第 5 卷第 3 期 2007 年 9 月
1672-6553/2007/05(3)/260-7
动 力 学 与 控 制 学 报
JOURNAL OF DYNAMICS AND CONTROL
Vol. 5 No. 3
Spe. 2007

压电热弹性体混合层合板的响应分析

贾立斌　卿光辉　刘艳红　杜洪增
（中国民航大学航空工程学院，天津　300300）

摘要　根据广义 Hamilton 变分原理推导出了压电热弹性体非齐次的 Hamilton 正则方程. 考虑热平衡方程与导热方程中变量的对偶关系，通过增加正则方程的维数，成功地将非齐次的正则方程转化为能独立求解的压电热弹性体耦合问题的齐次方程. 将非齐次方程转化为齐次方程不仅使问题变得大为简化，同时也减少了数值计算的工作量. 数值实例研究了温度载荷和力载荷作用下压电热弹性材料四边简支层合板的响应问题，部分算例与相关文献进行了比较.

关键词　压电热弹性体，　Hamilton 正则方程，　齐次方程，　层合板

引 言

压电弹性材料和磁电弹性材料，具有将某一或两种形式的能量转化为其他形式能量的功能，并且易于控制和操作，因此被广泛应用于空间工程、宇宙飞船、机动车和医疗设备等科学和工程领域[1]. 近年来，关于压电材料的研究已经获得长足的发展. Ray [2,3]等、Heyliger 和 Brooks[4,5]根据 Pagano[6]的薄板圆柱弯曲精确解方法，分析了压电层的精确解问题. 范家让[7]等人利用状态空间分析法研究了一般边界条件下复合材料层合板的系列力学问题. 目前，状态空间方法是求解任意厚度复合材料层合板或压电型层合结构有关问题的主要方法之一[8,9]，这种方法有很多优点.

以上所涉及到的对压电材料层合板的分析方法均没有考虑温度对压电材料力学性能的影响，在周围温度环境恶劣或温度剧烈变化的情况下，温度的变化将对材料的性能产生重大影响. Mindlin[10]最先提出了机、电、热场耦合的压电热弹性材料的控制方程，文献[11]提出了压电热弹性层合板的精确解，Vel 和 Batra[12]分析了压电热弹性材料层合板的广义应变问题. 以往文献研究热弹性材料层合板问题时，通常由基本方程推导出的控制方程是非齐次的，求解比较复杂[13,14].

本文利用混合变分原理[15-19]，结合热平衡方程和导热方程，根据对偶关系[20]的统一性成功地导出了压电热弹性体耦合问题的齐次状态方程. 数值实例研究了温度载荷和力载荷作用下压电热弹性材料四边简支层合板的响应问题.

1　压电热弹性材料的基本方程

假设所研究的压电热弹性材料为正交各向异性的，则其本构关系是

$$\boldsymbol{\sigma} = c\varepsilon - e^T\boldsymbol{E} - \lambda T, \quad \boldsymbol{D} = e\varepsilon + k\boldsymbol{E}\gamma T \qquad (1)$$

其中 $\boldsymbol{\sigma}$ 为应力向量（分量 $\sigma_{ij}, i,j = x,y,z$），ε 为应变向量分量（$\varepsilon_{ij}, i,j = x,y,z$），$\boldsymbol{\lambda}$ 为应力－温度向量（分量 $\lambda_i, i = 1,2,3,4,5,6$），$c$ 为刚度系数矩阵（分量 $c_{ij}, c_{ij} = c_{ji}, i,j = 1,2,3,4,5,6$），$\boldsymbol{e}$ 为压电系数矩阵（分量 $e_{ij}, i = 1,2,3, j = 1,2,3,4,5,6$），$\boldsymbol{E}$ 为电场强度向量（分量 $E_i, i = x,y,z$），$\boldsymbol{D}$ 为电位移向量（分量 $D_i, i = x,y,z$），$\boldsymbol{k}$ 为介电系数矩阵（分量 $k_{ij}, k_{ij} = k_{ji}$），$\boldsymbol{\gamma}$ 为热电系数向量（分量 $\gamma_i, i = 1,2,3$），T 为温度增量，“$\boldsymbol{T}$”表示转置.

弹性材料的应变－位移关系

$$\begin{aligned} &\varepsilon_{xx} = \alpha u \qquad \varepsilon_{yz} = \beta w + \partial v/\partial z \\ &\varepsilon_{yy} = \beta v \qquad \varepsilon_{xz} = \partial u/\partial z + \alpha w \\ &\varepsilon_{zz} = \partial w/\partial z \quad \varepsilon_{xy} = \alpha v + \beta u \end{aligned} \qquad (2)$$

其中 u,v,w 为位移分量，$\alpha = \partial/\partial x, \beta = \partial/\partial y$.

