第 6 卷第 2 期 2008 年 6 月
1672-6553/2008/06(2)/146-4
动 力 学 与 控 制 学 报
JOURNAL OF DYNAMICS AND CONTROL
Vol. 6 No. 2
Jun. 2008

用瑞利 - 里兹法求解失重液滴的自由晃动

夏恒新[1]　宝音贺西[2]　郑亚[1]
(1. 南京理工大学机械学院,南京　210000)(2. 清华大学航天航空学院,北京　100084)

摘要　失重作用可能在空间中构造理想的球形液滴,它在空间流体科学、空间材料合成等中均有应用. 在轨操纵中共振可能引起液滴的变形而影响实验质量,了解液滴晃动特性对空间实验的设计和避免与支撑结构的共振都有帮助. 用瑞利 - 里兹法研究了失重液滴的自由晃动问题,给出了液滴自由晃动的频率和模态函数. 可利用表面上的动力学条件研究自由液滴的晃动特性,但由于耦合系统复杂,往往用能量法加以研究. 该方法作为一种能量法,可为进一步研究失重环境中的液滴和支撑结构的耦合振动问题提供可行的途径.

关键词　瑞利 - 里兹法,　晃动,　自由液滴

引 言

随着空间技术的发展,在地球常重环境下不可实现的新材料合成、新型制造等技术在空间失重环境中成为可能[1,2]. 其中,因为重力的存在,在地球环境下液体物质形成理想的球形很困难,而在失重的空间环境中在表面张力的作用下容易形成理想的球体. 但由于周围力学环境或液滴支撑条件的特性可能引起变形或影响其形状,尤其共振可能是引起形状变形的重要因素. 本文利用瑞利 - 里兹法研究失重环境中球形液滴的自由振动特性. Bauer[3] 曾研究过球形容器中的液体和环绕球形物体液体的晃动问题;Bauer 和 Chiba[4] 研究了失重环境中的球形液滴的振动问题,并利用表面上的动力学条件给出了自由振动频率. 此外,作为空间中的晃动问题,在国内液体晃动与柔性附件的耦合运动[5] 和晃动本身的非线性特性[6,7] 等也是受到广泛关注. 因为瑞利 - 里兹在结构振动问题的常用,属于能量法. 用能量法研究该问题可为研究失重环境中的液滴与结构的耦合振动问题等提供一种可行的途径.

1　基本方程

假设液滴为不可压缩、无粘性液体. 由于研究自由晃动的特性,运动为小幅晃动,假设运动为无旋. 则液体运动存在势函数 $\Phi(r,\theta,\varphi,t)$,并连续性条件满足拉普拉斯方程,在球坐标系(如图 1 所示)下可描述为

$$\Delta\Phi=\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}=0 \tag{1}$$

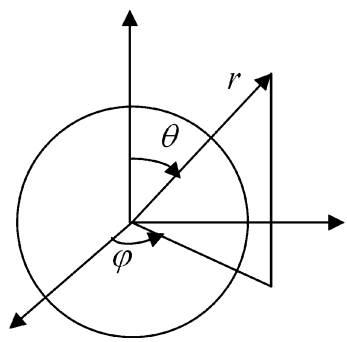

图 1　坐标系示意
Fig. 1　Coordinates definition

用能量法研究晃动问题,需要写出动能和势能表达式. 根据势流理论,其动能可写为

$$T=\frac{\rho}{2}\int_V(\nabla\Phi\cdot\nabla\Phi)r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi \tag{2}$$

其中 ρ 为液滴密度. 积分范围为整个液滴域.

当表面在平衡位置附近微小振动时,利用格林定理可把式(2)简化为

$$T=\frac{\rho a^2}{2}\int_0^{2\pi}\int_0^{\pi}\left(\Phi\frac{\partial\Phi}{\partial r}\right)_{r=a}\sin\theta\mathrm{d}\theta\mathrm{d}\varphi \tag{3}$$

其中 a 为平衡位置时的液滴半径.

2007-09-27 收到第 1 稿,2008-05-16 收到修改稿.

