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广义 Chaplygin 系统的形式不变性与 Noether 对称性*

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摘要 研究广义 Chaplygin 系统的形式不变性,利用广义 Chaplygin 方程在无限小变换下的变形形式,给出广义 Chaplygin 系统的形式不变性的定义和判据.给出了广义 Chaplygin 系统的 Noether 对称性判据,并研究形式不变性和 Noether 对称性的关系.结果表明形式不变性与 Noether 对称性是两种不同的对称性,广义 Chaplygin 方程的形式不变性有可能是 Noether 对称性,也可能不是 Noether 对称性.最后举例说明结果的应用.

关键词 广义 Chaplygin 系统,形式不变性,Noether 对称性,守恒量

引言

力学系统对称性与守恒量的研究在数学、力学、物理学中都有非常重要的意义.力学系统的对称性与守恒量的近代理论主要有 Noether 对称性理论和 Lie 对称性理论[1,2].形式不变性是不同于 Noether 对称性和 Lie 对称性的一种新的对称性理论,指运动方程中出现的动力学函数在经无限小变换后仍满足原来的方程.近年来,国内学者在形式不变性这方面做了大量的工作,梅凤翔、葛伟宽、李仁杰、乔永芬、方建会、罗绍凯等人已研究了形式不变性在 Chaplygin 系统[3,4]、Appell 方程[5]、Hamilton 正则方程[6]、Nielsen 方程[7]、Birkhorff 系统[8~12]等方面的应用.本文研究广义 Chaplygin 系统的形式不变性.给出广义 Chaplygin 系统的形式不变性的定义和判据,研究形式不变性和 Noether 对称性的关系,并举例说明结果的应用.

1 广义 Chaplygin 方程

假设力学系统的位形由 n 个广义坐标表示 $q_s(s=1,2,\cdots,n)$ 来确定,系统的运动受有 g 个理想 Chetaev 型非完整约束

$$\dot q_{\varepsilon+\beta}=\varphi_\beta(q_\sigma,\dot q_\sigma,t),(\beta=1,\cdots,g;\ \varepsilon=n-g;\sigma=1,\cdots,\varepsilon)\tag{1}$$

其中函数 $\varphi_\beta\in C^2$ 类.系统的动能不显含与不独立广义速度相应的广义坐标 $q_{\varepsilon+\beta}(\beta=1,\cdots,g)$,即有形式

$$T=T(q_\sigma,\dot q_s,t)\tag{2}$$

并设广义力不含 $q_{\varepsilon+\beta}$,即有形式

$$Q_s=Q_s(q_\sigma,\dot q_k,t),(s,k=1,\cdots,n)\tag{3}$$

则系统的运动方程为

$$\frac{\mathrm d}{\mathrm dt}\frac{\partial\tilde L}{\partial\dot q_\sigma}-\frac{\partial\tilde L}{\partial q_\sigma}-\frac{\partial L}{\partial\dot q_{\varepsilon+\beta}}\left(\frac{\mathrm d}{\mathrm dt}\frac{\partial\varphi_\beta}{\partial\dot q_\sigma}-\frac{\partial\varphi_\beta}{\partial q_\sigma}\right)=\tilde Q_\sigma,(\sigma=1,\cdots,n)\tag{4}$$

这儿及以后相同指标表示求和.式中 $L=T-V$ 为系统的 Lagrange 函数

$$\tilde L=\tilde L(q_\sigma,\dot q_\sigma,t)=L(q_\sigma,\dot q_\sigma,\varphi_\beta(q_\sigma,\dot q_\sigma,t),t)\tag{5}$$

$$\tilde Q_\sigma=\tilde Q_\sigma(q_\sigma,\dot q_\sigma,t)=Q_\sigma(q_\sigma,\dot q_\sigma,\varphi_\beta,t)+Q_{\varepsilon+\beta}(q_\sigma,\dot q_\sigma,\varphi_\beta,t)\frac{\partial\varphi_\beta}{\partial\dot q_\sigma}\tag{6}$$

式(4)称为广义 Chaplygin 方程.

2 广义 Chaplygin 系统的形式不变性

2.1 无限小群变换

取时间和广义坐标的无限小群变换

$$t^*=t+\Delta t,q_s^*(t^*)=q_s(t)+\Delta q_s$$

或其展开式

$$t^*=t+\varepsilon\xi_0(q,\dot q,t),\quad q_s^*=q_s+\varepsilon\xi_s(q,\dot q,t)\tag{7}$$

其中 ε 为无限小参数,ξ_0,ξ_s 为无限小生成元.

