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强非线性多自由度自治系统的内共振*

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摘要　基于改进的 KBM 法,研究了强非线性多自由度自治系统的内共振. 求出了极限环的振幅和近似解的表达式. 与 KBM 法比较,该方法的特点是:近似解中包含项中的不再是时间的线性函数,而是时间的非线性函数,它能提高近似解的精度,且应用更广. 最后给出一个具体实例,得到了近似解以及相图. 和数值结果比较,本文方法具有较高的精度.

关键词　强非线性多自由度自治系统,内共振,近似解

引 言

本文研究强非线性多自由度自治系统的内共振情况,其运动方程为

$$\ddot{x}_i + g_i(x_i) = \varepsilon f_i(x_1, x_2 \cdots, x_n, \dot{x}_1, \dot{x}_2, \cdots \dot{x}_n)$$
$$i = 1, 2, \cdots, n \tag{1}$$

其中 g_i 和 f_i 均为非线性函数,ε 为小参数,$0 < \varepsilon < < 1$,这个系统能够反映许多物理现象. 对于弱非线性振动,形如

$$g_i(x_i) = \omega_i^2 x_i, \quad i = 1, 2 \cdots, n \tag{2}$$

已有许多研究方法[1,2,3],对于强非线性系统(1),则研究得较少[4,5,6],主要原因是没有简单有效的分析方法. 本章将应用推广的平均化方法找出系统(1)的一个近似解,这种方法的优点是计算简单,便于应用.

1　未扰动周期解

先求未扰动方程的解,即 $\varepsilon = 0$,方程(1)化为

$$\ddot{x}_i + g_i(x_i) = 0 \quad i = 1, 2, \cdots, n \tag{3}$$

将上式对 $\dot{x}_i$ 积分,得

$$\frac{1}{2}\dot{x}_i^2 + v_i(x_i) = E_i \tag{4}$$

其中

$$v_i(x_i) = \int_0^{x} g_i(\xi)\,\mathrm{d}\xi \tag{5}$$

E_i 为积分常数.

方程(3)的周期解可写成

$$x_i(t) = a_i \cos\varphi_i + b_i \tag{6}$$

其中 φ_i 为 t 的函数,a_i 为振幅,b_i 为偏心距,且有

$$a_i + b_i = r_i, \qquad -a_i + b_i = s_i \tag{7}$$

现将 $\dot{x}_i = -a_i(d\varphi_i/dt)\sin\varphi_i$ 和方程(6)代入方程(4)得

$$\frac{1}{2}[a_i(d\varphi_i/dt)\sin\varphi_i]^2 + v_i(a_i\cos\varphi_i + b_i) = E_i \tag{8}$$

当 $\varphi_i = \pi$ 和 2π 时系统的势能相等,动能为零,分别代入上式得

$$v_i(-a_i + b_i) = v_i(a_i + b_i) = E_i \tag{9}$$

因此

$$\frac{d\varphi_i}{dt} = -\sqrt{\frac{2[v_i(a_i + b_i) - v_i(a_i\cos\varphi_i + b_i)]}{a_i^2\sin^2\varphi_i}} = \Phi_i(a_i, \varphi_i) \tag{10}$$

并有

$$\Phi_i(a_i, 0) = \lim_{\varphi_i \to \infty}\Phi_i(a_i, \varphi_i) = \sqrt{\frac{g_i(a_i + b_i)}{a_i}}$$
$$\Phi_i(a_i, \pi) = \lim_{\varphi_i \to \pi}\Phi_i(a_i, \varphi_i) = \sqrt{-\frac{g_i(-a_i + b_i)}{a_i}} \tag{11}$$

从上两式可知,条件(4)必须被满足. 从方程(10)可看出,$\Phi_i^{-1}(a_i, \varphi_i)$ 为偶函数,故将其展开成 Fourier 级数时,不包括 $\sin\varphi_i$ 项.

$$\Phi_i^{-1}(a_i, \varphi_i) = c_{i0}(a_i) + \sum_{n=1}^{m} c_{in}(a_i)\cos\varphi_i + R_m \tag{12}$$

其中 m 为正整数且 R_m 表示高阶项,未扰动方程(3)的周期解的周期 $T_i(a_i)$,可将方程(12)从 0 到 π 积分可得

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$$T_i(a_i)=2\pi c_{i0}(a_i) \tag{13}$$