电场强度分量与电势 ϕ 的关系

$$E_x = -\alpha\phi \quad E_y = -\beta\phi \quad E_z = -\partial\phi/\partial z \qquad (3)$$

对式(1)进行行列交换并写成分块矩阵形式

2007-03-06 收到第 1 稿，2007-03-15 收到修改稿.
* 天津市自然科学基金（07JCYBC02100）资助项目

$$\begin{Bmatrix}\boldsymbol{P}_1\\\boldsymbol{P}_2\end{Bmatrix}=\begin{bmatrix}\boldsymbol{\Gamma}_{11} & \boldsymbol{\Gamma}_{12}\\\boldsymbol{\Gamma}_{12} & \boldsymbol{\Gamma}_{22}\end{bmatrix}\begin{Bmatrix}\boldsymbol{D}_1\\\boldsymbol{D}_2\end{Bmatrix}+\begin{Bmatrix}\boldsymbol{B}_1\boldsymbol{T}\\\boldsymbol{B}_2\boldsymbol{T}\end{Bmatrix}\tag{4}$$

其中($\Gamma_{ij},i,j=1,2$)是本构关系进行行列交换后对应的分块矩阵

$\boldsymbol{P}_1=[\sigma_{xz}\quad\sigma_{yz}\quad\sigma_z\quad D_z]^T$,

$\boldsymbol{P}_2=[\sigma_x\quad\sigma_y\quad\sigma_{xy}\quad D_x\quad D_y]^T$,

$\boldsymbol{D}_1=[\varepsilon_{xz}\quad\varepsilon_{yz}\quad\varepsilon_z\quad -E_z]^T$,

$\boldsymbol{D}_2=[\varepsilon_x\quad\varepsilon_y\quad\varepsilon_{xy}\quad -E_x\quad -E_y]^T$,

$\boldsymbol{B}_1=[0\quad 0\quad -\lambda_{33}\quad r_3]^T$,

$\boldsymbol{B}_1=[\lambda_{11}\quad -\lambda_{22}\quad 0\quad 0\quad 0]$

将式(4)中的 $\boldsymbol{P}_2,\boldsymbol{D}_1$ 作为未知量求出,有

$$\begin{Bmatrix}\boldsymbol{P}_1\\\boldsymbol{P}_2\end{Bmatrix}=\begin{bmatrix}\boldsymbol{\Phi}_{11} & \boldsymbol{\Phi}_{12}\\\boldsymbol{\Phi}_{12} & \boldsymbol{\Phi}_{22}\end{bmatrix}\begin{Bmatrix}\boldsymbol{P}_1\\\boldsymbol{D}_2\end{Bmatrix}+\begin{Bmatrix}\boldsymbol{A}_1\boldsymbol{T}\\\boldsymbol{A}_2\boldsymbol{T}\end{Bmatrix}\tag{5}$$

其中

$\boldsymbol{\Phi}_{11}=\boldsymbol{\Gamma}_{11}^{-1},\quad \boldsymbol{\Phi}_{12}=-\boldsymbol{\Gamma}_{11}^{-1}\boldsymbol{\Gamma}_{12},\quad \boldsymbol{\Phi}_{21}=-\boldsymbol{\Phi}_{12}{}^{T}$

$\boldsymbol{\Phi}_{22}=\boldsymbol{\Phi}_{22}^T=\boldsymbol{\Gamma}_{22}-\boldsymbol{\Gamma}_{21}\boldsymbol{\Gamma}_{11}^{-1}\boldsymbol{\Gamma}_{12}$

$\boldsymbol{A}_1=-\boldsymbol{\Phi}_{11}\boldsymbol{B}_1,\quad \boldsymbol{A}_2=(\boldsymbol{B}_2-\boldsymbol{\Gamma}_{21}\boldsymbol{\Phi}_{11}\boldsymbol{B}_1)$

根据文献[15－19],压电热弹性材料的广义H－R变分原理为

$$\delta\Pi=\delta\int_v L_R\mathrm{d}V-\delta\int_{s_\sigma}\overline{T}^TQ\mathrm{d}S-\delta\int_{s_u}T^T(Q-\overline{Q})\mathrm{d}S=0\tag{6}$$

其中 L_R 为广义的 Reissner's 能密度函数

$\overline{\boldsymbol{T}}=[\overline{T}_x\quad\overline{T}_y\quad\overline{T}_z\quad\overline{T}_q]$,

$\boldsymbol{Q}=[u\quad v\quad w\quad\phi]^T$,

$\boldsymbol{F}=[f_x\quad f_y\quad f_z\quad f_q]$

为广义体积力;

$\boldsymbol{\Omega}=\mathrm{diag}[\rho\omega^2\quad\rho\omega^2\quad\rho\omega^2\quad 0]$,$\rho$ 是材料密度,ω 是频率.

将式(5)代入式(6)有消去其中的 $\boldsymbol{P}_2$ 和 $\boldsymbol{D}_1^T$,有

$$L_R=\boldsymbol{P}^T(\boldsymbol{D}_1+\boldsymbol{\Phi}_{21}^T\boldsymbol{D}_2)+\frac{1}{2}\boldsymbol{D}_2^T\boldsymbol{\Phi}_{22}\boldsymbol{D}_2-\frac{1}{2}\boldsymbol{P}^T\boldsymbol{\Phi}_{11}\boldsymbol{P}+\boldsymbol{A}_2^T\boldsymbol{D}_2\boldsymbol{T}-\boldsymbol{A}_1^T\boldsymbol{P}\boldsymbol{T}-\frac{1}{2}\boldsymbol{Q}^T\boldsymbol{\Omega}\boldsymbol{Q}-\boldsymbol{Q}^T\boldsymbol{F}\tag{7}$$

