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Benney 方程的势对称和不变解*

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摘要　利用微分形式的吴方法计算了 Benney 方程的势对称及其不变解. 由于 Benney 方程中包含不稳定项和耗散项,使得直接求其不变解较为困难,利用吴 - 微分特征列算法可大大降低其中确定方程组的计算难度. 本文全面讨论了 Benney 方程不同系数情况下的对称,并且得到了新的势对称,同时利用这些对称求得了相应的不变解,这些解对进一步研究 Benney 方程所描述的物理现象具有广泛的应用价值. 文章表明,对于守恒形式的偏微分方程,可通过其辅助系统求得的势对称来构造其不变解.

关键词　Benney 方程,　势对称,　不变解

引 言

Benney 方程作为物理学中非常重要的一类非线性波动方程[1](KdV 方程)的扩展,其精确解的研究相对较少. Benney[2]首先研究了具有耗散和不稳定因素作用下的 KdV 方程,提出了包含耗散和不稳定效应的一维波产生模型 - Benney 方程,这个方程在等离子体物理、流体动力学及其他领域有着广泛的应用,所以寻找其精确解具有重要意义. 由于该方程包含不稳定项和耗散作用项,因而使求解更为困难.

为了寻求上述偏微分方程的精确解,我们采用 Bluman[3]等提出的势对称理论进行计算. 势对称理论是扩充方程(组)的最有效且简便的方法,对于来自物理、化学、生物等现代科学产生的数学模型,通过势对称理论可以得到更多非常有意义的解[4],这些解不同于由经典对称得到的解. 这里以两个自变量为例,假设给定方程可以写成守恒形式:

$$D_x f(x,t,u,\underset{1}{u},\cdots,\underset{k-1}{u}) - D_t g(x,t,u,\underset{1}{u},\cdots,\underset{k-1}{u}) = 0 \tag{1}$$

引入势函数 v,得方程(1)的辅助系统:

$$\begin{cases} \dfrac{\partial v}{\partial t} = f(x,t,u,\underset{1}{u},\cdots,\underset{k-1}{u}) \\ \dfrac{\partial v}{\partial x} = g(x,t,u,\underset{1}{u},\cdots,\underset{k-1}{u}) \end{cases} \tag{2}$$

设辅助系统(2)的古典对称向量为

$$X = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta(x,t,u,v)\frac{\partial}{\partial u} + \varphi(x,t,u,v)\frac{\partial}{\partial v} \tag{3}$$

如果 $\xi_v^2 + \tau_v^2 + \eta_v^2 \neq 0$,则称 X 是方程(1)的势对称向量.

在势对称的计算过程中,对于求解难度很大的确定方程组这个关键问题,我们反复使用了微分形式的吴方法[5],也就是将吴微分特征列集应用在对称计算上,即吴 - 微分特征列算法[6-7]. 该算法是由 Mathematica 符号语言编制而成用于计算微分多项式组的特征列集的程序包,并已经成功地应用到一类非线性电报方程的守恒律和非局部对称的分类问题当中[8-9].

本文主要利用微分形式的吴方法计算了 *Benney* 方程的势对称和不变解,考虑到方程中包含不同的系数,每个系数都有具体的物理含义,我们分八种情况进行了讨论. 这对进一步研究非线性 *Benney* 方程所描述的物理现象具有重要的理论意义.

1　Benney 方程的势对称和不变解

设 Benney 方程

$$u_t + uu_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0 \tag{4}$$

写成守恒形式

$$(u)_t + (\frac{1}{2}u^2 + \alpha u_x + \beta u_{xx} + \gamma u_{xxx})_x = 0 \tag{5}$$

根据守恒律,引入势变量 v,得相应的辅助系统:

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$$\begin{cases} v_x = -u \\ v_t = \frac{1}{2}u^2 + \alpha u_x + \beta u_{xx} + \gamma u_{xxx} \end{cases} \quad (6)$$

