并考虑到(13)得

$$\begin{aligned}&(\ddot{u}_{1d}+c\dot{u}_{1d}+\omega_1^2u_{1d})(\gamma_1+\gamma_2u_{1d}+\beta_3u_{1d}^2)+\omega_1^2(\gamma_2u_{1s}u_{1d}+\\&\beta_3u_{1d}^2)=\frac{1}{2}\alpha_2\beta_1[4V_{dc}V_{ac}\cos(\Omega t)+\\&V_{ac}^2\cos(2\Omega t)+V_{ac}^2]+\alpha_1\beta_7(\gamma_3u_{1d}+\gamma_4u_{1d}^2+\\&\gamma_5u_{1d}^3+\gamma_6u_{1d}^4+\beta_6u_{1d}^5)\end{aligned} \tag{14}$$

其中

$$\gamma_1=1-2\beta_2u_{1s}+\beta_3u_{1s}^2,\gamma_2=-2\beta_2+2\beta_3u_{1s},$$

$$\gamma_3=3\beta_4u_{1s}^2-8\beta_5u_{1s}^3+5\beta_6u_{1s}^4,$$

$$\gamma_4=3\beta_4u_{1s}-12\beta_5u_{1s}^2+10\beta_6u_{1s}^3,$$

$$\gamma_5=\beta_4-8\beta_5u_{1s}+10\beta_6u_{1s}^2,\gamma_6=-2\beta_5+5\beta_6u_{1s}$$

忽略高于 3 次的项,得到

$$\begin{aligned}&\ddot{u}_{1d}+c\dot{u}_{1d}+\omega_c^2u_{1d}=p_1u_{1d}\ddot{u}_{1d}+p_2u_{1d}^2\ddot{u}_{1d}+\\&p_1cu_{1d}\dot{u}_{1d}+p_2cu_{1d}^2\dot{u}_{1d}+p_3u_{1d}^2+p_4u_{1d}^3+\\&p_5V_{ac}\cos(\Omega t)\end{aligned} \tag{15}$$

式中

$$\omega_c^2=\frac{\gamma_1\omega_1^2+\gamma_2\omega_1^2u_{1s}-\alpha_1\beta_7\gamma_3}{\gamma_1}>0,\quad p_1=-\frac{\gamma_2}{\gamma_1},$$

$$p_2=-\frac{\beta_3}{\gamma_1},\quad p_3=-\frac{\gamma_2\omega_1^2+\beta_3\omega_1^2-\alpha_1\beta_7\gamma_4}{\gamma_1},$$

$$p_4=\frac{\beta_3\omega_1^2-\alpha_1\beta_7\gamma_5}{\gamma_1},\quad p_5=\frac{2\alpha_2\beta_1V_{dc}}{\gamma_1}$$

2.2　非线性 Galerkin 方法降维

在式(10)中取 $M=3$,当 n 分别取 1,2,3 时可以得到 3 个自由度的常微分方程组,为了便于计算,在此我们取一个反映高频部分与低频部分关系的近似惯性流形. 考虑到高频部分对时间的导数很

小,令:$\dot{u}_2=0,\ddot{u}_2=0,\dot{u}_3=0,\ddot{u}_3=0$. 于是,3 自由度系统进一步近似为 1 个常微分方程和 2 个非线性代数方程. 忽略高阶小量进一步化简得到形式更为简单的关系式:

$$(\ddot{u}_1+c\dot{u}_1+\omega_1^2u_1)(1-2u_1\int_0^1\phi_1^3\mathrm{d}x-2u_3\int_0^1\phi_1^2\phi_3\mathrm{d}x+u_1^2\int_0^1\phi_1^4\mathrm{d}x+2u_1u_3\int_0^1\phi_1^3\phi_3\mathrm{d}x)=2\omega_3^2u_1u_3\int_0^1\phi_1^2\phi_3\mathrm{d}x-\omega_3^2u_1^2u_3\int_0^1\phi_1^3\phi_3\mathrm{d}x+\alpha_1[\int_0^1\phi_1\phi_1''\mathrm{d}x(u_1^3\Gamma(\phi_1,\phi_1)+2u_1^2u_3\Gamma(\phi_1,\phi_3))+u_1^2u_3\Gamma(\phi_1,\phi_1)\int_0^1\phi_1\phi_3''\mathrm{d}x]+\alpha_2[V_{dc}+V_{ac}\cos(\Omega t)]^2\int_0^1\phi_1\mathrm{d}x \tag{16}$$