将应变－位移关系和电场－电势关系表示成向量形式

$$\boldsymbol{D}_1=\partial\boldsymbol{Q}/\partial z+\boldsymbol{G}_1\boldsymbol{Q},\quad \boldsymbol{D}_2=\boldsymbol{G}_2\boldsymbol{Q}\tag{8}$$

其中 $\boldsymbol{G}_1$ 和 $\boldsymbol{G}_2$ 为操作算子矩阵, 将式(8)带入式(7)得

$$L_{MR}=\boldsymbol{P}^T\frac{\partial\boldsymbol{Q}}{\partial z}+\boldsymbol{P}^T(\boldsymbol{G}_1\boldsymbol{Q}+\boldsymbol{\Phi}_{21}^T\boldsymbol{G}_2\boldsymbol{Q})+\frac{1}{2}((\boldsymbol{G}_2\boldsymbol{Q})^T\boldsymbol{\Phi}_{22}\boldsymbol{G}_2\boldsymbol{Q}-\boldsymbol{Q}^T\boldsymbol{\Omega}\boldsymbol{Q})-\frac{1}{2}\boldsymbol{P}^T\boldsymbol{\Phi}_{11}\boldsymbol{P}-\boldsymbol{Q}^T\boldsymbol{F}+\boldsymbol{A}_2^T\boldsymbol{G}_2\boldsymbol{Q}\boldsymbol{T}-\boldsymbol{A}_1^T\boldsymbol{P}\boldsymbol{T}\tag{9}$$

将上式写成 $L_{MR}=\boldsymbol{P}_1^TQ_{,z}-H$,则式(6)可写成广义的 Hamilton 变分原理

$$\delta\Pi=\delta\int_v(\boldsymbol{P}_1^TQ_{,z}-H)\mathrm{d}V-\delta\int_{s_\sigma}\overline{T}^TQ\mathrm{d}S-\delta\int_{s_u}T^T(Q-\overline{Q})\mathrm{d}S=0\tag{10}$$

式中 H 是 Hamiltonian 函数.

设广义的应力边界条件和位移边界条件为 $\boldsymbol{T}=\overline{\boldsymbol{T}}$ 和 $\boldsymbol{Q}=\overline{\boldsymbol{Q}}$,以 $\boldsymbol{P}_1$ 和 $\boldsymbol{Q}$ 为相互独立的变量,对式(10)进行变分可得 Hamilton 正则方程(忽略体积力)

$$\frac{\mathrm{d}}{\mathrm{d}z}\begin{Bmatrix}\boldsymbol{P}\\\boldsymbol{Q}\end{Bmatrix}=\begin{bmatrix}\boldsymbol{G}_1^T+\boldsymbol{G}_2^T\boldsymbol{\Phi}_{21} & \boldsymbol{G}_2^T\boldsymbol{\Phi}_{22}\boldsymbol{G}_2\text{-}\boldsymbol{\Omega}\\\boldsymbol{\Phi}_{11} & \text{-}(\boldsymbol{G}_1+\boldsymbol{\Phi}_{21}^T\boldsymbol{G}_2)\end{bmatrix}\begin{Bmatrix}\boldsymbol{P}\\\boldsymbol{D}\end{Bmatrix}+\begin{Bmatrix}\boldsymbol{G}_2^T\boldsymbol{A}_2\\\boldsymbol{A}_1\end{Bmatrix}\boldsymbol{T}\tag{11}$$

式中 ,$\boldsymbol{P}=\boldsymbol{P}_1$,$\boldsymbol{P}$ 和 $\boldsymbol{Q}$ 为互相对偶的列向量.

很显然(11)式为一阶非齐次方程组,在文献[13,14]中,从基本方程中消去广义的平面内应力,也导出了相似的非齐次方程组,但是要想求解具体的稳态问题,还要联立从导热方程和热平衡方程导出的以增量温度 $T(x,y,z)$ 为变量的二阶微分方程.

导热方程和热平衡方程导如下

$$p_x=-k_{11}T_{,x},\quad p_y=-k_{22}T_{,y},\quad p_z=-k_{33}T_{,z},\quad p_{x,x}+p_{y,y}+p_{z,z}=0\tag{12}$$

现以方程(12)为基础结合边界条件导出以增量温度 $T(x,y,z)$ 为变量的二阶微分方程

$$\partial^2T(x,y,z)/\partial z^2-CT(x,y,z)=0\tag{13}$$

由方程(12)得

$$\partial p_z/\partial z=(k_{11}\alpha+k_{22}\beta^2)T\tag{14a}$$

$$\partial T/\partial z=-1/k_{33}p_z\tag{14b}$$

可证明上式中的热流量 $p_z(x,y,z)$ 和 $T(x,y,z)$ 温度增量是对偶变量,因此,可根据对偶关系[20],将式(14a)、(14b)与方程(11)中的 $\boldsymbol{P}$ 和 $\boldsymbol{Q}$ 扩展成统一的对偶列向量得到关于压电热弹性体的齐次方程

$$\mathrm{d}\boldsymbol{R}/\mathrm{d}z=\boldsymbol{D}\boldsymbol{R}\tag{15}$$

式中

$\boldsymbol{R}=[\sigma_{xz}\quad\sigma_{yz}\quad\sigma_{zz}\quad d_z\quad p_z\quad u\quad v\quad w\quad\phi\quad T]^T$

