第5卷第4期2007年12月
1672-6553/2007/05(4)/302-6

动力学与控制学报
JOURNAL OF DYNAMICS AND CONTROL

Vol.5 No.4
Dec. 2007

压电弹性厚板动力学的简化 Gurtin 型变分原理*

朱慧坚[1]　罗恩[2]

(1.广东技术师范学院计算机科学系,广州　510665)(2.中山大学应用力学与工程系,广州　510275)

摘要　根据古典阴阳互补和现代对偶互补的基本思想,通过罗恩提出的一条简单而统一的新途径,系统地建立了压电弹性厚板动力学的各类简化 Gurtin 型变分原理.这种简化 Gurtin 型变分原理能反映动力学初值-边值问题的全部特征.文中首先给出压电厚板动力学的广义虚功原理的表式,然后从该式出发,不仅能得到虚功原理,而且通过所给出的一系列广义 Legendre 变换,还能系统地成对导出压电弹性厚板动力学的8类、6类、4类变量简化 Gurtin 型变分原理的互补泛函以及3类和2类变量简化 Gurtin 型势能形式的泛函.同时,通过这条新途径还能清楚地阐明这些原理的内在联系.

关键词　压电弹性厚板,　动力学,　简化 Gurtin 型变分原理,　对偶互补,　初值-边值问题

引 言

对于压电弹性动力学,Tiersten 于1969年建立了二类变量 Hamilton 型变分原理[1];Sandhu 和 Pister 于1971年建立了六类变量 Gurtin 型变分原理[2];Oden 和 Reddy 于1976年建立了六类变量简化 Gurtin 型变分原理[3],但其泛函式中与初始条件有关的项有明显错误;罗恩等人于1997年系统地建立了虚功原理、互等定理和8类变量、6类变量、4类变量与2类变量简化 Gurtin 型变分原理[4],不仅改正了 Oden 和 Reddy 在专著[3]中的错误,而且给出了正确的泛函式.罗恩等人于2002年又系统地建立了有限变形压电弹性动力学的虚功原理和各类变量非传统 Hamilton 型变分原理[5],这种新的变分原理能反映这种动力学初值-边值问题的全部特征,从而克服了 Hamilton 型变分原理不能反映动力学初值-边值问题的全部特征的固有缺陷.文[6,7]建立了压电弹性薄板动力学非传统 Hamilton 型和简化 Gurtin 型变分原理.

压电弹性厚板在智能结构中有广泛的应用.但是,有关压电弹性厚板动力学的一些基本原理,如虚功原理和能反映动力学初值-边值问题的简化 Gurtin 型和非传统 Hamilton 型变分原理还没有系统建立.

根据古典阴阳互补和现代对偶互补[8]的基本思想,首先给出一个压电厚板动力学的广义虚功原理的表式,然后从该式出发,不仅能得到虚功原理和互等定理,而且通过所给出的一系列广义 Legendre 变换,还能系统地成对导出压电弹性厚板动力学的8类变量、6类变量、4类变量简化 Gurtin 型变分原理的互补泛函以及3类和2类变量简化 Gurtin 型变分原理势能形式的泛函.并且,通过这条途径还能清楚地阐明这些原理之间的内在联系.

1　基本方程

1.1　曲率和位移关系

对二维压电弹性矩形板,取与板中面重合的平面作为 x-y 坐标平面,忽略 σ_z 和 D_z 对变形及电场的影响[8],并设 $u_x=-z\psi_x$, $u_y=-z\psi_y$, $w=w(x,y,t)$. $w(x,y,t)$ 为挠度, ψ_x,ψ_y 分别为中面法线在 xz,yz 平面内的转角, ψ_x 以从 x 轴转90°到 z 轴的转向为正, ψ_y 以从 y 轴转90°到 z 轴的转向为正[9,10],因此应变位移关系为 $\varepsilon_x=\partial u_x/\partial x=-z\psi_{x,x}$, $\varepsilon_y=\partial u_y/\partial y=-z\psi_{y,y}$, $\gamma_{xy}=-z(\psi_{x,y}+\psi_{y,x})$, $\gamma_{yz}=w_{,y}-\psi_y$, $\gamma_{zx}=w_{,x}-\psi_x$. 由此可得曲率和位移关系

$$\kappa_x=-\psi_{x,x},\kappa_y=-\psi_{y,y},\kappa_{xy}=-(\psi_{x,y}+\psi_{y,x}),$$
$$\lambda_x=w_{,y}-\psi_y,\lambda_y=w_{,x}-\psi_x \qquad (1)$$

式中 $\kappa_x,\kappa_y,\kappa_{xy}$ 为两剖面间的相对转角, λ_x,λ_y 为变形前垂直中面的法线在变形后与中面夹角的变化.

2007-03-28 收到第1稿,2007-05-19 收到修改稿.
*高校博士点科研基金(20030558025)和国家自然科学基金(10172097)资助项目

1.2 速度位移关系

$$v=\partial w/\partial t=\dot{w},\omega_x=\partial\psi_x/\partial t=\dot{\psi}_x,\omega_y=\partial\psi_y/\partial t=\dot{\psi}_y \quad (2)$$

式中 ω_x,ω_y 分别为中面法线在 xz,yz 平面内的角速度.