另一方面,可以列出表面张力引起的势能表达式,由于液滴处于失重状态,不存在重力势能,故表面张力势能为其晃动的全部势能. 其表达式为[8]

$$U=\sigma\int_0^{2\pi}\int_0^{\pi}(f^2\sqrt{(1+\frac{f_\theta^2}{f^2}+\frac{1}{\sin^2\theta}\frac{f_\varphi^2}{f^2})}-a^2)\sin\theta\mathrm{d}\theta\mathrm{d}\varphi \tag{4}$$

其中下标表示偏导数,$f(\theta,\varphi,t)$为晃动后的表面函数,可表示为

$$f(\theta,\varphi,t)=a+z(\theta,\varphi,t) \tag{5}$$

由于运动为小幅,上式中的$z(\theta,\varphi,t)$为小量;从而$\frac{f_\theta^2}{f^2}+\frac{1}{\sin^2\theta}\frac{f_\varphi^2}{f^2}$也为小量,记$F=\frac{f_\theta^2}{f^2}+\frac{1}{\sin^2\theta}\frac{f_\varphi^2}{f^2}$后,可对式(4)根号项进行泰勒展开,得

$$\sqrt{1+F}=1+F/2+o(F^2) \tag{6}$$

考虑式(5)和(6),式(4)可简化为

$$U=\sigma\int_0^{2\pi}\int_0^{\pi}(2az+z^2+\frac{1}{2}(z_\theta^2+\frac{1}{\sin^2\theta}z_\varphi^2))\sin\theta\mathrm{d}\theta\mathrm{d}\varphi \tag{7}$$

由于液滴为不可压缩,可得到体积不变关系式

$$\frac{1}{3}\int_0^{2\pi}\int_0^{\pi}(a+z)^3\sin\theta\mathrm{d}\theta\mathrm{d}\varphi=V \tag{8}$$

其中 V,为液滴体积. 将式(8)代入式(7)得,

$$U=\sigma\int_0^{2\pi}\int_0^{\pi}(-z^2+\frac{1}{2}(z_\theta^2+\frac{1}{\sin^2\theta}z_\varphi^2))\sin\theta\mathrm{d}\theta\mathrm{d}\varphi \tag{9}$$

于是,系统的拉格朗日函数可写为

$$\tilde{L}=T-U \tag{10}$$

另外,在表面上的运动学条件为

$$\frac{\partial f}{\partial t}=\frac{\partial z}{\partial t}=\frac{\partial\Phi}{\partial r} \tag{11}$$

由于研究微小振动,表面函数和势函数可写为以下简谐运动形式,

$$z(\theta,\varphi,t)=\zeta(\theta,\varphi)\cos\omega t \tag{12a}$$

$$\Phi(r,\theta,\varphi,t)=-\omega\phi(r,\theta,\varphi)\sin\omega t \tag{12b}$$

将上式代入表面运动学条件得,

$$\zeta(\theta,\varphi)=\frac{\partial\phi(a,\theta,\varphi)}{\partial r} \tag{13}$$

再将式(13)代入系统拉格朗日函数,并将其在一个周期内积分,再乘以 ω/π,可得新的拉格朗日函数

$$L=\frac{\rho a^2\omega^2}{2}\int_\varphi\int_\theta(\phi\frac{\partial\phi}{\partial r})_{r=a}\sin\theta\mathrm{d}\theta\mathrm{d}\varphi-\sigma\int_\varphi\int_\theta((-\frac{\partial\phi}{\partial r})^2+\frac{1}{2}((\frac{\partial^2\phi}{\partial r\partial\theta})^2+\frac{1}{\sin^2\theta}(\frac{\partial^2\phi}{\partial r\partial\theta})^2))_{r=a}\sin\theta\mathrm{d}\theta\mathrm{d}\varphi \tag{14}$$

上述新的拉格朗日函数仍满足哈密顿原理[9,10].

2　求解

根据球形液滴的晃动特性,把球坐标下的拉普拉斯方程通过分离变量求解后,可得速度势函数形式为如下的形式[5]

$$\phi=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}e^{i\omega t}A_{mn}(\frac{r}{a})^nP_n^m(\cos\theta)\cos m\varphi \tag{15}$$

其中 $P_n^m(\cos\theta)$为第一类一般勒让德函数.

将式(15)代入式(14)得

$$\begin{aligned}L=&\frac{\rho a\omega^2}{2}\int_0^{2\pi}\int_0^{\pi}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{q=0}^{\infty}\sum_{s=1}^{\infty}A_{mn}A_{ms}sP_n^m(\cos\theta)\times P_s^m(\cos\theta)\cos m\varphi\cos q\varphi\sin\theta\mathrm{d}\theta\mathrm{d}\varphi-\\&\frac{\sigma}{a^2}\int_0^{2\pi}\int_0^{\pi}[-\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{q=0}^{\infty}\sum_{s=1}^{\infty}A_{mn}A_{ms}nsP_n^m(\cos\theta)\times P_s^m(\cos\theta)\cos m\varphi\cos q\varphi+\\&\frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{q=0}^{\infty}\sum_{s=1}^{\infty}A_{mn}A_{ms}ns\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\times\frac{\partial P_s^m(\cos\theta)}{\partial\theta}\cos m\varphi\cos q\varphi+\\&\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{q=0}^{\infty}\sum_{s=1}^{\infty}\frac{mnqs}{2\sin^2\theta}P_n^m(\cos\theta)\times P_s^m(\cos\theta)\sin m\varphi\sin q\varphi]\sin\theta\mathrm{d}\theta\mathrm{d}\varphi\end{aligned} \tag{16}$$