$$D=\begin{bmatrix}A & B\\ C & E\end{bmatrix}$$

$$A=\begin{bmatrix}0 & 0 & \alpha s_8 & \alpha s_{10} & 0\\ 0 & 0 & \beta s_9 & \beta s_{11} & 0\\ -\alpha & -\beta & 0 & 0 & 0\\ \alpha s_6 & \beta s_7 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$B=\begin{bmatrix}B_{11} & B_{12} & 0 & 0 & -\alpha s_{18}\\ B_{21} & B_{22} & 0 & 0 & -\beta s_{19}\\ 0 & 0 & -\rho\omega^2 & 0 & 0\\ 0 & 0 & 0 & B_{44} & 0\\ 0 & 0 & 0 & 0 & B_{55}\end{bmatrix}$$

$$C=\begin{bmatrix}s_1 & 0 & 0 & 0 & 0\\ 0 & s_2 & 0 & 0 & 0\\ 0 & 0 & s_3 & s_4 & 0\\ 0 & 0 & s_4 & s_5 & 0\\ 0 & 0 & 0 & 0 & s_{24}\end{bmatrix}$$

$$E=\begin{bmatrix}0 & 0 & -\alpha & \alpha s_6 & 0\\ 0 & 0 & -\beta & \beta s_7 & 0\\ \alpha s_8 & \beta s_9 & 0 & 0 & s_{20}\\ \alpha s_{10} & \beta s_{11} & 0 & 0 & s_{21}\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$B_{11}=-\alpha^2 s_{12}-\beta^2 s_{15}-\rho\omega^2$,

$B_{12}=-\alpha\beta(s_{13}+s_{15})$,

$B_{21}=-\alpha\beta(s_{13}+s_{15})$,

$B_{22}=-\beta^2 s_{14}-\alpha^2 s_{15}-\rho\omega^2$,

$B_{44}=-\alpha^2 s_{16}-\beta^2 s_{17}$,

$B_{44}=-\alpha^2 s_{22}-\beta^2 s_{23}$.

其中,$s_{(}i=1\sim24)$是与材料参数相关的常数.

2 控制方程的求解

以四边简支矩形压电材料层合板为例,侧面边界条件为:

$$\sigma_x=w=v=\phi=T=0\quad(x=0,a)$$

$$\sigma_y=w=u=\phi=T=0\quad(y=0,b)\tag{16}$$

设层合板任意一层满足边界条件(16)的解的形式如下

$$(\sigma_{xz},u)=\sum_m^\infty\sum_n^\infty(\sigma_{xz}^{mn}(z),u^{mn}(z))\cos(\eta x)\sin(\xi y)e^{\omega t},$$

$$(\sigma_{Yz},v)=\sum_m^\infty\sum_n^\infty(\sigma_{yz}^{mn}(z),v^{mn}(z))\sin(\eta x)\cos(\xi y)e^{\omega t},$$

$$(\sigma_z,d_z,p_z,w,\phi,T)=\sum_m^\infty\sum_n^\infty\psi(z)\sin(\eta x)\sin(\xi y)e^{\omega t},$$

$$\psi(z)=(\sigma_z^{mn}(z),d_z^{mn}(z),p_z^{mn}(z),w^{mn}(z),\phi^{mn}(z),T^{mn}(z))$$

其中

$$\eta=m\pi/a,\xi=n\pi/b$$

将上述解代入式(15),对于任意一对 $m-n$ 都有

$$\mathrm{d}\boldsymbol{R}^{mn}(z)/\mathrm{d}z=\boldsymbol{K}\boldsymbol{R}^{mn}(z)\tag{17}$$

$\boldsymbol{R}^{m}(z)=[\sigma_{xz}^{mn}(z)\ \sigma_{yz}^{mn}(z)\ \sigma_{zz}^{mn}(z)\ d_z^{mn}(z)\ p_z^{mn}(z)\ u^{mn}(z)\ v^{mn}(z)\ w^{mn}(z)\ \phi^{mn}(z)\ T^{mn}(z)]^T$

$$K=\begin{bmatrix}\bar{A} & \bar{B}\\ \bar{C} & \bar{E}\end{bmatrix}$$

$$\bar{A}=\begin{bmatrix}0 & 0 & \zeta s_8 & \zeta s_{10} & 0\\ 0 & 0 & \eta s_9 & \eta s_{11} & 0\\ \zeta & \eta & 0 & 0 & 0\\ -\zeta s_6 & -\eta s_7 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\bar{B}=\begin{bmatrix}\bar{B}_{11} & \bar{B}_{12} & 0 & 0 & -\zeta s_{18}\\ \bar{B}_{21} & \bar{B}_{22} & 0 & 0 & -\eta\beta s_{19}\\ 0 & 0 & -\rho\omega^2 & 0 & 0\\ 0 & 0 & 0 & \bar{B}_{44} & 0\\ 0 & 0 & 0 & 0 & \bar{B}_{55}\end{bmatrix}$$

$$\bar{C}=C$$

$$\bar{E}=\begin{bmatrix}0 & 0 & -\zeta & \zeta s_6 & 0\\ 0 & 0 & -\eta & \eta s_7 & 0\\ -\zeta s_8 & -\eta s_9 & 0 & 0 & s_{20}\\ -\zeta s_{10} & -\eta s_{11} & 0 & 0 & s_{21}\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