1.3 速度动量关系

$$p=\rho hv,L_x=\rho I_z\omega_x,L_y=\rho I_z\omega_y \quad (3)$$

式中 I_z 为惯性矩,$I_z=h^3/12$,L_x,L_y 分别为中面法线在 xz,yz 平面内的角动量,h 为板厚度.

动能密度 $K(v,\omega_x,\omega_y)$ 和余动能密度 $K^*(p,L_x,L_y)$ 分别为

$$K(v,\omega_x,\omega_y)=(\rho hvv+\rho I_z\omega_x\omega_x+\rho I_z\omega_y\omega_y)/2,$$

$$K^*(p,L_x,L_y)=(pp/h+L_xL_x/I_z+L_yL_y/I_z)/(2\rho)$$

1.4 本构方程

忽略 σ_z 和 D_z 对变形及电场的影响,由 σ_z,D_z 为零代入横观各向同性三维压电介质本构方程可导出 ε_z,E_z 有关 $\varepsilon_x,\varepsilon_y$ 的函数表达式,即 $\varepsilon_z=-\alpha(\varepsilon_x+\varepsilon_y)/r,E_z=\beta(\varepsilon_x+\varepsilon_y)/r$,因此压电弹性厚板的本构方程可表示为(带顶标^的量为未就压电弹性厚板的假定处理前的对应变量):

$$\sigma_x=\kappa_1\varepsilon_x+\kappa_2\varepsilon_y;\sigma_y=\kappa_2\varepsilon_x+\kappa_1\varepsilon_y;$$

$$\tau_{yz}=c_{44}^t\gamma_{yz}-e_{15}^t\hat{E}_x;\tau_{xz}=c_{44}^t\gamma_{xz}-e_{15}^t\hat{E}_x;\tau_{xy}=c_{66}^t\gamma_{xy}$$

$$\hat{D}_x=e_{15}^t\gamma_{xz}+g_{11}^t\hat{E}_x;\hat{D}_y=e_{15}^t\gamma_{yz}+g_{22}^t\hat{E}_y.$$

令

$$M_x=\int_{-h/2}^{h/2}z\sigma_x\mathrm{d}z,M_y=\int_{-h/2}^{h/2}z\sigma_y\mathrm{d}z,M_{xy}=\int_{-h/2}^{h/2}z\tau_{xy}\mathrm{d}z,$$

$$Q_x=\int_{-h/2}^{h/2}\tau_{yz}\mathrm{d}z,Q_y=\int_{-h/2}^{h/2}\tau_{xz}\mathrm{d}z,D_x=\int_{-h/2}^{h/2}\hat{D}_x\mathrm{d}z,$$

$$D_y=\int_{-h/2}^{h/2}\hat{D}_y\mathrm{d}z,E_x=\int_{-h/2}^{h/2}\hat{E}_x\mathrm{d}z,E_y=\int_{-h/2}^{h/2}\hat{E}_y\mathrm{d}z$$

可得压电弹性厚板的本构关系:

$$M_x=c_{11}\kappa_x+c_{12}\kappa_y;\quad M_y=c_{21}\kappa_x+c_{22}\kappa_y;$$

$$Q_x=c_{33}\lambda_x-e_{23}E_y;\quad Q_y=c_{44}\lambda_y-e_{14}E_x;$$

$$M_{xy}=2c_{55}\kappa_{xy} \quad (4)$$

或

$$\kappa_x=s_{11}M_x+s_{12}M_y;\quad \kappa_y=s_{21}M_x+s_{22}M_y;$$

$$\lambda_x=s_{33}Q_x-d_{23}D_y;\quad \lambda_y=s_{44}Q_y-d_{14}D_x;$$

$$\kappa_{xy}=s_{55}M_{xy}/2 \quad (5)$$

$$D_x=e_{14}\lambda_y+g_{11}E_x;\quad D_y=e_{23}\lambda_x+g_{22}E_y \quad (6)$$

或

$$E_x=d_{14}Q_y+\beta_{11}D_x;E_y=d_{23}Q_x+\beta_{22}D_y \quad (7)$$

式中,M_x,M_y,M_{xy},Q_x,Q_y 和 $\kappa_x,\kappa_y,\kappa_{xy},\lambda_x,\lambda_y$ 分别表示内力矩和挠曲面的曲率;$c_{ij},e_{ki},g_{kk},s_{ij},d_{ki},\beta_{kk}$ $(i,j=1,2\cdots5;k=1,2)$ 分别表示横观各向同性压电弹性厚板的弯曲刚度系数、压电应力系数、压电介电常数、弯曲柔度系数、压电电压系数和介电隔离常数,各分量与三维横观各向同性压电弹性体对应分量的关系为(带上标 t 的代表横观各向同性三维压电弹性体的对应系数分量):

$$c_{11}=I_z\kappa_1,c_{12}=c_{21}=I_z\kappa_2,c_{33}=c_{44}=hc_{44}^t,$$

$$c_{55}=I_zc_{66}^t,e_{14}=e_{23}=he_{15}^t,g_{11}=hg_{11}^t,g_{22}=hg_{22}^t.$$