2.2 形式不变性的定义和判据

假设广义 Chaplygin 方程(4)中的函数 $L,\tilde L$,

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φ_β，$\widetilde{Q}_\sigma$，经历无限小变换(7)后变为 L^*，$\widetilde{L}^*$，φ_β^*，$\widetilde{Q}_\sigma^*$，即

$$L^* = L\left(t^*, q_\sigma^*, \frac{\mathrm{d}q_s^*}{\mathrm{d}t^*}\right) = L(t, q, \dot{q}) + \varepsilon\left(\frac{\partial L}{\partial t}\xi_0 + \frac{\partial L}{\partial q_v}\xi_v + \frac{\partial L}{\partial \dot{q}_v}(\dot{\xi}_v - \dot{q}_v\dot{\xi}_0) + \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}(\dot{\xi}_{\varepsilon+\beta} - \dot{q}_{\varepsilon+\beta}\dot{\xi}_0)\right) + 0(\varepsilon^2)$$

$$\widetilde{L}^* = \widetilde{L}\left(t^*, q_\sigma^*, \frac{\mathrm{d}q_\sigma^*}{\mathrm{d}t^*}\right) = \widetilde{L}(t, q, \dot{q}) + \varepsilon\left(\frac{\partial \widetilde{L}}{\partial t}\xi_0 + \frac{\partial \widetilde{L}}{\partial q_v}\xi_v + \frac{\partial \widetilde{L}}{\partial \dot{q}_v}(\dot{\xi}_v - \dot{q}_v\dot{\xi}_0)\right) + 0(\varepsilon^2)$$

$$\varphi_\beta^* = \varphi_\beta\left(t^*, q_\sigma^*, \frac{\mathrm{d}q_\sigma^*}{\mathrm{d}t^*}\right) = \varphi_\beta(t, q, \dot{q}) + \varepsilon\left(\frac{\partial \varphi_\beta}{\partial t}\xi_0 + \frac{\partial \varphi_\beta}{\partial q_v}\xi_v + \frac{\partial \varphi_\beta}{\partial \dot{q}_v}(\dot{\xi}_v - \dot{q}_v\dot{\xi}_0)\right) + 0(\varepsilon^2)$$

$$\widetilde{Q}_\sigma^* = \widetilde{Q}_\sigma\left(t^*, q_\sigma^*, \frac{\mathrm{d}q_\sigma^*}{\mathrm{d}t^*}\right) = \widetilde{Q}_\sigma(t, q, \dot{q}) + \varepsilon\left(\frac{\partial \widetilde{Q}_\sigma}{\partial t}\xi_0 + \frac{\partial \widetilde{Q}_\sigma}{\partial q_v}\xi_v + \frac{\partial \widetilde{Q}_\sigma}{\partial \dot{q}_v}(\dot{\xi}_v - \dot{q}_v\dot{\xi}_0)\right) + 0(\varepsilon^2) \tag{8}$$

定义 如果在无限小变换(7)式下，(1)式和(4)式的形式保持不变，即

$$\frac{\mathrm{d}q_{\varepsilon+\beta}^*}{\mathrm{d}t^*} = \varphi_\beta\left(q_\sigma^*, \frac{\mathrm{d}q_\sigma^*}{\mathrm{d}t^*}, t^*\right) \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \widetilde{L}^*}{\partial \dot{q}_\sigma} - \frac{\partial \widetilde{L}^*}{\partial q_\sigma} - \frac{\partial L^*}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta^*}{\partial \dot{q}_\sigma} - \frac{\partial \varphi_\beta^*}{\partial q_\sigma}\right) = \widetilde{Q}_\sigma^* \tag{10}$$

成立，则相应的不变性称为广义 Chaplygin 方程的形式不变性.

将(8)式中第三式代入(9)式，舍去 ε^2 及更高阶小项，并利用约束方程(1)，得

$$\dot{\xi}_{\varepsilon+\beta} = \frac{\partial \varphi_\beta}{\partial t}\xi_0 + \frac{\partial \varphi_\beta}{\partial q_v}\xi_v + \frac{\partial \varphi_\beta}{\partial \dot{q}_v} \times (\dot{\xi}_v - \dot{q}_v\dot{\xi}_0), (\beta = 1, \cdots, g) \tag{11}$$