根据矩阵理论,方程(17)的精确解为

$$\boldsymbol{R}^{mn}(z)=e^{Kz}\boldsymbol{R}^{mn}(z)\tag{18}$$

对于 n 层的层合板来说,可令式(18)中的 $z=h_j$ 对应于第 j 层的厚度,则有

$$\boldsymbol{R}_j^{mn}(h_j)=\boldsymbol{K}(h_j)\boldsymbol{R}_j^{mn}(h_{j-1})\tag{19}$$

根据层间的连续关系有

$$\boldsymbol{R}_n^{mn}(h_j)=\boldsymbol{T}\boldsymbol{R}_1^{mn}(0)\tag{20}$$

其中 $\boldsymbol{T}=\boldsymbol{K}(h_k)\boldsymbol{K}(h_{k-1})\cdots\boldsymbol{K}(h_1)$,$\boldsymbol{R}_1^{mn}(0)$是第一层外表面($z=0$)的广义位移和广义应力的初始值,由作用在板上下表面的广义外力可求出 $\boldsymbol{R}_1^{mn}(0)$,进而可求得层合板任意空间位置的广义应力和广义位移.

3　实例分析

例 1:首先用静力问题[14]来验证本文的方法及分析程序(在经典解法中仅查到了有关静力学问题的文献),文献[14]给出了 7 层的混合层板 [0/0/90/0/0/90/0_p]算例,压电热弹性材料 PZT－5A 置于最下层,层合板厚度 $h = 1\mathrm{m}$,$a = b = 50\mathrm{m}$;PZT－5A 层的厚度 $h_p = 0.1h$,弹性材料 *Gr/Ep* 层的厚度相同 $h_s = 0.9h/6$;(下标 1 和 2 分别标记层合板上下表面);

工况 1:$\sigma_{zz1} = 0$,　$\sigma_{zz2} = \sin(\zeta x)\sin(\eta y)$,

$\phi_1 = \phi_2 = T_1 = T_2 = 0$

工况 2:$\sigma_{zz1} = \sigma_{zz2} = 0$,　$\phi_1 = \phi_2 = 0, T_1 = 0$,

$T_2 = \sin(\eta y)\sin(\zeta x)$

根据所给工况,选取了 $m = n = 1$ 进行求解,分析结果与比较列于表 1.

表 1　分析结果与比较

Table 1　Result and comparison

Case	1		2	
Parameter	W_{1b}	U_{1b}	W_{2b}	P_{2b}
Present	－641.340	－19.825	－8.0614	482.247
Ref.[8]	－641.342	－20.044	－8.0528	481.211

其中

$W_{1b} = w(a/2,b/2,0)/K_1, U_{1b} = u(0,b/2,0)/K_1$,

$W_{2b} = 10 \times w(a/2,b/2,0)/K_2$,

$P_{2b} = pb(a/2,b/2,0)/K_2$;

K_1, K_2 为无量纲参量且 $K_1 = a/h/10.3 \times 10^9$,

$K_2 = a/h \times 22.5 \times 10^{-6}$.

例 2:分析三层的混合层板[0/0/90],压电热弹性材料 PZT－5A 置于最下层, PZT－5A 层的厚度 $0.1h$,密度 $7500\mathrm{kg/m^3}$;弹性材料 *Gr/Ep* 的厚度相同为 $0.9h/2$,密度 $1500\mathrm{Pkg/m^3}$,层合板厚度 $h = 1\mathrm{m}$,$a = b = 10\mathrm{m}$,$\sigma_{zz1} = \sin(\eta y)\sin(\zeta x)\sin 50t$,取 $m = n = 1$ 求解. 以下工况下标 1、2 分别表示上、下表面.

工况 1:$T_1 = \sin(\eta y)\sin(\zeta x)\sin 50t$,

$\sigma_{xz1} = \sigma_{xz2} = \sigma_{yz1} = \sigma_{yz2} = \sigma_{zz2} = \phi_1 = \phi_2 = T_2 = 0$

工况 2:$T_1 = 2\sin(\eta y)\sin(\zeta x)\sin 50t$,

$\sigma_{xz1} = \sigma_{xz2} = \sigma_{yz1} = \sigma_{yz2} = \sigma_{zz2} = \phi_1 = \phi_2 = T_2 = 0$

工况 3:$T_1 = 3\sin(\eta y)\sin(\zeta x)\sin 50t$,

$\sigma_{xz1} = \sigma_{xz2} = \sigma_{yz1} = \sigma_{yz2} = \sigma_{zz2} = \phi_1 = \phi_2 = T_2 = 0$

层合板上下表面和中性面的部分广义应力和广义位移的幅值分别列于表 2 和表 3.