设方程组(6)对应的古典对称向量为:

$$X = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta(x,t,u,v)\frac{\partial}{\partial u} + \varphi(x,t,u,v)\frac{\partial}{\partial v} \quad (7)$$

下面对方程组(6)的系数 α,β,γ 分八种情况进行讨论:

1.1 $\alpha\neq 0,\beta\neq 0,\gamma\neq 0$

利用吴-微分特征列集程序包计算得确定方程组的特征列集所对应的方程组为[6-7]

$$\eta_v = \eta_u = \eta_t = \eta_x = 0, \xi_v = \xi_u = \xi_x, \eta - \xi_t = 0$$
$$\tau_v = \tau_u = \tau_t = \tau_x = 0, \varphi_v = \varphi_u = \varphi_t, \eta + \varphi_x = 0 \quad (8)$$

解方程组(8)得 $\xi = c_1 t + c_2, \tau = c_3, \eta = c_1, \varphi = -c_1 x + c_4$,其中 c_1,c_2,c_3,c_4 为任意常数,则对称向量为 $X_1 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} - x\frac{\partial}{\partial v}, X_2 = \frac{\partial}{\partial x}, X_3 = \frac{\partial}{\partial t}, X_4 = \frac{\partial}{\partial v}$ 显然无势对称. 我们可以计算对称向量 X_1 的不变解, X_1 的特征方程为 $\frac{dx}{t} = \frac{dt}{0} = \frac{du}{1} = \frac{dv}{-x}$,得不变量为 $t = k_1, u = \frac{x}{t} + k_2, v = -\frac{x^2}{2t} + k_3$. 设 $k_1 = \theta, k_2 = h_1(\theta)$, $k_3 = h_2(\theta)$,则 $u = \frac{x}{t} + h_1(t), v = -\frac{x^2}{2t} + h_2(t)$. 将 u, v 代入(6)得 $h_1 = 0, h_2 = \alpha\ln t + a$,故 X_1 对应的不变解为 $u = \frac{x}{t}, v = -\frac{x^2}{2t} + \alpha\ln t + a$,若将 $u = \frac{x}{t} + h_1(t)$ 代入(4)中,得 $h_1 = \frac{c}{t}$,则又可得一解 $u = \frac{x+b}{t}$.

1.2 $\alpha\neq 0,\beta\neq 0,\gamma = 0$

方程组(6)变为

$$\begin{cases} v_x = -u \\ v_t = \frac{1}{2}u^2 + \alpha u_x + \beta u_{xx} \end{cases} \quad (9)$$

利用吴-微分特征列集程序包计算得确定方程组的特征列集所对应的方程组为

$$\eta_v = \eta_u = \eta_t = \eta_x = 0, \xi_v = \xi_u = \xi_x = 0, \eta - \xi_t = 0$$
$$\tau_v = \tau_u = \tau_t = \tau_x = 0, \varphi_v = \varphi_u = \varphi_t = 0, \eta + \varphi_x = 0 \quad (10)$$

方程组(10)与方程组(8)相同,故其不变解与1.1类同.

1.3 $\alpha\neq 0,\beta = 0,\gamma = 0$

方程组(6)变为

$$\begin{cases} v_x = -u \\ v_t = \frac{1}{2}u^2 + \alpha u_x \end{cases} \quad (11)$$

利用吴-微分特征列集程序包计算得确定方程组的特征列集所对应的方程组为

$$\eta_t + u\eta_x = 0, \eta_{xx} = \eta_{tt} = \eta_{xt} = \eta_{uu} = 0,$$
$$\xi_{xx} = \xi_{tt} = \xi_u = \xi_v = 0, \tau_x = \tau_u = \tau_v = 0$$
$$\eta_t + u\xi_{xt} = 0, \eta - u\eta_u - \xi_t = 0,$$
$$\eta_x - \xi_{xt} = 0, \eta - 2\alpha\eta_v - \xi_t + u\xi_x = 0, \varphi_u = 0$$
$$\eta - \xi_t + u\xi_x - u\varphi_v = 0, \alpha\eta_x - \varphi_t = 0,$$
$$\xi_t + \varphi_x = 0, 2\xi_x - \tau_t = 0 \quad (12)$$