其中

$$\kappa_1=c_{11}^t-\alpha c_{13}^t/r-\beta e_{31}^t/r,\kappa_2=c_{12}^t-\alpha c_{13}^t/r-\beta e_{31}^t/r,$$

$$\alpha=c_{13}^tg_{33}^t+e_{31}^te_{33}^t,\beta=c_{13}^te_{33}^t-c_{33}^te_{31}^t,r=e_{33}^{t\,2}+g_{33}^tc_{33}^t$$

因此压电弹性厚板的电焓 $U(\kappa_x,\kappa_y,\kappa_{xy},\lambda_x,\lambda_y,E_x,E_y)$ 和余电焓 $U^*(M_x,M_y,M_{xy},Q_x,Q_y,D_x,D_y)$ 为

$$U=(c_{11}\kappa_x\kappa_x+c_{12}\kappa_x\kappa_y+c_{21}\kappa_y\kappa_x+c_{22}\kappa_y\kappa_y+c_{33}\lambda_x\lambda_x+c_{44}\lambda_y\lambda_y+2c_{55}\kappa_{xy}\kappa_{xy})/2-(e_{14}E_x\lambda_y+e_{23}E_y\lambda_x)-(g_{11}E_xE_x+g_{22}E_yE_y)/2 \quad (8)$$

$$U^*=(s_{11}M_xM_x+s_{12}M_xM_y+s_{21}M_yM_x+s_{22}M_yM_Y+s_{33}Q_xQ_x+s_{44}Q_yQ_y+2s_{55}M_{xy}M_{xy})/2-(d_{14}D_xQ_y+d_{23}D_yQ_x)-(\beta_{11}D_xD_x+g_{22}D_yD_y)/2 \quad (9)$$

1.5 运动方程

将横观各向同性压电板结构的运动方程中 x 和 y 方向两方程乘 z 并沿厚度方向积分,z 方向运动方程沿厚度方向 z 积分,可得

$$M_{x,x}+M_{xy,y}-Q_x+\rho J\ddot{\psi}_x=m_x \quad (10)$$

或 $$\dot{L}_x+M_{x,x}+M_{xy,y}-Q_x=m_x \quad (11)$$

$$M_{y,y}+M_{xy,x}-Q_y+\rho J\ddot{\psi}_y=m_y \quad (12)$$

或 $$\dot{L}_x+M_{y,y}+M_{xy,x}-Q_y=m_y \quad (13)$$

$$\rho J\ddot{w}-Q_{x,x}-Q_{y,y}=f \quad (14)$$

或 $$\dot{p}-Q_{x,x}-Q_{y,y}=f \quad (15)$$

式中 m_x,m_y 分别为作用在单位中面面积内的载荷在 xz 和 yz 平面内的合力矩,f 为 z 轴方向的合力,m_x,m_y 的正负规定分别与 ψ_x,ψ_y 相同.

1.6 Maxwell 方程

拟静电场中,将 Maxwell 方程中电位移和电荷关系式沿 z 积分,可得

$$D_{x,x}+D_{y,y}+D_{zz}=q \quad (16)$$

而电场强度和电势关系为

$$E_x=-\varphi_{,x},E_y=-\varphi_{,y} \quad (17)$$

式中 $D=_{zz}=\int_{-h/2}^{h/2}\hat{D}_z\mathrm{d}z=\hat{D}_z(x,y,h/2)-\hat{D}_z(x,y,-h/2)$, $q=\int_{-h/2}^{h/2}\hat{q}\mathrm{d}z$, $\varphi=\int_{-h/2}^{h/2}\hat{\varphi}(x,y,z)\mathrm{d}z/h$.

1.7 边界条件

(1)位移边界条件为

在 $\partial\Omega_1$ 上:$w=\bar{w},\psi_n=\bar{\psi}_n,\psi_s=\bar{\psi}_s$;

在 $\partial\Omega_2$ 上:$w=\bar{w},\psi_s=\bar{\psi}_s$ (18)

(2) 力的边界条件为

在 $\partial\Omega_2$ 上:$M_n=\bar{M}_n$;

在 $\partial\Omega_3$ 上:$M_n=\bar{M}_n,M_{ns}=\bar{M}_{ns},Q_n=\bar{Q}_n$ (19)

这里 n 是边界的外法线方向,s 是边界的切线方向,并规定 n 到 s 的转向与 x 轴到 y 轴的转向相同;$\psi_n,\psi_s,\psi_n,\psi_s,M_n,M_{ns},Q_n$ 与 $\psi_x,\psi_y,Q_x,Q_y,M_x,M_y,M_{xy}$有下面的关系

$$\psi_n=\psi_x\cos\theta+\psi_y\sin\theta,\psi_s=-\psi_x\sin\theta+\psi_y\cos\theta,$$

$$Q_n=Q_x\cos\theta+Q_y\sin\theta,$$

$$M_n=M_x\cos^2\theta+2M_{xy}\cos\theta\sin\theta+M_y\sin^2\theta,$$

$$M_{ns}=(M_y-M_x)\cos\theta\sin\theta+M_{xy}(\cos^2\theta-\sin^2\theta)$$

式中 θ 是边界的外法线与轴的夹角.