表 2　层合板上下表面广义位移和应力值

Table 2　Generalized stress and displacement values of laminated plate top/bottom surface

Parameter	Case1	Case 2	Case 3
u_1	$-2.3509E-6$	$-4.70036E-6$	$-7.05013E-6$
v_1	$-7.64729E-6$	$-1.52929E-5$	$-2.29385E-5$
w_1	$-2.5906E-5$	$-5.18025E-5$	$-7.7699E-5$
Φ_1	$2.24991E-5$	$4.49981E-5$	$6.74971E-5$
T_1	0.694127	1.38825	2.08238
u_2	$3.34686E-6$	$6.69177E-6$	$1.00367E-5$
v_2	$-3.31795E-7$	$-6.64433E-7$	$-9.97071E-7$
w_2	$-1.74663E-5$	$-3.4923E-5$	$-5.23797E-5$
Φ_2	$2.43459E-5$	$4.86917E-5$	$7.30374E-5$
T_2	0.774945	1.54989	2.32483

表 3　层合板中性面广义位移和应力值

Table 3　Generalized stress and displacement values of laminated plate neuter surface

Parameter	Case 1	Case 2	Case 3
u_m	$-4.96879E3$	$-9.93591E3$	$-1.4903E4$
v_m	$9.31433E3$	$1.86311E4$	$2.79478E4$
w_m	$-3.58398E2$	$-7.1723E2$	$-1.07606E3$
Φ_m	$1.2196E3$	$2.43921E3$	$3.65882E3$
T_m	0.268291	0.536583	0.804874
σ_{xzm}	$8.24077E-7$	$1.6476E-6$	$2.47112E-6$
σ_{yzm}	$-3.54283E-6$	$-7.08516E-6$	$-1.06275E-5$
σ_{zzm}	$-1.9028E-5$	$-3.80465E-5$	$-5.70649E-5$
D_{zm}	$2.37322E-5$	$4.74642E-5$	$7.11962E-5$
p_{zm}	0.757234	1.51447	2.2717

部分广义平面外应力幅值和广义位移幅值沿厚度方向的变化趋势列于图 1-图 10(为温度增量):