利用三角函数的正交关系

$$\left\{\begin{aligned}&\int_0^{2\pi}\cos m\varphi\cos q\varphi\\&\int_0^{2\pi}\sin m\varphi\sin q\varphi\end{aligned}\right.=\begin{cases}0 & \text{when}m\neq q\\ \pi & \text{when}m=q\end{cases} \tag{17}$$

可把式(16)进一步简化为

$$\begin{aligned}L=&\frac{\rho a\omega^2}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}A_{mn}A_{ms}n\times\int_\theta P_n^m(\cos\theta)P_s^m(\cos\theta)\sin\theta\mathrm{d}\theta-\frac{\pi\sigma}{a^2}[-\\&\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}A_{mn}A_{ms}ns\int_\theta P_n^m(\cos\theta)P_s^m(\cos\theta)+\\&\frac{1}{2}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{s=1}^{\infty}A_{mn}A_{ms}ns\int_\theta(\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\times\frac{\partial P_s^m(\cos\theta)}{\partial\theta}+\frac{m^2}{\sin^2\theta}P_n^mP_n^m(\cos\theta)\times\\&P_s^m(\cos\theta))]\sin\theta\mathrm{d}\theta\end{aligned} \tag{18}$$

根据勒让德多项式的正交性,可得下两式

$$\int_0^\pi P_n^m(\cos\theta)P_s^m(\cos\theta)\sin\theta d\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{ns} \tag{19a}$$

$$\int_0^\pi \Big(\frac{\partial P_n^m(\cos\theta)}{\partial\theta}\frac{\partial P_s^m(\cos\theta)}{\partial\theta} + \frac{m^2}{\sin^2\theta}P_n^m P_n^m(\cos\theta)P_s^m(\cos\theta)\Big)\sin\theta d\theta = \frac{2n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{ns} \tag{19b}$$

其中 δ 为克罗内克符号. 式(19a)可从手册直接查得,式(19b)证明如下:记 $x=\cos\theta$,则 $\sin\theta d\theta = -dx$,式(19b)可写为

$$\int_{-1}^1 \{[P_n^m(x)']\,[P_n^m(x)]'(1-x^2) + \frac{m^2}{1-x^2}P_n^m(x)P_n^m(x)\}dx = \frac{2n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{ns}$$

其中$(\ )'$为 $d(\ )/dx$. 运用分部积分法,左半部分变为

$$P_n^m(x)[P_s^m(x)]'(1-x^2)\,|_{-1}^1 - \int_{-1}^1 P_n^m(x)\{[P_s^m(x)]''(1-x^2) - 2x[P_s^m(x)]'\}dx + \int_{-1}^1 \frac{m^2}{1-x^2}P_n^m(x)P_s^m(x)dx$$

注意到 $P_n^m(x)$ 是微分方程 $(1-x^2)\frac{d^2P}{dP^2} - 2x\frac{dP}{dP} + [s(s+1) - \frac{m^2}{1-x^2}]P = 0$ 的解, 故有

$$[P_s^m(x)]''(1-x^2) - 2x[P_s^m(x)]' = -[s(s+1) - \frac{m^2}{1-x^2}]P_s^m(x)$$

从而得

$$P_n^m(x)[P_s^m(x)]'\,|_{-1}^1 - \int_{-1}^1 P_n^m(x)\{[P_s^m(x)]'' \times (1-x^2) - 2x[P_s^m(x)]'\}dx + \int_{-1}^1 \frac{m^2}{1-x^2}P_n^m(x)P_s^m(x)dx = \int_{-1}^1 P_n^m(x)s(s+1)P_s^m(x)dx = \frac{2n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{ns}$$

证毕.