$$u_2=0 \tag{17}$$

$$u_3=(\omega_3^2+d_1\ddot{u}_1+d_2u_1\ddot{u}_1+d_3u_1+d_4u_1^2)^{-1}\times\{(\ddot{u}_1+c\dot{u}_1+\omega_1^2u_1)(d_5u_1-d_6u_1^2)+d_7u_1^3+\alpha_2[V_{dc}+V_{ac}\cos(\Omega t)]^2\int_0^1\phi_3\mathrm{d}x\} \tag{18}$$

式中

$$d_1=-2\int_0^1\phi_1\phi_3^2\mathrm{d}x,d_2=2\int_0^1\phi_1^2\phi_3^2\mathrm{d}x,$$

$$d_3=-2(\omega_1^2+\omega_3^2)\int_0^1\phi_1\phi_3^2\mathrm{d}x,$$

$$d_4=(2\omega_1^2+\omega_3^2)\int_0^1\phi_1^2\phi_3^2\mathrm{d}x-2\alpha_1\Gamma(\phi_1,\phi_3)\int_0^1\phi_1''\phi_3\mathrm{d}x-\alpha_1\Gamma(\phi_1,\phi_1)\int_0^1\phi_3''\phi_3\mathrm{d}x,$$

$$d_5=2\int_0^1\phi_1^2\phi_3\mathrm{d}x,\quad d_6=\int_0^1\phi_1^3\phi_3\mathrm{d}x,$$

$$d_7=\alpha_1\Gamma(\phi_1,\phi_1)\int_0^1\phi_1''\phi_3\mathrm{d}x.$$

由于$\frac{u_1}{\omega_3},\frac{\ddot{u}_1}{\omega_3}$为小量,将$(\omega_3^2+d_1\ddot{u}_1+d_2u_1\ddot{u}_1+d_3u_1+d_4u_1^2)^{-1}$展开,忽略三阶及以上小量,将 u_3 代入(16),化简得

$$(\ddot{u}_1+c\dot{u}_1+\omega_1^2u_1)(n_0+n_1u_1+n_2u_1^2)=m_1u_1^3+m_2cu_1^2\dot{u}_1+m_2u_1^2\ddot{u}_1+m_3u_1+m_4V_{dc}^2+2m_4V_{dc}V_{ac}\cos(\Omega t) \tag{19}$$

其中

$$n_0=1-\frac{2\alpha_2V_{dc}^2}{\omega_3^2}\int_0^1\phi_1^2\phi_3\mathrm{d}x\int_0^1\phi_3\mathrm{d}x,$$

$$n_1=-2\int_0^1\phi_1^3\mathrm{d}x,$$

$$n_2=-4(\int_0^1\phi_1^2\phi_3\mathrm{d}x)^2\frac{\omega_1^2}{\omega_3^2}+\int_0^1\phi_1^4\mathrm{d}x,$$

$$m_1=\alpha_1\Gamma(\phi_1\phi_1)\int_0^1\phi_1\phi_1''\mathrm{d}x+4\omega_1^2(\int_0^1\phi_1^2\phi_3\mathrm{d}x)^2,$$

$$m_2=4(\int_0^1\phi_1^2\phi_3\mathrm{d}x)^2,$$

$$m_3=2\alpha_2V_{dv}^2\int_0^1\phi_1^2\phi_3\mathrm{d}x\int_0^1\phi_3\mathrm{d}x,$$

$$m_4=\alpha\int_0^1\phi_1\mathrm{d}x$$

令 $u_1=u_{1s}+u_{1d}$,其中 u_{1s} 为直流电压产生的静位移,u_{1d} 为交流电压产生的位移,则

$$\omega_1^2u_{1s}(n_0+n_1u_{1s}+n_2u_{1s}^2)=m_1u_{1s}^3+m_3u_{1s}+m_4V_{dc}^2 \tag{20}$$

考虑到(20)式,(19)式可进一步写成

$$\ddot{u}_{1d}+c\dot{u}_{1d}+\omega_f^2u_{1d}=q_1u_{1d}\ddot{u}_{1d}+q_2u_{1d}^2\ddot{u}_{1d}+q_1cu_{1d}\dot{u}_{1d}+q_2cu_{1d}^2\dot{u}_{1d}+q_3u_{1d}^2+q_4u_{1d}^3+q_5V_{ac}\cos(\Omega t) \tag{21}$$