解方程组(12)得 $\xi = c_1 xt + c_2 x + c_3 t + c_4, \tau = c_1 t^2 + 2c_2 t + c_5, \eta = c_1(x - tu) - c_2 u + c_6 u e^{\frac{v}{2\alpha}} + c_3, \varphi = c_1(\alpha t - \frac{x^2}{2}) - c_3 x + 2\alpha c_6 e^{\frac{v}{2\alpha}} + c_7$,其中 c_1,c_2,c_3,c_4,c_5,c_6, c_7 为任意常数,则对称向量为

$$X_1 = xt\frac{\partial}{\partial x} + 2t^2\frac{\partial}{\partial t} + (x - tu)\frac{\partial}{\partial u} + (\alpha t - \frac{x^2}{2})\frac{\partial}{\partial v},$$
$$X_2 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}, X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} - x\frac{\partial}{\partial v},$$
$$X_4 = \frac{\partial}{\partial x}, X_5 = \frac{\partial}{\partial t}, X_6 = ue^{\frac{v}{2\alpha}}\frac{\partial}{\partial u} + 2\alpha e^{\frac{v}{2\alpha}}\frac{\partial}{\partial v},\ X_7 = \frac{\partial}{\partial v}$$

显然 X_6 为势对称向量,则我们可利用势对称向量 $X_4 + X_6$ 求得方程组的不变解.

$X_4 + X_6$ 的特征方程为 $\frac{dx}{1} = \frac{dx}{1} = \frac{dt}{0} = \frac{dv}{2\alpha e^{\frac{v}{2\alpha}}}$,不变量 $t = k_1, u = k_2 e^{\frac{v}{2\alpha}}, x + e^{-\frac{v}{2\alpha}} = k_3$,设 $k_1 = \theta, k_2 = h_1(\theta), k_3 = h_2(\theta)$,则 $u = h_1(t)e^{\frac{v}{2\alpha}}, v = -2\alpha\ln(h_2(t) - x)$,将 u,v 代入(11),得 $h_1 = -2\alpha, h_2 = a$,故 $X_4 + X_6$ 对应的不变解为 $u = \frac{-2\alpha}{a - x}, v = -2\alpha\ln(a - x)$.

1.4 $\alpha = 0,\beta\neq 0,\gamma = 0$

方程组(6)变为

$$\begin{cases} v_x = -u \\ v_t = \frac{1}{2}u^2 + \beta u_{xx} \end{cases} \quad (13)$$

利用吴-微分特征列集程序包计算得确定方程组的特征列集所对应的方程组为

$$\eta_v = \eta_t = \eta_x = 0, \xi_v = \xi_u = \xi_{xx} = 0, \eta_u + 2\xi_x = 0,$$
$$\eta - \xi_t + 2u\xi_x = 0, \tau_v = \tau_u = \tau_x = 0, 3\xi_x - \tau_t = 0,$$
$$\xi_x + \varphi_v = 0, \varphi_u = \varphi_t = 0, \eta + \varphi_x + 2u\xi_x = 0 \quad (14)$$