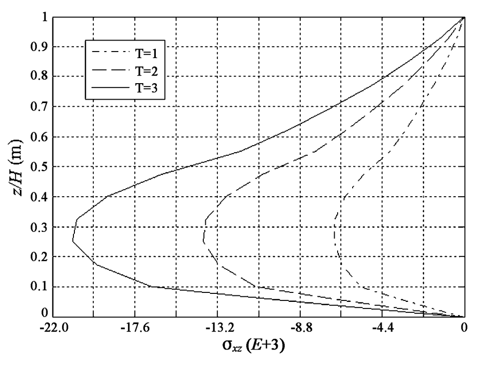

图 1　σ_{xz}沿厚度方向的分布

Fig. 1　σ_{xz} distribution along thickness

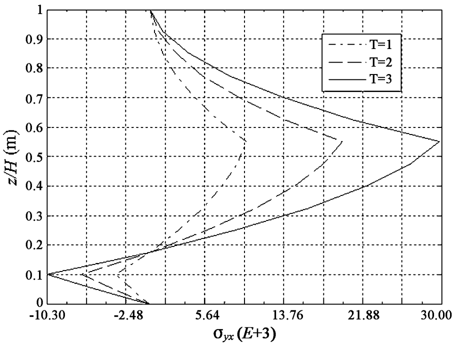

图 2　σ_{yz}沿厚度方向的分布

Fig. 2　σ_{yz} distribution along thickness

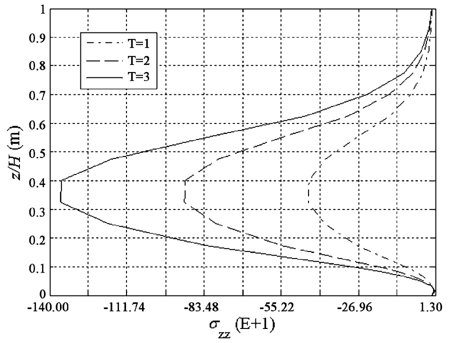

图 3　σ_{zz}沿厚度方向的分布

Fig. 3　σ_{zz} distribution along thickness

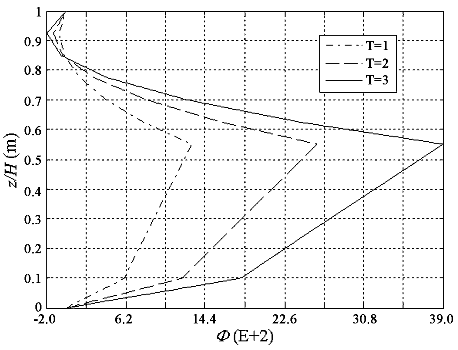

图 4　ϕ 沿厚度方向的分布

Fig. 4　ϕ distribution along thickness

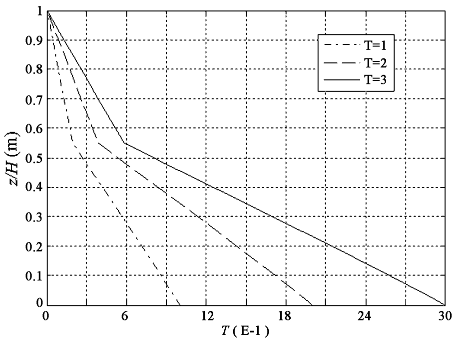

图 5　T 沿厚度方向的分布

Fig. 5　T distribution along thickness

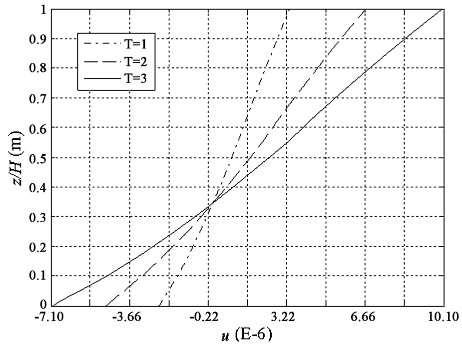

图 6　u 沿厚度方向的分布

Fig. 6　u distribution along thickness

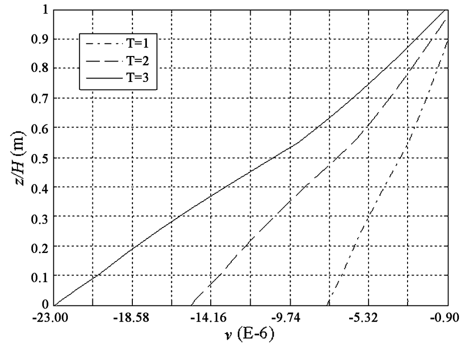

图 7　v 沿厚度方向的分布

Fig. 7　v distribution along thickness

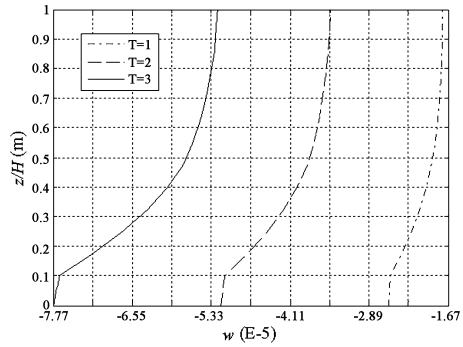

图 8　w 沿厚度方向的分布

Fig. 8　w distribution along thickness

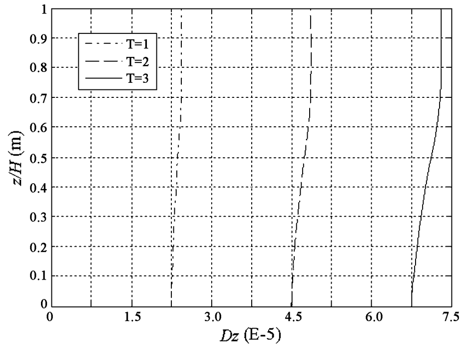

图 9　D_z 沿厚度方向的分布

Fig. 9　D_z distribution along thickness

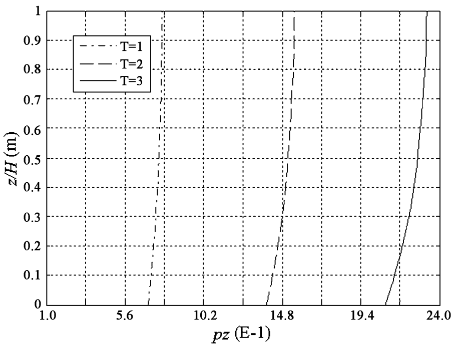

图 10　p_z 沿厚度方向的分布

Fig. 10　p_z distribution along thickness

图 11 给出了温度增量 $T=1$ 时中性面上广义应力 σ_{xz}；图 12 给出了温度增量 $T=1$ 时中性面上位移 u 随时间的变化趋势,其它平面外应力 – 时间和广义位移 – 时间关系与此类似.

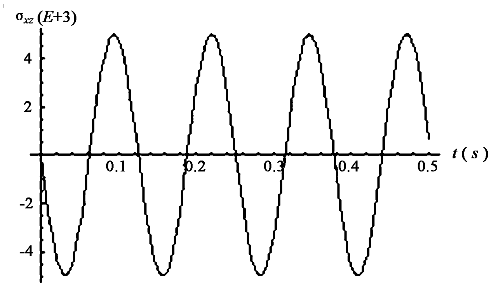

图 11　中性面处应力 σ_{xz}

Fig. 11　neuter surface stress σ_{xz}

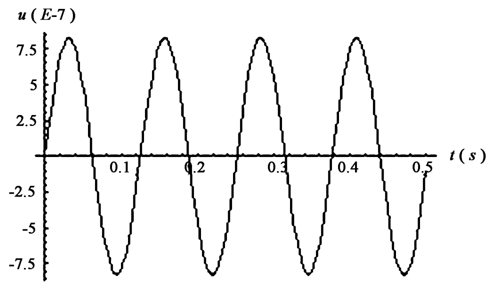

图 12　中性面处位移 u

Fig. 12　neuter surface displacement u

4　结束语

本文联立导热方程和热平衡方程并结合边界条件,根据对偶关系的统一性将非齐次方程扩展成齐次控制方程,从而大大减少了运算工作量. 具体实例中分析了压电热弹性材料四边简支层合板在温度载荷和动力载荷作用下的响应的精确解问题,为建立压电热弹性材料板的近似数值方法提供了参考依据.

对于一般的复合材料,如果也考虑温度梯度问题,也能导出类似的齐次方程.