式中,

$$\omega_f^2=\frac{(n_0+2n_1u_{1s}+3n_2u_{1s}^2)\omega_1^2-3m_1u_{1s}^2-m_3}{n_0+n_1n_{1s}+(n_2-m_2)u_{1s}^2}>0,$$

$$q_1=\frac{2(m_2-n_2)u_{1s}-n_1}{n_0+n_1u_{1s}+(n_2-m_2)u_{1s}^2},q_2=\frac{m_2-n_2}{n_0+n_1u_{1s}+(n_2-m_2)u_{1s}^2},$$

$$q_3=\frac{3m_1u_{1s}-3\omega_1^2u_{1s}n_2-n_1\omega_1^2}{n_0+n_1u_{1s}+(n_2-m_2)u_{1s}^2},q_4=\frac{m_1-n_2\omega_1^2}{n_0+n_1u_{1s}+(n_2-m_2)u_{1s}^2},$$

$$q_5=\frac{2m_4V_{dc}}{n_0+n_1u_{1s}+(n_2-m_2)u_{1s}^2}$$

3 多尺度法求解稳态解

3.1 多尺度法求解

由上一节的推导可以看出,两种情况下所得方程(15)与方程(21)具有相同的形式,写成通式为

$$\ddot{v}+c\dot{v}+\omega_g^2v=\lambda_1v\ddot{v}+\lambda_2v^2\ddot{v}+\lambda_1cv\dot{v}+\lambda_2cv^2\dot{v}+\lambda_3v^2+\lambda_4v^3+\lambda_5V_{ac}\cos(\Omega t) \tag{22}$$

考虑主共振情况,取

$$\Omega=\omega_g+\varepsilon^2\sigma \tag{23}$$

$\sigma=O(1)$为调谐参数. 令:$c=\varepsilon^2\tilde{c},V_{ac}=\varepsilon^3\tilde{V}_{ac}$,设解的形式为

$$v=\varepsilon v_1+\varepsilon^2v_2+\varepsilon^3v_3+\cdots \tag{24}$$

其中,$T_n=\varepsilon^nt(n=0,1,\cdots)$. 代入方程(22)得

$$\varepsilon^1:D_0^2v_1+\omega_g^2v_1=0 \tag{25}$$

$$\varepsilon^2:D_0^2v_2+\omega_g^2v_2=-2D_0D_1v_1+\lambda_1v_1D_0^2v_1+\lambda_3v_1^2 \tag{26}$$

$$\varepsilon^3:D_0^2v_3+\omega_g^2v_3=-D_1^2v_1-2D_0D_2v_1+\lambda_1v_2D_0^2v_1+2\lambda_1v_1D_0D_1v_1+\lambda_2v_1^2D_0^2v_1-\tilde{c}D_0v_1+\lambda_1v_1D_0^2v_2-$$

$$2D_0D_1v_2 + 2\lambda_3 v_1 v_2 + \lambda_4 v_1^3 + \lambda_5 \tilde{V}_{ac}\cos(\Omega t) \tag{27}$$

由方程(25)求得

$$v_1 = A(T_1, T_2)\exp(i\omega_g T_0) + \bar{A}(T_1, T_2)\exp(-i\omega_g T_0) \tag{28}$$

将(28)式代入方程(26)得

$$D_0^2 v_2 + \omega_g^2 v_2 = (\lambda_3 - \lambda_1\omega_g^2)(A^2\exp(2i\omega_g T_0) + A\bar{A}) - 2i\omega D_1 A\exp(i\omega_g T_0) + cc \tag{29}$$

消去久期项得

$$D_1 A = 0 \tag{30}$$

即 A 与 T_1 无关. 于是由(29)式可解得

$$v_2 = -\frac{1}{3\omega_g^2}(\lambda_3 - \lambda_1\omega_g^2)A^2\exp(2i\omega_g T_0) + \frac{1}{\omega_g^2}(\lambda_3 - \lambda_1\omega_g^2)A\bar{A} + cc \tag{31}$$