因此其频率为

$$\Omega_i(a_i)=1/c_{i0}(a_i) \qquad i=1,2,\cdots n \tag{14}$$

2 求系统周期解的平均法

下面用平均化方法找出系统(1)的一个近似解,考虑主共振情况,设.

$$\Omega_1(a_1):\Omega_2(a_2):\cdots\Omega_n(a_n)\approx k_1:k_2\cdots:k_n \tag{15}$$

其中 $k_1,k_2,\cdots,k_n$ 是没有公约数的正整数,下面假设

$$\Omega_i^2(a_i)=c^2k_i^2+\varepsilon\sigma_i \qquad i=1,2,\cdots,n \tag{16}$$

其中 σ_i 为解谐参数,并设正整数 k_1 为奇数. 引入新的时间变量 $\tau=ct$,方程(15)化为

$$\frac{d^2x_i}{d\tau^2}+g_i(x_i)=\varepsilon\tilde{f}_i(x_1,x_2,\cdots,x_n,\tau\frac{dx_1}{d\tau},\tau\frac{dx_2}{d\tau},\cdots,\tau\frac{dx_n}{d\tau}) \tag{17}$$

其中

$$\tilde{f}(x_1,\cdots,x_n,\tau\frac{dx_1}{d\tau},\cdots,\tau\frac{dx_n}{d\tau})=\frac{1}{c^2}[f_i(x_1,x_2,\cdots,x_n,\dot{x}_1,\dot{x}_2,\cdots,\dot{x}_n,)-\sigma x_i] \tag{18}$$

类似非共振情形的 KBM 法,把方程(1)的解写成如下形式

$$x_1=a_1\cos k_1\varphi+b_1+\varepsilon x_{11}(a_1,a_2,\cdots,a_n,\theta_1,\theta_2,\cdots,\theta_n)+\cdots \tag{19}$$

$$x_s=a_s\cos(k_s\varphi+\theta_s)+b_s+\varepsilon x_{s1}(a_1,a_2,\cdots,a_n,\theta_1,\theta_2,\cdots,\theta_n,\varphi)+\cdots \tag{20}$$

其中 x_{1k} 与 φ 无关,而 x_{sk} 是 φ 的以 2π 为周期的函数,$a_i\neq 0$($S=2,3,\cdots,n,k=1,2,\cdots,n,i=1,2,\cdots,n$)假设 a_i,θ_s 和 φ 为 τ 的函数,满足以下方程

$$\frac{da_i}{d\tau}=\varepsilon A_{i1}(a_1,a_2,\cdots,a_n,\theta_1,\theta_2,\cdots,\theta_n)+\varepsilon A_{i2}+\cdots \tag{21}$$

$$\frac{d\theta_s}{d\tau}=\varepsilon\Theta_{s1}(a_1,a_2,\cdots,a_n,\theta_1,\theta_2,\cdots,\theta_n)+\varepsilon^2\Theta_{s2}+\cdots \tag{22}$$

$$\frac{d\varphi}{d\tau}=1+\varepsilon\Phi_1(a_1,a_2,\cdots,a_n,\theta_1,\theta_2,\cdots,\theta_n)+\varepsilon^2\Phi_2+\cdots \tag{23}$$

其中 $\Phi_k(k=1,2,\cdots)$ 是 φ 的以 2π 为周期的函数,当 a_i 和 θ_s 为常数时,式(19),(20)的 φ 不再是时间的线性函数,而是时间的非线性函数,这是本方法与 KBM 法的重要差别,它能够提高近似解的精度,且应用更广.

