非线性损伤粘弹性薄板准静态力学行为分析*

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摘要　基于损伤粘弹性材料的一种卷积型本构关系和大挠度薄板的 von Kármán 假设,给出了损伤粘弹性薄板准静态问题的数学模型,其控制方程为一组非线性积分－偏微分型方程. 采用 Galerkin 截断技术,将原积分－偏微分系统化为积分系统. 然后采用四阶的 Runge－Kutta 法在数值上得到了损伤粘弹性薄板的准静态问题的解.

关键词　损伤粘弹性薄板,　von Kármán 假设,　Galerkin 方法,　准静态问题,　积分－偏微分方程

引 言

随着高分子聚合物等粘弹性材料在国防和民用工业中的广泛使用,粘弹性力学已成为国内外研究的热点之一. 如今,粘弹性力学已成为固体力学的基础内容,成为现代连续介质力学的一个重要组成部分. 由于粘弹性结构的大变形和/或非线性本构关系将导致其数学模型中出现非线性项而成为非线性系统,对这类问题很难求得精确解,一般只能利用各种数值的方法(如有限元法,边界元法)求得数值解[1,2]. 张能辉等[3]基于线性粘弹性材料的 Boltzmann 叠加原理和大挠度薄板的 von Kármán 假设,给出了粘弹性薄板准静态问题分析中的一种时域算法,即在空域上采有 Galerkin 方法,在时域上采用一种新的算法,得到了横向阶跃载荷作用下的粘弹性简支板的准静态响应.

本文首先利用线性损伤粘弹性理论和薄板大挠度理论,建立损伤粘弹性准静态分析的非线性数学模型,其控制方程是积分－偏微分型方程. 对这类问题的求解,可在空间上应用 Galerkin 截断技术,把积分－偏微分方程化为积分－常微分方程. 最后对简化的非线性动力系统进行研究,在数值上得到了损伤粘弹性薄板的准静态响应.

1　损伤粘弹性薄板准静态分析的数学模型

假定损伤粘弹性薄板的厚度为 h,板承受横向谐载荷 $q(x,y,t)$. 设板沿三个坐标轴 x、y、z 方向的位移分别为 u、v 和 w,由 *von Kármán* 理论,薄板的应变可以分解为两部分,即平均应变和弯曲应变

$$\varepsilon_x=\varepsilon_x^0-z\frac{\partial^2 w}{\partial x^2},\varepsilon_y=\varepsilon_y^0-z\frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy}=\gamma_{xy}^0-2z\frac{\partial^2 w}{\partial x\partial y}\tag{1}$$

其中 ε_x^0、ε_y^0 和 γ_{xy}^0 是平均应变,且有

$$\varepsilon_x^0=\frac{\partial u}{\partial x}+\frac{1}{2}(\frac{\partial w}{\partial x})^2,\varepsilon_y^0=\frac{\partial v}{\partial y}+\frac{1}{2}(\frac{\partial w}{\partial y})^2,$$

$$\gamma_{xy}^0=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\tag{2}$$

为了便于分析,可以假设损伤增量与坐标成正比,即有

$$D(x_i,t)-D^0(x_i)=D(x_\alpha,t)z\tag{3}$$

其中 D^0 是初始损伤. 这里和今后,凡下标为希腊字母者,取值为 x 和 y. 根据损伤粘弹性力学的动力学方程[4,5],容易得到线性各向同性损伤粘弹性板的应力应变关系

$$\sigma_x=(C_1+C_2)\otimes(\varepsilon_x^0-z\frac{\partial^2 w}{\partial x^2})+C_2\otimes(\varepsilon_y^0-z\frac{\partial^2 w}{\partial y^2})-\beta Dz$$

$$\sigma_y=C_2\otimes(\varepsilon_x^0-z\frac{\partial^2 w}{\partial x^2})+(C_1+C_2)\otimes(\varepsilon_y^0-z\frac{\partial^2 w}{\partial y^2})-\beta Dz$$

$$\tau_{xy}=\frac{1}{2}C_1\otimes(\gamma_{xy}^0-2z\frac{\partial^2 w}{\partial x\partial y})\tag{4}$$