(3) 电势和电位移边界条件为

$\varphi=\bar{\varphi}$,在 $\partial\Omega_\varphi$ 上; (20)

$D=D_in_i=\bar{D}(i=x,y)$,在 $\partial\Omega_D$ 上 (21)

式中 $\bar{w},\bar{\psi}_n,\bar{\psi}_s,\bar{M}_n,\bar{M}_{ns},\bar{Q}_n,\bar{\varphi},\bar{D}$ 为已知函数.

1.8 初始条件

$$w_0=\tilde{w}_0,\psi_{x0}=\tilde{\psi}_{x0},\psi_{y0}=\tilde{\psi}_{y0}; \quad (22)$$

$$p_0=\tilde{p}_0,L_{x0}=\tilde{L}_{x0},L_{y0}=\tilde{L}_{y0} \quad (23)$$

式中 $\tilde{w}_0,\tilde{\psi}_{x0},\tilde{\psi}_{y0},\tilde{p}_0,\tilde{L}_{x0},\tilde{L}_{y0}$为已知初始值.

2 广义虚功原理,虚功原理,互等定理

可以证明,对于互不相干的任意函数 $p,L_x,L_y,M_x,M_y,M_{xy},Q_x,Q_y,w,\psi_x,\psi_y,\varphi,D_x,D_y$ 下列卷积表示的积分关系式恒成立

$$\iint_\Omega[p*\dot{w}+L_x*\dot{\psi}_x+L_y*\dot{\psi}_y-M_x*\psi_{x,x}-M_y*\psi_{y,y}-M_{xy}*(\psi_{x,y}+\psi_{y,x})+Q_y*(w_{,x}-\psi_x)+Q_x*(w_{,y}-\psi_y)+D_x*\varphi_{,x}+D_y*\varphi_{,y}]\mathrm{d}x\mathrm{d}y+\iint_\Omega[w*(Q_{x,x}+Q_{y,y}-\dot{p})+\psi_x*(-M_{x,x}-M_{xy,y}+Q_x-\dot{L}_x)+\psi_y*(-M_{xy,x}-M_{y,y}+Q_y-\dot{L}_y)+(D_{x,x}+D_{y,y})*\varphi]\mathrm{d}x\mathrm{d}y+\int_{\partial\Omega}(M_n*\psi_n+M_{ns}*\varphi_s-Q_n*w-D*\varphi)\mathrm{d}S+\iint_\Omega(pw_0+L_x\psi_{x0}+L_y\psi_{y0}-p_0w-L_{x0}\psi_x-L_{y0}\psi_y)\mathrm{d}x\mathrm{d}y=\Pi_1+\Pi_2+\Pi_3+\Pi_4=0 \quad (24)$$

式中 Π_1,Π_2,Π_3,Π_4 分别代表第一、二、三、四项积分,"$*$"代表卷积.

(24)式是本文给出的一个重要关系式,在力学上可以认为是压电厚板动力学的广义虚功原理的表式,从该式出发,不仅能系统地建立虚功原理和压电弹性厚板动力学的各类简化 Gurtin 型变分原理,而且能清晰地阐明这些原理之间的内在联系.

当动量和内力矩满足方程(11),(13),(15)及条件(19)和(23);电位移满足方程(16)及条件(21);位移满足方程(1),(2)及条件(18),(22);电势满足方程(17)及条件(20)时,则由(24)式可得

$$\iint_\Omega[f*w+m_x*\psi_x+m_y*\psi_y-q*\varphi]\mathrm{d}x\mathrm{d}y+\int_{\partial\Omega}(Q_n*w-M_n*\psi_n-M_{ns}*\psi_s+D*\varphi)\mathrm{d}S+\iint_\Omega(\tilde{P}_0w+\tilde{L}_{x0}\varphi_x+\tilde{L}_{y0}\varphi_y-p\tilde{w}_0-L_x\tilde{\varphi}_{x0}-L_y\tilde{\varphi}_{y0})\mathrm{d}x\mathrm{d}y=\iint_\Omega[M_x*\kappa_x+M_y*\kappa_y+2M_{xy}*\kappa_{xy}+Q_x*\lambda_x+Q_y*\lambda_y+p*v+L_x*\omega_x+L_y*\omega_y-D_x*E_x-D_y*E_y]\mathrm{d}x\mathrm{d}y \quad (25)$$

(25)式可以看成为压电厚板动力学的虚功原理的表式,它反映广义动力可能状态与广义运动

可能状态之间最一般关系,或者说,它反映 $f,m_x,m_y,q,p,L_x,L_y,M_x,M_y,M_{xy},D_x,D_y$ 与 $w,\psi_x,\psi_y,\varphi,v,\omega_x,\omega_y,\kappa_x,\kappa_y,\kappa_{xy},E_x,E_y$ 这两组对偶变量之间的最一般关系.