将 v_1, v_2 代入方程(27)得

$$D_0^2 v_3 + \omega_g^2 v_3 = [\frac{1}{3}(\lambda_3 - \lambda_1\omega_g^2)(5\lambda_1 - \frac{2\lambda_3}{\omega_g^2}) + \lambda_4 - \lambda_2\omega_g^2]A^3\exp(3i\omega_g T_0) - [2i\omega_g D_2 A + \tilde{c}i\omega_g A - \frac{\lambda_5}{2}\tilde{V}_{ac}\exp(i\sigma T_2)]\exp(i\omega_g T_0) - [\frac{1}{3}(\lambda_3 - \lambda_1\omega_g^2)(\lambda_1 - \frac{10\lambda_3}{\omega_g^2}) + 3\lambda_2\omega_g^2 - 3\lambda_4]A^2\bar{A}\exp(i\omega_g T_0) + cc \tag{32}$$

上式消去久期项得

$$2i\omega_g D_2 A + \tilde{c}i\omega_g A - \frac{\lambda_5}{2}\tilde{V}_{ac}\exp(i\sigma T_1) + [\frac{1}{3}(\lambda_3 - \lambda_1\omega_g^2)(\lambda_1 - \frac{10\lambda_3}{\omega_g^2}) + 3\lambda_2\omega_g^2 - 3\lambda_4]A^2\bar{A} = 0 \tag{33}$$

令

$$\kappa_1 = \frac{1}{3}(\lambda_3 - \lambda_1\omega_g^2)(\lambda_1 - \frac{10\lambda_3}{\omega_g^2}) + 3\lambda_2\omega_g^2 - 3\lambda_4 \tag{34}$$

则(33)式简化成

$$2i\omega_g D_2 A + \tilde{c}i\omega_g A + \kappa_1 A^2\bar{A} - \frac{\lambda_5}{2}\tilde{V}_{ac}\exp(i\sigma T_2) = 0 \tag{35}$$

令

$$A = \frac{1}{2}a\exp(i\varphi) \tag{36}$$

其中 a, φ 是 T_2 的实函数. 将(36)式代入(35)式得

$$a' = -\frac{1}{2}\tilde{c}a + \frac{1}{2}\frac{\lambda_5\tilde{V}_{ac}}{\omega_g}\sin(\sigma T_2 - \varphi) \tag{37}$$

$$a\varphi' = \frac{1}{8}\frac{\kappa_1 a^3}{\omega_g} - \frac{1}{2}\frac{\lambda_5\tilde{V}_{ac}}{\omega_g}\cos(\sigma T_2 - \varphi) \tag{38}$$

a, φ 右上角的撇表示对 T_2 的导数. 取定常解,则得频幅特性方程:

$$[\frac{1}{4}\tilde{c} + (\sigma - \frac{1}{8}\frac{\kappa_1 a^2}{\omega_g})^2]a^2 = \frac{\lambda_5^2\tilde{V}_{ac}^2}{4\omega_g^2} \tag{39}$$

3.2　幅频特性分析

取 $V_{dc} = 2.0V, \tilde{c} = 2.46$. $\tilde{V}_{ac}$ 分别为 1.0,1.5,2.0 时绘出两种方法下系统的幅频特性曲线(图 2 和图 3). 从图中可以看出,在相同的电压幅值下,随着 Ω 的变化,振幅所能达到的最大值差不多大小. 但是,在相同参数情况下,两种方法得出的动态响应的频率却相差较大. 通过图 2 和图 3 的对比可以看出,用传统 Galerkin 方法所得幅频特性曲线各峰值点对应的频率与非线性 Galerkin 方法所求得的结果均有较大的偏离.

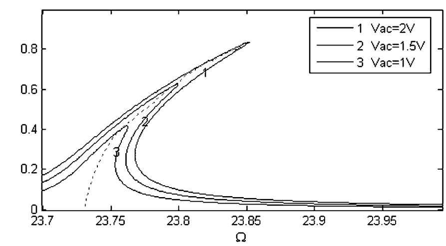

图 2　幅频特性曲线(传统 Galerkin 方法)

Fig. 2　Amplitude – frequency curve (Traditional Galerkin method)