将(21)–(23)代入(1)并且令两边 ε 的同次幂系数相等,得到 ε 阶项为

$$a_1k_1\frac{\partial}{\partial\phi}(\phi_1\sin^2k_1\varphi)=-\tilde{f}_1(x_{10},x_{20},\cdots,x_{n0},x_{10}',x_{20}',\cdots,x_{n0}')\sin k_1\varphi-2k_1A_{11}\sin^2k_1\varphi+x_{11}k_1^2\sin k_1\varphi \tag{24}$$

$$\frac{\partial^2x_{s1}}{\partial\varphi_2}+g_{ix_i}'x_{s1}=\tilde{f}_1(x_{10},x_{20},\cdots,x_{n0},x_{10}',x_{20}',\cdots,x_{n0}')+2k_sA_{s1}\sin(k_s\varphi+\theta_s)+2k_sa_s\Theta_{s1}\cos(k_s\varphi+\theta_s)+2a_sk_s^2\phi_1\cos(k_s\varphi+\theta_s)+a_sk_s\frac{\partial\phi_1}{\partial\varphi}\sin(k_s\varphi+\theta_s) \tag{25}$$

下面先确定 x_{11},A_{11} 和 ϕ,对方程(24)积分得

$$a_1k_1\Phi_1\sin^2k_1\varphi=-\int_0^{\varphi}\tilde{f}_1(x_{10},x_{20},\cdots,x_{n0},x_{10}',x_{20}',\cdots,x_{n0}')\sin k_1\varphi d\varphi. \tag{26}$$

按假设条件 k_1 为奇数,$\int_0^{\pi}g_{ix_1}'(x_{10})\sin k_1\varphi d\varphi=-\frac{1}{a_1}[g_1(-a_1+b_1)-g_1(a_1+b_1)]$,令 $\varphi=2\pi,\pi$ 分别代入式(26)得

$$A_{11}=-\frac{1}{2\pi k_1}\int_0^{2\pi}\tilde{f}_1(x_{10},x_{20},\cdots,x_{n0},x_{10}',x_{20}',\cdots,x_{n0}')\sin k_1\varphi d\varphi. \tag{27}$$

$$x_{11}=-\frac{1}{a_1}[g_1(-a_1+b_1)-g_1(a_1+b_1)]\times\int_0^{\pi}[\tilde{f}_1(x_{10},x_{20},\cdots,x_{n0},x_{10}',x_{20}',\cdots,x_{n0}')+2k_1A_{11}\sin k_1\varphi]\sin k_1\varphi d\varphi \tag{28}$$

再由式(26)和以上两式便可以确定 Φ_1.

下面从式(23)来确 $x_{s1}(a_1,a_2,\cdots,a_n,\theta_2,\cdots,\theta_n,\varphi)$,$A_{s1}(a_1,a_2,\cdots,a_n,\theta_2,\cdots,\theta_n)$ 和 $\Theta_{s1}(a_1,a_2,\cdots,a_n,\theta_2,\cdots,\theta_n)$ 为了使 x_{s1} 是 φ 的周期函数,方程组(26)右端必须不含 $\cos k_s\varphi$ 和 $\sin k_s\varphi$ 项. 由此得

$$A_{s1}=-\frac{1}{2\pi k}\int_0^{2\pi}\hat{f}_s(a,\theta,\varphi)\sin(k_s\varphi+\theta_n)d\varphi \tag{29}$$

$$\Theta_{s1}=-\frac{1}{2\pi a_sk_s}\int_0^{2\pi}\hat{f}_s(a,\theta,\varphi)\cos(k_s\varphi+\theta_s)d\varphi \tag{30}$$

$$x_{s1}=\frac{C_{s0}}{-\frac{1}{a}[g_s(-a_s+b_s)-g_s(a_s+b_s)]}+\sum_{\substack{n=1\\n\neq k_s}}^{\infty}\frac{1}{k_s^2-n^2}(C_{sn}\cos n\varphi+D_{sn}\sin\varphi) \tag{31}$$

其中

$$\hat{f}_s(a,\theta,\varphi) = \tilde{f}_s(x_{10},x_{20},\cdots,x_{n0},x'_{10},x'_{20},\cdots,x'_{n0})$$
$$+2a_sk_s^2\Phi_1\cos(k_s\varphi+\theta_s)+a_sk_s\frac{\partial\Phi_1}{\partial\varphi}\sin(k_s+\theta_s) \quad (32)$$

$$C_{sn}=\frac{1}{2\pi}\int_0^{2\pi}\hat{f}_s(a,\theta,\varphi)\cos n\varphi d\varphi \quad (33)$$

$$D_{sn}=\frac{1}{2\pi}\int_0^{2\pi}\hat{f}_s(a,\theta,\varphi)\sin n\varphi d\varphi \quad (34)$$

类似地可求 $s=2,3,\cdots$ 时的高阶近似解.