3 各类简化 Gurtin 型变分原理

3.1 8类变量广义变分原理

当 M_x,M_y,M_{xy},Q_x,Q_y 和 $\kappa_x,\kappa_y,\kappa_{xy},\lambda_x,\lambda_y$、$D_x,D_y$ 和 E_x,E_y 是互不相关的任意函数时,可以得到下列关系式

$$M_x*\kappa_x+M_y*\kappa_y+2M_{xy}*\kappa_{xy}+Q_x*\lambda_x+Q_y*\lambda_y-D_x*E_x-D_y*E_y=\overset{*}{U}+\overset{*}{U}{}^*+\overset{*}{A} \quad (26)$$

式中

$$\overset{*}{U}=(c_{11}\kappa_x*\kappa_x+c_{12}\kappa_x*\kappa_y+c_{21}\kappa_y*\kappa_x+c_{22}\kappa_y*\varepsilon_y+c_{33}\lambda_x*\lambda_x+c_{44}\lambda_y*\lambda_y+2c_{55}\kappa_{xy}*\kappa_{xy})/2-(e_{14}E_x*\lambda_y+e_{23}E_y*\lambda_x)-(g_{11}E_x*E_x+g_{22}E_y*E_y)/2$$

$$\overset{*}{U}{}^*=(s_{11}M_x*M_x+s_{12}M_x*M_y+s_{21}M_y*M_x+s_{22}M_y*M_y+s_{33}Q_x*Q_x+s_{44}Q_y*Q_y+$$

$$2s_{55}M_{xy} * M_{xy})/2 - (d_{14}D_x * Q_y + d_{23}D_y * Q_x) - (\beta_{11}D_x * D_x + g_{22}D_y * D_y)/2$$

$$\overset{*}{A} = \{[M_x - (c_{11}\kappa_x + c_{12}\kappa_y)] * [\kappa_x - (s_{11}M_x + s_{12}M_y)] + [M_y - (c_{21}\kappa_x + c_{22}\kappa_y)] * [\kappa_y - (s_{21}M_x + s_{22}M_y)] + [Q_x - (c_{33}\lambda_x - e_{23}E_y)] * [\lambda_x - (s_{33}Q_x - d_{23D_y})] + [Q_y - (c_{44}\lambda y - e_{14}E_x)] * [\lambda_y - (s_{33}Q_y - d_{14}D_x)] + (M_{xy} - 2c_{55}\kappa_{xy}) * (2\kappa_{xy} - s_{55}M_{xy}) + [D_x - (e_{14}\lambda_y + g_{11}E_x)] * [E_x - (d_{14}Q_y + \beta_{11}D_x)] + [D_y - (e_{23}\lambda_x + g_{22}E_y)] * [E_y - (d_{23}Q_x + \beta_{22}D_y)]\}/2$$

只有当 $M_x, M_y, M_{xy}, Q_x, Q_y, \kappa_x, \kappa_y, \kappa_{xy}, \lambda_x, \lambda_y, D_x, D_y, E_x, E_y$ 满足(4)-(7)式时,才有

$$M_x * \kappa_x + M_y * \kappa_y + 2M_{xy} * \kappa_{xy} + Q_x * \lambda_x + Q_y * \lambda_y - D_x * E_x - D_y * E_y = \overset{*}{U} + \overset{*}{U}^* \quad (27)$$

于是,(24)式第一项积分 Π_1 中的被积函数

$$-M_x * \psi_{x,x} - M_y * \psi_{y,y} - M_{xy} * (\psi_{x,y} + \psi_{y,x}) + Q_y * (w_{,x} - \psi_x) + Q_x * (w_{,y} - \psi_y) + D_x * \varphi_{,x} + D_y * \varphi_{,y}$$

可变换为

$$-M_x * \psi_{x,x} - M_y * \psi_{y,y} - M_{xy} * (\psi_{x,y} + \psi_{y,x}) + Q_y * (w_{,x} - \psi_x) + Q_x * (w_{,y} - \psi_y) + D_x * \varphi_{,x} + D_y * \varphi_{,y} = \overset{*}{U} + \overset{*}{U}^* + \overset{*}{A} - M_x * (\kappa_x + \psi_{x,x}) - M_y * (\kappa_y + \psi_{y,y}) - M_{xy} * [2\kappa_{xy} + (\psi_{x,y} + \psi_{y,x})] - Q_x * [\lambda_x - (w_{,y} - \psi_y)] - Q_y * [\lambda_y - (w_{,x} - \psi_x)] + D_x * (E_x + \varphi_{,x}) + D_y * (E_y + \varphi_{,y}) \quad (28)$$

当 p, L_x, L_y 和 v, ω_x, ω_y 是互不相关的任意函数时,可以得到下列关系式

$$p * v + L_x * \omega_x + L_y * \omega_y = \overset{*}{K} + \overset{*}{K}^* - \overset{*}{B} \quad (29)$$

式中

$$\overset{*}{K} = (\rho h v * v + \rho I_z \omega_x * \omega_x + \rho I_z \omega_y * \omega_y)/2,$$

$$\overset{*}{K}^* = (p * p/H + L_x * L_x/I_z + L_y * L_y/I_z)/(2\rho),$$

$$\overset{*}{B} = [(\rho h v - p) * (\rho h v - p)/h + (\rho I_z \omega_x - L_x) * (\rho I_z \omega_x - L_x)/I_z + (\rho I_z \omega_y - L_y) * (\rho I_z \omega_y - L_y)/I_z]/(2\rho)$$