并且内力分量有表示式

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$$N_x = \int_{-h/2}^{h/2} \boldsymbol{\sigma}_x \mathrm{d}z = h\{(C_1 + C_2) \otimes [\frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^2] + C_2 \otimes [\frac{\partial \upsilon}{\partial y} + \frac{1}{2}(\frac{\partial w}{\partial y})^2]\}$$

$$N_y = \int_{-h/2}^{h/2} \boldsymbol{\sigma}_y \mathrm{d}z = h\{C_2 \otimes [\frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^2] + (C_1 + C_2) \otimes [\frac{\partial \upsilon}{\partial y} + \frac{1}{2}(\frac{\partial w}{\partial y})^2]\}$$

$$N_{xy} = \int_{-h/2}^{h/2} \boldsymbol{\tau}_{xy} \mathrm{d}z = \frac{h}{2} C_1 \otimes (\frac{\partial u}{\partial y} + \frac{\partial \upsilon}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y})$$

$$M_x = \int_{-h/2}^{h/2} \boldsymbol{\sigma}_x \mathrm{d}z = -\frac{h^3}{12}[(C_1 + C_2) \otimes \frac{\partial^2 w}{\partial x^2} + C_2 \otimes \frac{\partial^2 w}{\partial y^2} - \beta D]$$

$$M_y = \int_{-h/2}^{h/2} \boldsymbol{\sigma}_y \mathrm{d}z = -\frac{h^3}{12}[C_2 \otimes \frac{\partial^2 w}{\partial x^2} + (C_1 + C_2) \otimes \frac{\partial^2 w}{\partial y^2} - \beta D]$$

$$M_{xy} = \int_{-h/2}^{h/2} \boldsymbol{\tau}_{xy} \mathrm{d}z = -\frac{h^3}{12} C_1 \otimes \frac{\partial^2 w}{\partial x \partial y} \tag{5}$$

式中,N_x、N_y、N_{xy}为板的薄膜内力,Q_x、Q_y、M_x、M_y、M_{xy}分别为板的横向剪力、弯矩和扭矩.

假设板的质量密度为ρ,并忽略惯性力,则由薄板微元体的平衡,可得到用三个位移分量,即u,υ,w和损伤D表示的运动微分方程组.

$$(C_1+C_2)\otimes(u_{,xx}+w_{,x}w_{,xx})+C_2\otimes(\upsilon_{yx}+w_{,y}w_{,yx})+\frac{1}{2}C_1\otimes(u_{,yy}+\upsilon_{,xy}+w_{,xy}w_{,y}+w_{,x}w_{,yy})=0$$

$$C_2\otimes(u_{,xy}+w_{,x}w_{,xy})+(C_1+C_2)\otimes(\upsilon_{yy}+w_{,y}w_{,yy})+\frac{1}{2}C_1\otimes(u_{,yx}+\upsilon_{,xx}+w_{,xx}w_{,y}+w_{,x}w_{,yx})=0$$

$$-\frac{h^3}{12}[(C_1+C_2)\otimes\nabla^4 w-\beta\nabla^2 D]+h[(C_1+C_2)\otimes(u_{,x}+w_{,x}^2/2)+C_2\otimes(\upsilon_{,y}+w_{,y}^2/2)]w_{,xx}+h[C_2\otimes(u_{,x}+w_{,x}^2/2)+(C_1+C_2)\otimes(\upsilon_{,y}+w_{,y}^2/2)]w_{,yy}+hC_1\otimes(u_{,y}+\upsilon_{,x}+w_{,x}w_{,y})w_{,xy}=-q \tag{6}$$

损伤运动微分方程[4]为

$$\alpha D_{,\alpha\alpha}-\xi D-\beta(w_{,xx}+w_{,yy})=0 \tag{7}$$

其中α,ξ,β为材料的特征常数.假定板为四边简支矩形板,则有下面的边界条件

$$u^0=\upsilon^0=w=M_x=0(x=0,a)$$

$$u^0=\upsilon^0=w=M_y=0(y=0,a)$$

$$\underset{\sim}{n}\cdot\underset{\sim}{\nabla}D=0 \tag{8}$$

方程(6)-(8)是控制损伤粘弹性薄板准静态行为的控制方程,是一组非线性的积分-偏微分方程组.初始条件为

$$u_i(x_i,0)=u_i^0(x_i),\dot{u}_i(x_i,0)=\dot{u}_i^0(x_i)$$

$$D(x_i,0)=D^0(x_i),\dot{D}(x_i,0)=\dot{D}^0(x_i)(t=0) \tag{9}$$

2 数值求解

为了得到这个边值问题的解,我们采用Galerkin平均化的方法将问题进行简化.根据边界条件(8),方程(6)-(7)的解可取为如下的形式

$$u(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\bar{u}(t)_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