为了使近似解(19),(20)成为周期解,式(21),(22)必须有平衡点,略去 ε^2 以上的项,在(21),(22)中令 $da_i/d\tau$ 和 $d\theta_s/d\tau=0$,于是求周期解的问题便化为求解 $2n-1$ 个函数方程

$$\begin{aligned}A_{i1}&=0\\ \Theta_{i1}&=0\end{aligned} \quad (i=1,2,\cdots,n \quad s=2,3,\cdots,n) \quad (35)$$

并注意到(16)式,便可将全部未知量求出.

3 应用举例

作为例子,研究 Van der pol 振子

$$\frac{d^2x_1}{dt^2}+\alpha_1x_1+\beta_1x_1^3=\varepsilon(1-x_1^2)\frac{dx_1}{dt}+\varepsilon\mu_1x_2$$
$$\frac{d^2x_2}{dt^2}+\alpha_2x_2+\beta_2x_2^3=\varepsilon(1-x_2^2)\frac{dx_2}{dt}+\varepsilon\mu_2x_1 \quad (36)$$

由式(9)知,$b_i=0$,由式(10),(14)知

$$\begin{aligned}\Omega_1(a_1)&\approx\frac{1}{2}(4\alpha_1+3\beta_1a_1^2)^{\frac{1}{2}}\\ \Omega_2(a_2)&\approx\frac{1}{2}(4\alpha_2+3\beta_1a_2^2)^{\frac{1}{2}}\end{aligned} \quad (37)$$

由式(27)~(34),通过简单积分便可得

$$x_{11}=0$$
$$\Phi_1=\frac{\mu}{2a_1c^2}(a_1-a_2\cos\theta_2)-\frac{1}{8c_1}a_1^2\sin2\varphi$$
$$A_{11}=\frac{1}{2c^2}[c(a_1-\frac{1}{4}a_1^3)+\mu_2a_2\sin\theta_2]$$
$$A_{21}=\frac{1}{2c^2}[c(a_2-\frac{1}{4}a_2^3)-\mu_1a_1\sin\theta_2]$$
$$\Theta_{21}=\frac{1}{2c^2}[\mu_1+\mu_2-\mu_2(\frac{a_1}{a_2}-\frac{a_2}{a_1})\cos\theta_2]$$
$$x_{21}=\frac{1}{32c^2}a_2[a_1^2\sin(3\varphi+\theta_2)-a_2^2\sin(3\varphi+3\theta_2)] \quad (38)$$

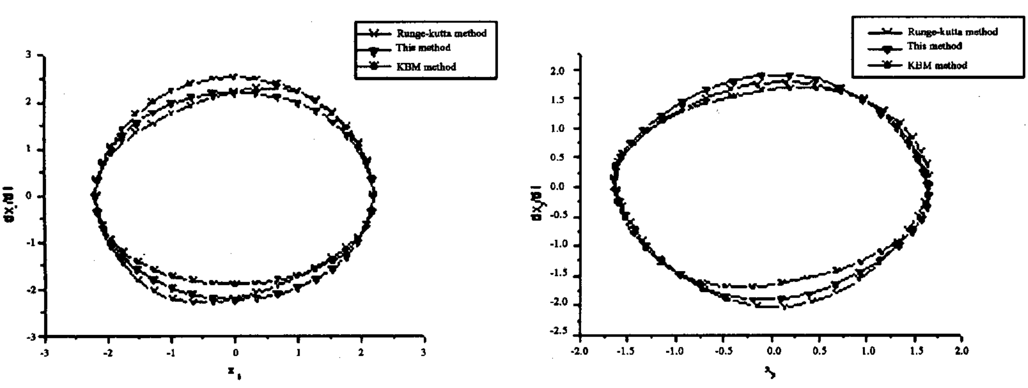
现取参数 $\alpha_1=0.54$,$\beta_1=0.3$,$\alpha_2=2.56$,$\beta_2=0.3$,$\mu_1=0.4$,$\mu_2=0.56$,$\varepsilon=0.15$,则求得其平衡点为