只有当 p, L_x, L_y 和 v, ω_x, ω_y 满足(3)式时,才有

$$p * v + L_x * \omega_x + L_y * \omega_y = \overset{*}{K} + \overset{*}{K}^* \quad (30)$$

于是,(24)式第一项积分 Π_1 中的被积函数 $p * \dot{w} + L_x * \dot{\psi}_x + L_y * \dot{\psi}_y$ 可变换为

$$p * \dot{w} + L_x * \dot{\psi}_x + L_y * \dot{\psi}_y = \overset{*}{K} - p * (v - \dot{w}) - L_x * (\omega_x - \dot{\psi}_x) - L_y * (\omega_x - \dot{\psi}_y) + \overset{*}{K}^* - \overset{*}{B} \quad (31)$$

上述的(26),(29)式是本文给出的广义 Legendre 变换式.

而(24)式第二、三、四项积分可变换为

$$\Pi_2 + \Pi_3 + \Pi_4 = \iint_\Omega [(-\dot{p} + Q_{x,x} + Q_{y,y} + f) * w + (-\dot{L}_x - M_{x,x} - M_{xy,y} + Q_x + m_x) * \psi_x + (-\dot{L}_y - M_{xy,x} - M_{y,y} + Q_y + m_y) * \psi_y + (D_{x,x} + D_{y,y} + D_{zz} - q) * \varphi] \mathrm{d}x\mathrm{d}y + \Gamma_B + \Gamma_I - \iint_\Omega (f * w + m_x * \psi_x + m_y * \psi_y - q * \varphi + D_{zz} * \varphi) \mathrm{d}x\mathrm{d}y + \Pi_B + \Pi_I \quad (32)$$

式中

$$\Pi_B = \int_{\partial\Omega_2 + \partial\Omega_3} \bar{M}_n * \psi_n \mathrm{d}S + \int_{\partial\Omega_3} (\bar{M}_{ns} * \psi_s - \bar{Q}_n * w) \mathrm{d}S + \int_{\partial\Omega_1} (\psi_n - \bar{\psi}_n) * M_n \mathrm{d}S + \int_{\partial\Omega_1 + \partial\Omega_2} [(\psi_s - \bar{\psi}_s) * M_{ns} - (w - \bar{w}) * Q_n] \mathrm{d}S - \int_{\partial\Omega_D} \bar{D} * \varphi dS - \int_{\partial\Omega_\varphi} (\varphi - \bar{\varphi}) * D \mathrm{d}S$$

$$\Gamma_B = \int_{\partial\Omega_1} \bar{\psi}_n * M_n \mathrm{d}S + \int_{\partial\Omega_1 + \partial\Omega_2} (\bar{\psi}_s * M_{ns} - \bar{w} * Q_n) \mathrm{d}S + \int_{\partial\Omega_2 + \partial\Omega_3} (M_n - \bar{M}_n) * \psi_n \mathrm{d}S$$

$$\Pi_I = \iint_\Omega [(w_0 - \tilde{w})p + (\psi_{x0} + \tilde{\psi}_{x0})L_x + (\psi_{y0} + \tilde{\psi}_{y0})L_y - \tilde{p}_0 w - \tilde{L}_{x0}\psi_x - \tilde{L}_{y0}\psi_y] \mathrm{d}x\mathrm{d}y$$

$$\Gamma_I = \iint_\Omega [\tilde{w}_0 p + \tilde{\psi}_{x0} L_x + \tilde{\psi}_{y0} L_y - (p_0 - \tilde{p}_0) w - (L_{x0} - \tilde{L}_{x0})\psi_x - (L_{y0} - \tilde{L}_{y0})\psi_y] \mathrm{d}x\mathrm{d}y$$

将(28),(31)和(32)代入(24)式,经整理后可得

$$\Pi_8(p, L_x, L_y; v, \omega_x, \omega_y; w, \psi_x, \psi_y; M_x, M_y, M_{xy}, Q_x, Q_y; \kappa_x, \kappa_y, \kappa_{xy}, \lambda_x, \lambda_y; \varphi, D_x, D_y, E_x, E_y) + \Gamma_8(p, L_x, L_y; v, \omega_x, \omega_y; w, \psi_x, \psi_y; M_x, M_y, M_{xy}, Q_x, Q_y; \kappa_x, \kappa_y, \kappa_{xy}, \lambda_x, \lambda_y; \varphi, D_x, D_y, E_x, E_y) = 0 \quad (33)$$

而其势能形式的泛函和余能形式的泛函分别为

$$\Pi_8 = \iint_\Omega \{\overset{*}{K} - p * (v - \dot{w}) - L_x * (\omega_x - \dot{\psi}_x) - L_y * (\omega_y - \dot{\psi}_y) + \overset{*}{U} - M_x * (\kappa_x + \psi_{x,x}) -$$