$$\upsilon(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\bar{\upsilon}(t)_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

$$w(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\bar{w}(t)_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

$$D(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\bar{D}(t)_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \tag{10}$$

假定横向载荷q为

$$q(x,y,t)=q_0H(t)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \tag{11}$$

这里$H(t)$是Heaviside函数.将(10)、(11)代入到(6)中,分别令$n=1,1$、$m=1,3$,可得二阶Galerkin截断模型如下

$$A_1\otimes\bar{u}_{11}=0,A_2\otimes\bar{u}_{13}=0,$$

$$B_1\otimes\bar{\upsilon}_{11}=0,B_2\otimes\bar{\upsilon}_{13}=0$$

$$A_3\otimes\bar{w}_{11}+\bar{w}_{11}(A_4\otimes\bar{w}_{11}^2)+\bar{w}_{11}[A_5\otimes(\bar{w}_{11}\bar{w}_{13})]+\bar{w}_{11}(A_6\otimes\bar{w}_{13}^2)+\bar{w}_{13}(A_7\otimes\bar{w}_{11}^2)+\bar{w}_{13}[A_8\otimes(\bar{w}_{11}\bar{w}_{13})]+A_9D_{11}+\frac{ab}{4}q_0=0$$

$$B_3\otimes\bar{w}_{13}+\bar{w}_{11}(B_4\otimes\bar{w}_{11}^2)+\bar{w}_{11}[B_5\otimes(\bar{w}_{11}\bar{w}_{13})]+\bar{w}_{13}(B_6\otimes\bar{w}_{11}^2)+\bar{w}_{13}(B_7\otimes\bar{w}_{13}^2)+B_8D_{13}=0$$

$$A_{10}D_{11}=A_{11}\bar{w}_{11},B_9D_{13}=B_{10}\bar{w}_{13} \tag{12}$$

限于篇幅,(12)中的系数未给出.在(12)中消去D_{11}、D_{13},可得

$$A_3\otimes\bar{w}_{11}+\bar{w}_{11}(A_4\otimes\bar{w}_{11}^2)+\bar{w}_{11}[A_5\otimes(\bar{w}_{11}\bar{w}_{13})]+\bar{w}_{11}(A_6\otimes\bar{w}_{13}^2)+\bar{w}_{13}(A_7\otimes\bar{w}_{11}^2)+\bar{w}_{13}[A_8\otimes(\bar{w}_{11}\bar{w}_{13})]+A_9A_{11}\bar{w}_{11}/A_{10}+\frac{ab}{4}q_0=0$$

$$B_3 \otimes \bar{w}_{13} + \bar{w}_{11}(B_4 \otimes \bar{w}_{11}^2) + \bar{w}_{11}[B_5 \otimes (\bar{w}_{11}\bar{w}_{13})] + \bar{w}_{13}(B_6 \otimes \bar{w}_{11}^2) + \bar{w}_{13}(B_7 \otimes \bar{w}_{13}^2) + B_8 D_{13}\bar{w}_{13}/B_9 = 0 \quad (13)$$

方程(13)的系数由于篇幅所限,未给出. 方程(13)是一个积分方程,可知其解析表达式是相当困难的,下面我们在数值上进行求解. 设板的尺寸如下:$a = b = 10\text{m}$,高 $h = 1\text{m}$,材料参数对于标准线性固体,材料的松弛函数 $c_1(t)$ 满足下面的条件:

$$c_1(t) = 0.5 + 0.5\exp(-0.06t)$$

这里,已引入 $c_1(\tau) = C_1(\tau)/C_1(0)$,同时取材料的常数为 $\alpha = 8\text{Gpam}^2$,$\xi = 12\text{Gpa}$,$\beta = 10\text{Gpa}$,$C_1(0) = 15\text{Gpa}$,材料的泊松比 $\mu = 0.23$,即 $C_2(t)/C_1(t) = \mu/(1-\mu) = 1/3$,引入无量纲化参量,并作如下的变量变换:

$$w = \bar{w}/h,\ y_0 = t,\ y_1 = w_{11},\ y_2 = w_{13},$$

$$y_3 = \int_0^t \dot{c}_1(t-\tau)w_{11}(\tau)\mathrm{d}\tau,\ y_4 = \int_0^t \dot{c}_1(t-\tau)w_{13}(\tau)\mathrm{d}\tau,$$

$$y_5 = \int_0^t \dot{c}_1(t-\tau)w_{11}^2(\tau)\mathrm{d}\tau,\ y_6 = \int_0^t \dot{c}_1(t-\tau)w_{13}^2(\tau)\mathrm{d}\tau,$$

$$y_7 = \int_0^t \dot{c}_1(t-\tau)w_{11}(\tau)w_{13}(\tau)\mathrm{d}\tau \quad (14)$$

显见方程(13)是一个非线性自治微分方程,其相应的自治系统为

$$\dot{Y} = F(Y) \quad (15)$$

其中

$$Y = \{y_0, y_1, y_2, \cdots, y_7\}^T,\ F = \{F_0, F_1, \cdots, F_7\}^T$$

$$F_0 = 1,$$

$$F_1 = -(0.00411y_1^3 + 1.18717y_2 + 0.282183y_1^2y_2 + 1.089y_2^3 + 2.37434y_4 + 0.00822y_1y_5 + 0.547926y_2y_5 + 2.178y_2y_6 + 0.01644y_1y_7)/(-0.411y_1^2\text{-}18.8122y_1y_2\text{-}0.137y_5\text{-}0.274y_7) + \{(-49.7724\text{-}9.4061y_1^2\text{-}108.9y_2^2\text{-}9.1321y_5\text{-}36.3y_6)[-(-2.18078\text{-}2.6025y_1^2\text{-}2.2834y_1y_2\text{-}7.945y_2^2\text{-}0.8675y_5\text{-}10.228y_6 + 3.196y_7)(0.00411y_1^3 + 1.18717y_2 + 0.282183y_1^2y_2 + 1.089y_2^3 + 2.37434y_4 + 0.00822y_1y_5 + 0.547926y_2y_5 + 2.178y_2y_6 + 0.01644y_1y_7) + (-0.411y_1^2\text{-}18.8122y_1y_2\text{-}0.137y_5\text{-}0.274y_7)(0.04749y_1 + 0.026025y_1^5 + 0.034251y_1^2y_2 + 0.23835y_1y_2^2 + 0.09498y_3 + 0.05205y_1y_5 + 0.260262y_2y_5 + 0.61368y_1y_6\text{-}0.19176y_1y_7\text{-}0.13698y_2y_7)]\}/\{(-0.411y_1^2\text{-}18.8122y_1y_2\text{-}0.137y_5\text{-}0.274y_7)[(-0.411y_1^2\text{-}18.8122y_1y_2\text{-}0.137y_5\text{-}0.274y_7)(-1.1417y_1^2\text{-}15.89y_1y_2\text{-}4.3377y_5 + 2.283y_7)\text{-}(-49.7724\text{-}9.4061y_1^2\text{-}108.9y_2^2\text{-}9.1321y_5\text{-}36.3y_6)(-2.18078\text{-}2.6025y_1^2\text{-}2.2834y_1y_2\text{-}7.945y_2^2\text{-}0.8675y_5\text{-}10.228y_6 + 3.196y_7)]\}$$