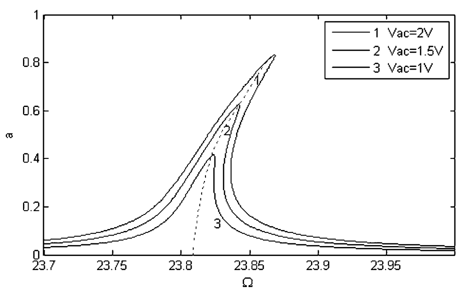

图 3　幅频特性曲线(非线性 Galerkin 方法)

Fig. 3　Amplitude – frequency curve (Nonlinear Galerkin method)

4　数值验证

截取前三阶模态,利用(10)式可得 3 自由度的非线性耦合微分方程. 取 $V_{dc} = 2V, V_{ac} = 0.002V, \Omega = 23.74$. 采用数值积分法求解,可得梁中点的位移时间历程(图 4). 再分别对采用传统 Galerkin 方法和非线性 Galerkin 方法降维得到的系统(15)和(21)进行数值求解,由 $w = u_1\phi_1$ 得出梁中点的位移时间历程(图 5 和图 6).

由图4,图5和图6可知,采用非线性Galerkin方法所得结果比采用传统Galerkin方法更能逼近直接求解前三阶模态的结果.

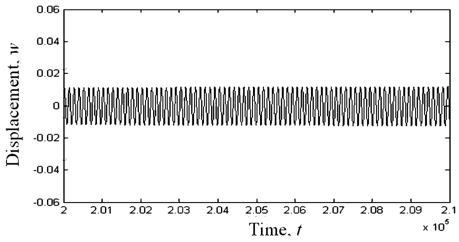

图4　$x=0.5$处的位移时间历程(直接积分求解)

Fig.4　Displacement time history at $x=0.5$

(Numerical integration method)

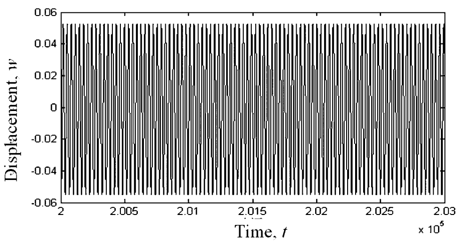

图5　$x=0.5$处的位移时间历程(传统Galerkin方法)

Fig.5　Displacement time history at $x=0.5$

(Traditional Galerkin method)

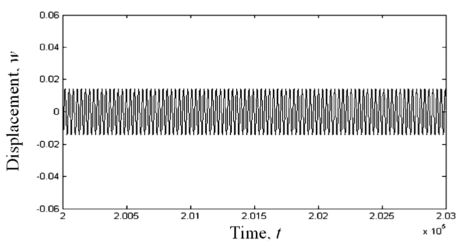

图6　$x=0.5$处的位移时间历程(非线性Galerkin方法)

Fig.6　Displacement time history at $x=0.5$

(Nonlinear Galerkin method)

5　结论

从本文计算结果可以看出,采用文中所选的参数,两种Galerkin方法降维后得到的系统幅频特性响应在定性方面一致,但量上差异很大.通过直接数值积分得出原系统响应作为参考,可以看出非线性Galerkin方法所得响应更为接近原系统真实响应,验证了非线性Galerkin方法处理非线性梁动力学问题的优越性.值得指出的是,由于微梁的非线性大振幅动态特性的特征,采用传统的Galerkin方法会带来较大的误差,尤其是响应的频率特征.因此,在微梁动力学响应分析中采用非线性Galerkin方法是必要的.此外,多场耦合是微机电系统的另一个特点[16],如何将非线性Galerkin方法应用于多场耦合作用下的微梁的非线性动力学分析是值得进一步研究的问题.

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DYNAMICAL RESPONSES OF MICROBEAMS USING NONLINEAR GALERKIN METHOD *

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Abstract　The nonlinear dynamic behaviour of a micro-resonator driven by electric force was investigated. The first three modes were used to get a model with single-degree-of-freedom in terms of the nonlinear Galerkin method. The multiple scales method was employed to obtain the dynamic responses of the reduced order model. The amplitude-frequency curve of stationary responses was depicted out and used to compare with the results obtained by the traditional Galerkin method where only the first mode is considered. Taking the dynamic responses obtained numerically from the model with three-degrees-of-freedom as a reference, it can be concluded that the nonlinear Galerkin method has higher accuracy in comparison with the traditional Galerkin method.

Key words　MEMS,　micro-beam,　inertial manifolds,　nonlinear Galerkin method

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