利用上两个勒让德函数正交关系,可把式(18)进一步简化为

$$L = \pi\rho a\omega^2\sum_{m=0}^\infty\sum_{n=1}^\infty A_{mn}^2\frac{2}{2n+1}\times\frac{(n+m)!}{(n-m)!} - \frac{\pi\sigma}{a^2}\Big[\sum_{m=0}^\infty\sum_{n=1}^\infty A_{mn}^2(n(n+1)-2)\frac{n^2}{2n+1}\frac{(n+m)!}{(n-m)!}\Big] \tag{20}$$

利用瑞利-里兹法,系统满足

$$\frac{\partial L}{\partial A_{mn}} = 0 \tag{21}$$

将式(20)代入上式,并利用 A_{mn} 不能同时为零的条件,可得

$$\omega^2 = n(n-1)(n+2)\frac{\sigma}{\rho a^3} \tag{22}$$

上式为失重球形液滴的自由晃动频率,与文献[3]中的结果一致. 考虑上式和式(15)速度势函数可得各阶模态,在此不再赘述.

3 结论

本文用瑞利-里兹法研究了失重环境中液滴的自由晃动问题,得到了其自由晃动的频率和模态,与利用表面动力学条件得到的结果一致. 由于能量的法的优点,本文的方法可用于研究失重环境中液滴和结构的耦合振动问题.

参 考 文 献

1 Egry, D. A. Diefenbach et al. Containerless processing in space-thermophysical property measurement using electromagnetic levitation. *Journal of Thermophysics*, 2001, 22: 569 ~ 578

2 E. H. Trinh, J. Depew. Solid surface wetting and the deployment of drop in microgravity. *Microgravity Science and Technology*, 1995, 7(4): 299 ~ 306

3 H. F. Bauer. Oscillations of immiscible liquids in free space or in spherical containers in zero-gravity environment. *Archive of Applied Mechanics*, 1982, 51(6): 363 ~ 381

4 H. F. Bauer and M. Chiba. Oscillations of captured spherical drop of frictionless liquid. *Journal of Sound and Vibration*, 2004, 274: 725 ~ 746

5 吕敬,李俊峰,王天舒等. 充液箱位置对航天器耦合特性影响分析. 动力学与控制学报, 2007, 5(2): 159 ~ 164 (Lu Jing, Li Junfeng, Wang Tianshu et al. Analysis on effects of fuel position on the attitude of spacecraft with e-

lastic appendages. *Journal of Dynamics and Control*,2007,5(2):159 ~ 164(in Chinese))

6 贺元军,王萍萍,王本利等. 谐波平衡法求解俯仰运动矩形贮箱中液体非线性晃动. 动力学与控制学报,2004,2(4):29 ~ 34(He Yuanjun,Wang Pingping,Wang Benli et al. Nonlinear sloshing of liquid in rectangular container under pitching excitation by harmonic balance method. *Journal of Dynamics and Control*,2004,2(4):29 ~ 34(in Chinese))

7 夏恒新,宝音贺西,郑亚. 多腔充液晃动的等效特性. 动力学与控制学报,2007,5(4):346 ~ 349(Xia Hengxin,Baoyin Hexi,Zheng Ya. Equivalent characteristics of liquid sloching in multi-tank. *Journal of Dynamics and Control*,2007,5(4):346 ~ 349(in Chinese))

8 R. Natarajan and R. Brown. Quadratic resonance in the three-dimensional oscillations of inviscid drops with surface tension. *Physics of Fluids*,1986,29(9):2788 ~ 2797

9 R. A. Ibrahim. Liquid sloshing dynamics. Cambridge University Press,2005

10 N. N. Moiseyev and V. V. Rumyantsev. Dynamics stability of bodies containing fluid. Springer-Verlag,Berlin,1968

USING RAYLEIGH-RITZ METHOD TO SOLVE FREE OSCILLATION OF A LIQUID DROP IN ZERO GRAVITY

Xia Hengxin[1]　Baoyin Hexi[2]　Zheng Ya[1]

(1. *School of Mechanical Engineering*, *Nanjing University of Science and Technology*,*Nanjing* 210000,*China*)

(2. *School of Aerospace*, *Tsinghua University*, *Beijing* 100084 *China*)

Abstract Due to absence the gravity, an ideal spherical liquid drop can be formed in the space environment that is useful for some experiments like space liquid science and on-orbit material composing. But the coupled vibration with its supporting structure may cause an undesirable distortion of the drop, so understanding its oscillation characteristic is essential. Using Rayleigh-Ritz method, this paper investigated the free oscillation of a liquid drop in zero gravity environment, and presented the natural frequencies and modes of the drop oscillation. Although some researchers have studied the same problem using dynamical condition on a drop surface, the energy method can provide a way to further study the liquid-structure coupled vibration in the zero gravity space environment because the energy method is usually adopted to deal with the vibration of more complicated coupled systems.

Key words Rayleigh-Ritz method, oscillation, free liquid drop

Received 27 September 2007, revised 16 May 2008.