解方程组(14)得 $\xi = c_1 x + c_3 t + c_4, \tau = 3c_1 t + c_2, \eta =$

$-2c_1u+c_3$, $\varphi=-c_1v-c_3x+c_5$,其中 c_1,c_2,c_3,c_4,c_5 为任意常数,则对称向量为 $X_1=x\frac{\partial}{\partial x}+3t\frac{\partial}{\partial t}-2u\frac{\partial}{\partial u}-v\frac{\partial}{\partial v}$, $X_2=\frac{\partial}{\partial t}$, $X_3=t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}-x\frac{\partial}{\partial v}$, $X_4=\frac{\partial}{\partial x}$, $X_5=\frac{\partial}{\partial v}$,显然无势对称. 计算对称向量 X_4 的不变解,由 X_4 的特征方程$\frac{dx}{1}=\frac{dt}{0}=\frac{du}{0}=\frac{dv}{0}$,得不变量:$t=k_1$, $u=k_2$, $v=k_3$. 设 $k_1=\theta$, $k_2=h_1(\theta)$, $k_3=h_2(\theta)$,则 $u=h_1(t)$, $v=h_2(t)$. 将 u,v 代入(13),得 $h_1=0$, $h_2=a$,故 X_4 对应的不变解为 $u=0$, $v=a$.

1.5　$\alpha=0,\beta=0,\gamma\neq0$

方程组(6)变为

$$\begin{cases}v_x=-u\\ v_t=\frac{1}{2}u^2+\gamma u_{xxx}\end{cases}\tag{15}$$

利用吴-微分特征列集程序包计算得确定方程组的特征列集所对应的方程组为

$$\begin{gathered}\eta_v=\eta_t=\eta_x=0,\xi_v=\xi_u=\xi_{xx}=0,\eta_u+3\xi_x=0,\\ \eta-\xi_t+3u\xi_x=0,\tau_v=\tau_u=\tau_x=0,4\xi_x-\tau_t=0,\\ 2\xi_x+\varphi_v=0,\varphi_u=\varphi_t=0,\eta+\varphi_x+2u\xi_x=0\end{gathered}\tag{16}$$

解方程组(16)得 $\xi=c_1x+c_3t+c_4$, $\tau=4c_1t+c_2$, $\eta=-3c_1u+c_3$, $\varphi=-2c_1v-c_3x+c_5$,其中 c_1,c_2,c_3,c_4,c_5 为任意常数,则对称向量为 $X_1=x\frac{\partial}{\partial x}+4t\frac{\partial}{\partial t}-3u\frac{\partial}{\partial u}-2v\frac{\partial}{\partial v}$, $X_2=\frac{\partial}{\partial t}$, $X_3=t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}-x\frac{\partial}{\partial v}$, $X_4=\frac{\partial}{\partial x}$, $X_5=\frac{\partial}{\partial v}$,显然无势对称. 此种情况类同于 1.4.

1.6　$\alpha=0,\beta\neq0,\gamma\neq0$

方程组(6)变为

$$\begin{cases}v_x=-u\\ v_t=\frac{1}{2}u^2+\beta u_{xx}+\gamma u_{xxx}\end{cases}\tag{17}$$

利用吴-微分特征列集程序计算得确定方程组的特征列集所对应的方程组为

$$\begin{gathered}\eta_v=\eta_u=\eta_t=\eta_x=0,\eta-\xi_t=0,\xi_v=\xi_u=\xi_x=0,\\ \tau_x=\tau_t=\tau_u=\tau_v=0,\varphi_t=\varphi_u=\varphi_v=0,\eta+\varphi_x=0\end{gathered}\tag{18}$$

解方程组(18)得 $\xi=c_1t+c_2$, $\tau=c_3$, $\eta=c_1$, $\varphi=-c_1x+c_4$,其中 c_1,c_2,c_3,c_4 为任意常数,则对称向量为 $X_1=t\frac{\partial}{\partial x}+\frac{\partial}{\partial u}-x\frac{\partial}{\partial v}$, $X_2=\frac{\partial}{\partial x}$, $X_3=\frac{\partial}{\partial t}$, $X_4=\frac{\partial}{\partial v}$,显然无势对称. 下面计算对称向量 X_1 的不变解,X_1 的特征方程为$\frac{dx}{t}=\frac{dt}{0}=\frac{du}{1}=\frac{dv}{-x}$,得不变量 $t=k_1$, $u=\frac{x}{t}+k_2$, $v=-\frac{x^2}{2t}+k_3$,设 $k_1=\theta$, $k_2=h_1(\theta)$, $k_3=h_2(\theta)$,则 $u=\frac{u}{t}+h_1(t)$, $v=-\frac{x^2}{2t}+h_2(t)$,将 u,v 代入(17),得 $h_1=0$, $h_2=a$,故 X_1 对应的不变解为 $u=\frac{x}{t}$, $v=-\frac{x^2}{2t}+a$.