$$F_2 = -\{[-(-2.18078\text{-}2.6025y_1^2\text{-}2.2834y_1y_2\text{-}7.945y_2^2\text{-}0.8675y_5\text{-}10.228y_6 + 3.196y_7)(0.00411y_1^2 + 1.18717y_2 + 0.282183y_1^2y_2 + 1.089y_2^3 + 2.37434y_4 + 0.00822y_1y_5 + 0.547926y_2y_5 + 2.178y_2y_6 + 0.01644y_1y_7) + (-0.411y_1^2\text{-}18.8122y1y_2\text{-}0.137y_5\text{-}0.274y_7)(0.04749y_1 + 0.026025y_1^3 + 0.034251y_1^2y_2 + 0.23835y_1y_2^2 + 0.09498y_3 + 0.05205y_1y_5 + 0.260262y_2y_5 + 0.61368y_1y_6\text{-}0.19176y_1y_7\text{-}0.13698y_2y_7)]\}/[(-0.411y_1^2\text{-}18.8122y_1y_2\text{-}0.137y_5\text{-}0.274y_7)(-1.1417y_1^2\text{-}15.89y_1y_2\text{-}4.3377y_5 + 2.283y_7)\text{-}(-49.7724\text{-}9.4061y_1^2\text{-}108.9y_2^2\text{-}9.1321y_5\text{-}36.3y_6)(-2.18078\text{-}2.6025y_1^2\text{-}2.2834y_1y_2\text{-}7.945y_2^2\text{-}0.8675y_5\text{-}10.228y_6 + 3.196y_7)],$$

$$F_3 = -\alpha(c_1y_1 + y_3),\ F_4 = -\alpha(c_1y_2 + y_4),$$

$$F_5 = -\alpha(c_1y_1^2 + y_5),\ F_6 = -\alpha(c_1y_2^2 + y_6),$$

$$F_7 = -\alpha(c_1y_1y_2 + y_7) \quad (16)$$

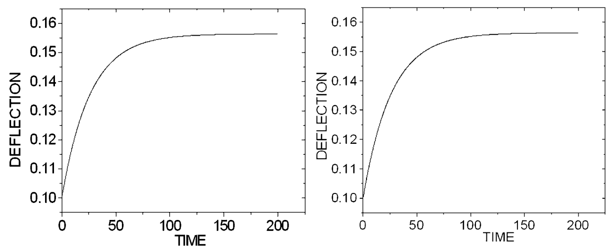

图 1　$x = a/2, y = b/2$ 处挠度随时间的变化曲线

Fig. 1　Time - path curves of deflection at $x = a/2, y = b/2$

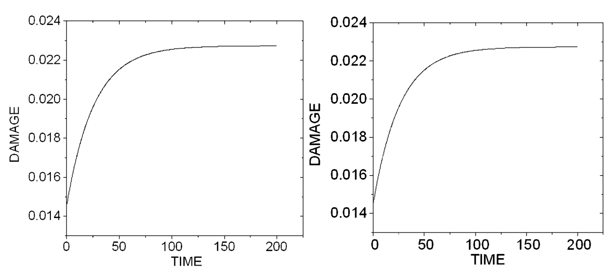

图 2　$x = a/2, y = b/2$ 处损伤随时间的变化曲线

Fig. 2　Time - path curves of damage at $x = a/2, y = b/2$

方程(13)的初值,从初始条件(9),可写为

$$\{y_1(0), y_2(0), \cdots, y_7(0)\} =$$

$$\{w_1^0, w_3^0, 0, 0, 0, 0, 0\} \quad (17)$$

应用四阶的龙格－库塔(Runge－Kutta)法,求解上面的方程组,可以得到 $x = a/2, y = b/2$ 处挠度和损伤随时间的变化曲线. 它们分别由图 1 和 2 表示.

3 结论

图 1 和图 2 分别给出了损伤粘弹性方板在阶跃载荷作用下中心处的挠度和损伤的时程曲线,由图可知,随着时间的增大,挠度和损伤都渐近达到最大值而趋于稳定. 说明材料的稳态响应为弹性响应. 从图 1 和 2 可以看出,一阶截断系统和二阶截断系统的动力学性质定性基本一致.

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QUASI-STATIC BEHAVIORS OF NONLINEAR VISCOELATIC THIN PLATES WITH DAMAGE*

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Abstract Based on convolution-type constitutive equations for linear viscoelastic materials with damage and von Kármán's hypotheses, the equations governing quasi-static behavior of thin plates with damage were derived under large deformation. It can be seen that the derived equations are a set of nonlinear integro- partial-differential equations. In order to analyze the equations, the Galerkin method was firstly applied to simplify the set of equations into a set of integral equations.

Key words thin plates with damage, von Kármán hypotheses, Galerkin methods, quasi-static problems, integro-partial-differential equations

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