$$a_1^0=2.1820 \quad a_2^0=1.7034 \quad \theta_2^0=0.5608 \quad (39)$$

对应于此平衡点的周期解为

$$x_1=a_1^0\cos\varphi+\varepsilon x_{11}=2.1820\cos\varphi$$
$$\dot{x}_1=-a_1^0(1+\varepsilon\varphi_1)\sin\varphi=-2.1820(1.0165-0.1462\sin\varphi)\sin\varphi$$
$$x_2=a_2^0\cos(\varphi+\theta_2^0)+\varepsilon x_{21}=1.7034\cos(\varphi+0.5608)+0.0624\sin(3\varphi+0.5608)-0.039\sin(3\varphi+1.678)$$
$$\dot{x}_2=[-1.7034\sin(\varphi+0.5608)+0.1872\cos(3\varphi+0.5608)-0.1173\cos(3\varphi+1.6780)](1.0165-0.146\sin2\varphi) \quad (40)$$

由式(40)作出的相图如下,图中还给出了用数值方法和 KBM 法求解的结果作为比较,本文方法具有较高的精度.



(a) phase – space trajectories (b) phase – space trajectories

图 1 系统相图

Fig. 1 phase – space trajectories of the system

参 考 文 献

1 D W stortiard,R H Rand. Dynamics of two strongly coupled van der pol scillators. *International Journal of non - linear Mechanics* ,1982,17 :143 ~ 150.

2 S L Lan and Z xu. On internal resonance of nonlinear vibrating systems with many degrees of freedom. *Applied Mathematics and Mechanics*,1992,13 :29 ~ 37.

3 刘世龄,徐兆. 多自由度非线性振动系统的内共振. 应用数学和力学,1992,13(1) :27 ~ 34 (S. L Lau and Z Xu. On internal resonance of nonlinear vibrating systems with many degree of freedom. *Applied mathematics and mechanics*,1992,13(1) :27 ~ 34 (in Chinese))

4 梁建术,陈予恕. 一类含平方、立方非线性性质项 1 :2 内共振两自由度系统得全局分叉. 应用力学学报,2002,19(4) :62 ~ 67. (Liang Jianshu,Chen Yushu. Global bifurcations in a class of two - degree of freedom systems with quadratic and cubic nonlinearity and 1 :2internal resonance. *Journal of applied mechanics*,2002,19(4) :62 ~ 67 (in Chinese))

5 李欣业,陈予恕. 两自由度内共振系统非线性模态的分叉特性. 力学学报,2002,34(3) :401 ~ 407 (Lin Xinye, Chen Yushu. Bifurcation of nonlinear normal modes of multi - degree of freedom systems with internal resonance. *ACTA Mechanica Sinica*, 2002,34(3) :401 ~ 407(in Chinese))

6 唐驾时. 求强非线性系统次谐共振解的 MLP 方法. 应用数学与力学,2000,21(10) :1039 ~ 1045 (Tang Jiashi. The MLP methpd for subharmonic and ultraharmonic resonance solution of strongly nonlinear systems. *Applied Mathematics and Mechanics*,2000,21(10) :1039 ~ 1045)

7 孙中奎, 徐伟, 杨晓丽. 求解强非线性动力系统响应的一种新方法. 动力学与控制学报, 2005, 3(2) :29 ~ 35 (Sun zhong kui, Xu wei, Yang xiao li. A new analytic approximate technique for strongly nonlinear dynamic systems. *Journal of Dynamics and Control*, 2005,3(2) :29 ~ 35) (in Chinese))

INTERNAL RESONANCE OF STRONGLY NONLINEAR AUTONOMOUS SYSTEMS WITH MULTI - DEGREES OF FREEDOM *

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Abstract The internal resonance of strongly non - linear autonomous systems with multi - degrees of freedom was analyzed on the basis of modifying the KBM method, and the amplitude of limit cycles and the approximate solution were obtained. Compared with KBM method, the characteristic of the present method was that the term included in the approximate solution was a nonlinear function of time instead of a linear function, which could increase the accuracy and be used extensively. An example was given, whose approximate solution and phase - space trajectories were obtained. The results computed by this method were in pretty good agreement with the numerical results, and the accuracy of the present method was very good.

Key words strongly non - linear autonomous systems with multi - degrees of freedom, internal resonance, approximate solution

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