1.7　$\alpha\neq0,\beta=0,\gamma\neq0$

方程组(6)变为

$$\begin{cases}v_x=-u\\ v_t=\frac{1}{2}u^2+\alpha u_x+\gamma u_{xxx}\end{cases}\tag{19}$$

利用吴-微分特征列集程序包计算得确定方程组的特征列集所对应的方程组为

$$\begin{gathered}\eta_v=\eta_u=\eta_t=\eta_x=0,\eta-\xi_t=0,\xi_v=\xi_u=\xi_x=0,\\ \tau_x=\tau_t=\tau_u=\tau_v=0,\varphi_t=\varphi_u=\varphi_v=0,\eta+\varphi_x=0\end{gathered}\tag{20}$$

方程组(20)与方程组(18)相同,故其不变解与 1.6 类同.

1.8　$\alpha=0,\beta=0,\gamma=0$

方程组(6)变为

$$\begin{cases}v_x=-u\\ v_t=\frac{1}{2}u^2\end{cases}\tag{21}$$

利用吴-微分特征列集程序计算得确定方程组的特征列集所对应的方程组为:

$$\begin{gathered}u^2\eta_v-2\eta_t-2u\eta_x=0,\ -2u\xi_u+u^2\tau_u-2\varphi_u=0\\ 2\eta+u^2\xi_v-2\xi_t-2u\xi_x-u^3\tau_v+2u\tau_t+2u^2\tau_x=0\\ 2\eta_u+u\eta_{uu}-2\xi_{tu}-2\xi_x-u\xi_{xu}+2\tau_t+2u\tau_{tu}+3u\tau_x+u^2\tau_{xu}=0\\ 2\eta_t+2u\eta_{tu}+u^2\eta_{xu}-4\xi_{tt}-4u\xi_{xt}-u^2\xi_{xx}+4u\tau_{tt}+4u^2\tau_{xt}+u^3\tau_{xx}=0\\ 6\eta+2u\eta_u-u^2\xi_v-6\xi_t-4u\xi_x+6u\tau_t+4u^2\tau_x-2u\varphi_v=0\\ u^2\eta_u-u^2\xi_x+u^2\tau_t+u^3\tau_x+2\varphi_t=0\\ 2\eta+2u\eta_u-4\xi_t-4u\xi_x+4u\tau_t+3u^2\tau_x-2\varphi_x=0\end{gathered}\tag{22}$$

方程组(22)求解困难,有待于进一步研究.

2　结论

本文主要对 Benney 方程不同系数的各种情况

的对称进行了讨论,并用之求得其对应的不变解.结果表明:可写成守恒形式的偏微分方程并不一定都允许势对称,在本文中只有第三种和第八种情况存在势对称.在什么情况下偏微分方程允许势对称,这是有待于研究的问题.

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POTENTIAL SYMMETRIES AND INVARIANT SOLUTIONS OF BENNEY EQUATION *

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Abstract The potential symmetries and invariant solutions of Benney equation were calculated by differential form Wu's method. It was difficult to obtain the invariant solutions of Benney equation directly due to the unstable term and dissipative term involved in Benney equation. The calculating difficulty of determining equations can be greatly decreased by using Wuwen-tsun-Differential Characteristic Algorithm. In this paper, the symmetries of Benney equation with different coefficients were fully discussed, and new potential symmetries were obtained. Furthermore, the corresponding invariant solutions can be obtained by using the above symmetries. The solutions had extensive application value for further researching the physical phenomena described by Benney equation. The research indicates that the invariant solutions of conserved partial differential equations can be constructed by using the potential symmetries, which are derived from their auxiliary system.

Key words Benney equation, potential symmetries, invariant solutions

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