$M_y*(\kappa_y+\psi_{y,y})-M_{xy}*[2\kappa_{xy}+(\psi_{x,y}+\psi_{y,x})]-Q_x*[\lambda_x-(w_{,y}-\psi_y)]-Q_y*[\lambda_y-(w_{,x}-\psi_x)]+D_x*(E_x+\varphi_{,x})+D_y*(E_y+\varphi_{,y})-D_{zz}*\varphi-f*w-m_x*\psi_x-m_y*\psi_y+q*\varphi\}\mathrm{d}x\mathrm{d}y+\Pi_B+\Pi_I$

$$\Gamma_8=\iint_\Omega[\overset{*}{K}{}^*-\overset{*}{B}+\overset{*}{U}{}^*+\overset{*}{A}+(Q_{x,x}+Q_{y,y}+f-\dot{p})*w+(-M_{x,x}-M_{y,y}+Q_x+m_x-\dot{L}_x)*\psi_x+(-M_{xy,x}-M_{y,y}+Q_y+m_y-\dot{L}_y)*\psi_y+(D_{x,x}+D_{y,y}+D_{zz}-q)*\varphi]\mathrm{d}x\mathrm{d}y+\Gamma_B+\Gamma I$$

定理 1 当且仅当 $p,L_x,L_y,v,\omega_x,\omega_y,w,\psi_x,\psi_y,M_x,M_y,M_{xy},Q_x,Q_y,\kappa_x,\kappa_y,\kappa_{xy},\lambda_x,\lambda_y,\varphi,D_x,D_y,E_x,E_y$ 是混合问题(1)-(4),(6),(11),(13),(15)-(23)式的解,则必定满足下列变分式

$$\delta\Pi_8=0 \text{ 或 } \delta\Gamma_8=0\ . \tag{34}$$

Π_8 和 Γ_8 分别是压电弹性厚板动力学8类变量简化Gurtin型广义变分原理的势能形式和余能形式的泛函,对于任意无关的 $p,L_x,L_y,v,\omega_x,\omega_y,w,\psi_x,\psi_y,M_x,M_y,M_{xy},Q_x,Q_y,\kappa_x,\kappa_y,\kappa_{xy},\lambda_x,\lambda_y,\varphi,D_x,D_y,E_x,E_y$,它们之间存在互补关系式(33).

当 $p,L_x,L_y,v,\omega_x,\omega_y,w,\psi_x,\psi_y$ 满足(2)和(3)式时,由8类变量变分原理的泛函式可导出6类变量变分原理的泛函式;当 M_x,M_y,M_{xy},Q_x,Q_y 与 $\kappa_x,\kappa_y,\kappa_{xy},\gamma_x,\gamma_y$、$D_x,D_y$ 和 E_x,E_y 满足(4)-(7)式时,可由6类变量变分原理的泛函式导出4类变量变分原理的泛函式; 当 p、L_x,L_y 和 v,ω_x,ω_y、$\kappa_x,\kappa_y,\kappa_{xy},\lambda_x,\lambda_y$ 和 w,ψ_x,ψ_y、E_x,E_y 和 φ 分别满足(3),(1)和(17)式时, 由 Π_8 可导出3类变量广义变分原理. 因篇幅所限,略去具体推导过程.

3.2 2类变量变分原理

当 $p,L_x,L_y,v,\omega_x,\omega_y,w,\psi_x,\psi_y$ 满足(2)和(3)式时, 当 M_x,M_y,M_{xy},Q_x,Q_y 与 $\kappa_x,\kappa_y,\kappa_{xy},\gamma_x,\gamma_y$、$D_x,D_y$ 和 E_x,E_y 满足(4)-(7)式时,$\kappa_x,\kappa_y,\kappa_{xy},\lambda_x,\lambda_y$ 和 w,ψ_x,ψ_y、E_x,E_y 和 φ 分别满足(3),(1)和(17)式时, 由 Π_8 可导出

$$\Pi_2(w,\psi_x,\psi_y;\varphi)=\iint_\Omega(\overset{*}{K}+\overset{*}{U}{}'-f*w-m_x*\psi_x-m_y*\psi_y-D_{zz}*\varphi+q*\varphi)\mathrm{d}x\mathrm{d}y+\Pi_B'+\Pi_I^* \tag{35}$$

定理 2 当且仅当 w,ψ_x,ψ_y,φ 是混合问题(18)-(20),(22),(23)及下列各式

$$-(c_{11}\psi_{x,x}+c_{12}\psi_{y,y})_{,x}-c_{55}(\psi_{x,y}+\psi_{y,x})_{,y}-c_{33}(w_{,y}-\psi_y)+e_{23}\varphi_{,y}+\rho I_z\ddot{\psi}_x=m_x$$

$$-(c_{21}\psi_{x,x}+c_{22}\psi_{y,y})_{,y}-c_{55}(\psi_{x,y}+\psi_{y,x})_{,x}-c_{44}(w_{,x}-\psi_x)+e_{14}\varphi_{,x}+\rho I_z\ddot{\psi}_y=m_y$$

$$\rho h\ddot{w}-[c_{33}(w_{,y}-\psi_y)-e_{23}\varphi_{,y}]_{,x}-[c_{44}(w_{,x}-\psi_x)-e_{14}\varphi_{,x}]_{,y}=f$$

的解,则必定满足变分式 $\delta\Pi_2=0$.

Π_2 是压电弹性厚板动力学2类变量简化Gurtin型广义变分原理势能形式的泛函.