参　考　文　献

1　Dunn M. L. Green's functions for transversely isotropic piezoelectric solids. *International Journal Solids and Structures*, 1996, 33:4571～4581

2　Ray M. C., Rao. K. M. and Samanta. B. Exact analysis of coupled electroelastic behavior of a piezoelectric plate under cylindrical bending. *Computers & Structures*, 1993, 47:1031～1042

3　Ray Mc, Bhattacharya R, Samanta B. Exact solutions for static analysis of intelligent structures. *AIAA Journal*, 1993, 31:1684～1691

4　Heyliger P., Brooks S. Exact solutions for laminated piezoelectric plates in cylindrical bending. *Journal of Applied Mechanics*, 1996, 63:903～910

5　Heyliger P., Brooks S. Free vibration of piezoelectric laminates in cylindrical bending. *International Journal Solids and Structures*, 1995, 32:2945～2960

6　Pagano N. J. Exact solutions for rectangular bidirectional composites and sandwich plates. *Journal of Composite Material*, 1970, 4:20～24

7　范家让. 强厚度叠层板壳的精确理论(第 2 版). 北京:科学出版社, 1998 (Fan Jiarang. Exact theory of thick laminated plate and shell. Beijing: Science Publishers, 1996 (in chinese))

8　丁皓江,陈伟球,徐荣桥. 压电板壳自由振动的三维精确解分析. 力学季刊, 2001, 2:1～9 (Ding Haojiang, Chen Weiqiu, Xu Rongqiao. Three-dimensional exact solution analysis of piezoelectric plate and shell for free vibration. *Chinese Quarterly of Mechanics*, 2001, 2:1～9 (in chinese))

9　盛宏玉,张伟林,高荣誉. 一般边界条件下压电层合厚板的精确解. 安徽建筑工业学院学报, 2002, 4:1～6 (Sheng Hongyu, Zhang Weilin, Gao Rongyu. Exact solution of thick laminated piezoelectric plate under general boundary conditions. *Journal of Anhui Institute of Architecture*, 2002, 10(4):1～6. (in chinese))

10　Mindlin R. D. Equations of high frequency vibrations of thermopiezoelectric crystal plates. *International Journal Solids and Structures*, 1974, 1: 625 ~ 632

11 Zhang. C., Cheung Y. K., Di S., Zhang N. The exact solution of coupled thermoelectroelastic behavior of piezoelectric laminates. *Computers and Structures*, 2002, 80: 1201 ~ 1202

12 Vel S. S., Batra R. C. Generalized plane strain thermopiezoelectric analysis of multilayered plates. *Journal of Thermal Stresses*, 2003, 26: 353 ~ 377

13 Ootao Y., Tanigawa Y. Control of transient thermoelastic displacement of a two-layered composite plate constructed of isotropic elastic and piezoelectric layers due to nonuniform heating. *Archive of Applied Mechanics*, 2001, 71: 207 ~ 230

14 Kapuria S., Dumir P. C. and Sengupta S. Three-dimensional solution for shape control of a simply supported rectangular hybrid plate. *Journal of Thermal Stresses*, 1999, 22: 159 ~ 176

15 Steele C. R. and Kim Y Y. Modified mixed variational principle and the state-vector equation for elastic bodies and shells of revolution. *Journal of Applied Mechanics*, 1992, 59: 587 ~ 598

16 邹贵平. 层合板壳问题哈密顿体系与哈密顿型广义变分原理. 上海大学学报(自然科学版), 1996, 2: 119 ~ 128 (Zou Guiping. Hamilton system for laminated plate and shell problems and generalized Hamilton variation principle. *Journal of Shanghai University (Natural Science)*, 1996, 2: 119 ~ 128 (in chinese))

17 卿光辉, 邱家俊, 塔娜. 压电体的变分原理及叠层板的自由振动分析. 振动工程学报, 2004, 17: 285 ~ 290 (Qing Guanghui, Qiu Jiajun, Ta Na. The variation principle of piezoelectric bodies and free vibration analysis of laminated plate. *Journal of Vibration Engineering*, 2004, 17: 285 ~ 290 (in chinese))

18 Qing Guanghui, Qiu Jiaujun, Liu YanHong. A semi-analytical solution for dynamic analysis of plate with piezoelectric patches. *International Journal of Solids and Structures*, 2006, 43: 1388 ~ 1403

19 Qing Guanghui, Qiu Jiaujun, Liu YanHong. Free vibration analysis of stiffened laminated plates. *International Journal of Solids and Structures*, 2006, 43: 1357 ~ 1371

20 钟万勰著. 应用力学对偶体系. 北京: 科学出版社, 2003 (Zhong Wanxie. Symplectic system of applied mechanics. Beijing: Science Publishers, 2003 (in chinese))

RESPONSE ANALYSIS OF HYBRID LAMINATED PIEZOTHERMOELASTIC PLATE *

Jia Libin Qing Guanghui Liu Yanhong Du Hongzeng

(*Aviation Engineering College, Civil Aviation University of China, Tianjin* 300300, *China*)

Abstract Based upon the generalized Hamilton variation principle, the non – homogeneous Hamilton canonical equation for piezothermoelastic bodies was derived. Considering the symplectic relations of variations in the thermal equilibrium formulations and gradient equations, the non – homogeneous Hamilton canonical equation was transformed into homogeneous equation for solving independently the coupling problem of piezothermoelastic bodies by increasing the dimensions of the canonical equation. The treatment, by which the non – homogeneous equation was transformed into homogeneous equation, not only simplifies greatly the solution programs, but also reduces the workload of numerical computation. The numerical response of the laminated piezothermoelastic plate with four simply supported edges under thermal and dynamic load was studied. Part of examples was compared with relative reference.

Key words piezothermoelastic body, Hamilton canonical equation, homogeneous equation, laminated plate

Received 6 March 2007, revised 15 March 2007.

* The project supported by the Natural Sciene Foundation of Tianjin (07JCYBC02100)