4 结论

本文所建立的虚功原理和各类简化Gurtin型变分原理是压电弹性厚板动力学一般理论中最核心的部分. 这些简化Gurtin型变分原理能反映这种动力学初值－边值问题的全部特征. 因此,本文所建立的这些新的变分原理,无论在有关压电弹性厚板的理论研究方面,还是在建立其动力分析的各种近似解法和工程实用理论方面都有重要作用.

参 考 文 献

1 Tiersten H F. Linear piezoelectric plate vibration. New York: Plenum Press,1969

2 Sandhu R S, Pister K S. Variational principles for boundary value and intial－boundary value problems in continuum mechanics. *International Journal of Solids & Structures*, 1971,7:639～654

3 Oden J T, Reddy J N. Variational Methods in Theoretical Mechanics. 2nd ed. Berlin: Springer－Verlag,1983:169～173

4 罗恩,张贺忻. 压电弹性动力学的一些基本原理. 现代力学与科技进步,1997, 1272～1275 (Luo En, Zhang Hexin. Some basic principles for Elastodynamics of Piezoelectricity. *Moden Mechanics and Technological Progress*, 1997,1272～1275(in Chinese))

5 罗 恩,罗志国. 有限变形压电弹性动力学的非传统Hamilton型变分原理. 北京:国防工业出版社,2001 (Luo En,Luo Zhiguo. Unconventional hamilton－type variational principles for elastodynamics of finite deformational piezoelectricity. Beijing: Defense Industry Press (in Chinese))

6 朱慧坚,罗恩. 压电弹性薄板动力学非传统Hamilton型变分原理. 中山大学学报(自然科学版),2006,45(2): 16～19((Zhu Huijian, Luo En. Unconventional hamilton－type variational principles for elastodynamics of piezoelectric thin plate. *ACTA Scientiarum naturaltium universita-*

tis Sunyatseni,2006,45(2):16 ~ 19(in Chinese))

7 朱慧坚,罗恩. 压电弹性薄板动力学简化 Gurtin 型变分原理. 汕头大学学报(自然科学版),2006,21(4):64 ~ 69(Zhu Huijian, Luo En. Simplified gurtin - type variational principles for elastodynamics of piezoelectric thin plate. *Journal of Shantou University* (*Natural Science*), 2006,21(4):64 - 69(in Chinese))

8 罗恩,邓鹏. 浅论力学研究中的一些思维方法与力学美. 力学与实践,2004,26(1):76 ~ 78(Luo En, Deng Peng. On thinking ways and beauty in Mechanics Research. *Mechanics and Engineering*,2004,26(1):76 ~ 78 (in Chinese))

9 陈伟球,徐荣桥,丁皓江. 压电矩形板的自由振动. 浙江大学学报(自然科学版),1998,32(5):584 ~ 589(Chen Weiqiu, Xu Rongqiao, Ding Haojiang. Free vibration of piezoelectric rectangular plate. Journal Of Zhejiang University (Natural Science),1998,32(5):584 ~ 589(in Chinese))

10 曹志远,杨昇田. 厚板动力学理论及其应用. 北京:科学出版社, 1984(Cao Zhiyuan, Yang Shengtian. Dynamic theory of thick plate with application. Beijing: Science Press,1984(in Chinese))

11 刘正兴,孙雁,王国庆. 计算固体力学. 上海:上海交通大学出版社,2000(Liu Zhengxing, Sun Yan, Wang Guoqing. Computational solid mechanics. Shang Hai: Shang Hai Jiao Tong University Press,2000(in Chinese))

SIMPLIFIED GURTIN-TYPE VARIATIONAL PRINCIPLES FOR ELASTODYNAMICS OF PIEZOELECTRIC THICK PLATE *

Zhu Huijian[1]　Luo En[2]

(1. *Department of Computer Science*, *Guangdong Polytechnic Normal University*, *Guangzhou* 510665, *China*)

(2. *Department of Applied Mechanics and Engineering*, *Sun Yat-sen University*, *Guangzhou* 510275, *China*)

Abstract　According to the basic idea of classical Yin-Yang complementarity and modern dual-complementarity, in a new, simple and unified way proposed by Luo, the simplified Gurtin-type variational principles for elastodynamics of piezoelectric thick plate can be established systematically. The simplified Gurtin-type variational principles can fully characterize the initial-boundary problem of elastodynamics of piezoelectric thick plate. An important integral relation is given, which can be considered as the generalized principle of virtual work for dynamics of piezoelectric thick plate. Based on this relation, it is possible not only to obtain the principles of virtual work, but also to derive systematically the complementary functionals for eight-field, six-field, four-field simplified Gurtin-type variational principles and the potential energy functional for three-field and two-field simplified Gurtin-type principle for elastodynamics of piezoelectric thick plate by the generalized Legendre transformation given in this paper . Furthermore , with this approach, the intrinsic relation among various principles can be explained clearly.

Key words　piezoelectric thick plate,　dynamics,　simplified Gurtin-type variational principles,　dual-complementary relation,　initial-boundary-value problem

Received 28 March 2007, revised 19 May 2007.

* The project supported by the Scientific Foundation of the Ministry of Education of China for Doctoral Program (20030558025) and the National Natural Science Foundation of China (10172097)