将(8)式代入方程(10)，舍去 ε^2 及更高阶小项，并利用方程(4)，得到

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_\sigma} - \frac{\partial}{\partial q_\sigma}\right)\left\{\varepsilon\left[\frac{\partial \widetilde{L}}{\partial t}\xi_0 + \frac{\partial \widetilde{L}}{\partial q_v}\xi_v + \frac{\partial \widetilde{L}}{\partial \dot{q}_v}(\dot{\xi}_v - \dot{q}_v\dot{\xi}_0)\right]\right\} - \frac{\partial}{\partial \dot{q}_{\varepsilon+\beta}} \times \left\{\varepsilon\left[\frac{\partial \widetilde{L}}{\partial t}\xi_0 + \frac{\partial \widetilde{L}}{\partial q_v}\xi_v + \frac{\partial \widetilde{L}}{\partial \dot{q}_v}(\dot{\xi}_v - \dot{q}_v\dot{\xi}_0) + \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}}(\dot{\xi}_{\varepsilon+\beta} - \dot{q}_{\varepsilon+\beta}\dot{\xi}_0)\right]\right\} \times \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_\sigma} - \frac{\partial}{\partial q_\sigma}\right)\left\{\varphi_\beta + \varepsilon\left[\frac{\partial \varphi_\beta}{\partial t}\xi_0 + \frac{\partial \varphi_\beta}{\partial q_v}\xi_v + \frac{\partial \varphi_\beta}{\partial \dot{q}_v}(\dot{\xi}_v - \dot{q}_v\dot{\xi}_0)\right]\right\} - \frac{\partial \widetilde{L}}{\partial \dot{q}_{\varepsilon+\beta}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_\sigma} - \frac{\partial}{\partial q_\sigma}\right)\left\{\varepsilon\left[\frac{\partial \varphi_\beta}{\partial t}\xi_0 + \frac{\partial \varphi_\beta}{\partial q_v}\xi_v + \frac{\partial \varphi_\beta}{\partial \dot{q}_v}(\dot{\xi}_v - \dot{q}_v\dot{\xi}_0)\right]\right\} = \varepsilon\left[\frac{\partial \widetilde{Q}_\sigma}{\partial t}\xi_0 + \frac{\partial \widetilde{Q}_\sigma}{\partial q_v}\xi_v + \frac{\partial \widetilde{Q}_\sigma}{\partial \dot{q}_v}(\dot{\xi}_v - \dot{q}_v\dot{\xi}_0)\right], (\sigma = 1, \cdots, \varepsilon) \tag{12}$$

判据 如果无限小生成元 ξ_0，ξ_s，满足方程(11)和(12)，则相应变换对应广义 Chaplygin 系统的形式不变性.

3 形式不变性和 Noether 对称性

按照 Noether 对称性理论，对 Lagrange 函数为 L，非势广义力为 Q_s 的力学系统，如果无限小变换生成元 ξ_0，ξ_s 和规范函数 $G = G(q, \dot{q}, t)$ 满足如下广义 Noether 等式[2]

$$L\dot{\xi}_0 + \frac{\partial L}{\partial t}\xi_0 + \frac{\partial L}{\partial q_\sigma}\xi_\sigma + \frac{\partial L}{\partial \dot{q}_\sigma}(\dot{\xi}_0 - \dot{q}_\sigma\dot{\xi}_0) + \frac{\partial L}{\partial q_{\varepsilon+\beta}}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma}(\xi_\sigma - \dot{q}_\sigma\xi_0) + \frac{\partial L}{\partial \dot{q}_{\varepsilon+\beta}} \times \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \varphi_\beta}{\partial \dot{q}_\sigma} - \frac{\partial \varphi_\beta}{\partial q_\sigma} - \frac{\partial \varphi_\beta}{\partial q_{\varepsilon+\gamma}}\frac{\partial \varphi_\gamma}{\partial q_\sigma}\right) \times (\xi_\sigma - \dot{q}_\sigma\xi_0) + Q_\sigma(\xi_\sigma - \dot{q}_\sigma\xi_0) + \dot{G} = 0 \tag{13}$$

则广义 Chaplygin 系统存在如下形式的守恒量

$$I = L\xi_0 + \frac{\partial L}{\partial \dot{q}_\sigma}(\xi_\sigma - \dot{q}_\sigma\xi_0) + G = const \tag{14}$$

广义 Chaplygin 方程的形式不变性可能是 Noether 对称性，也可能不是 Noether 对称性.若存在无限小变换(7)和规范函数 $G = G(q, \dot{q}, t)$ 同时满足式(11)，(12)，(13)，则形式不变性是 Noether 对称性，否则形式不变性不是 Noether 对称性.

4　算例

研究质量为 m,半径为 r 的匀质圆球在完全粗糙水平面上的纯滚动运动问题的 Noether 对称性与形式不变性.问题的 Lagrange 函数和约束方程分别为

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{2}{5}ma^2(\dot{\psi}^2 + \dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\psi}\dot{\phi}\cos\theta),$$
$$\dot{x} = a(\dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi),$$
$$\dot{y} = -a(\dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi)$$

其中 x, y 是球心坐标,ψ, θ, ϕ 为 Euler 角.令 $q_1 = \psi, q_2 = \theta, q_3 = \phi, q_4 = x, q_5 = y$,则有

$$L = \frac{1}{2}m(\dot{q}_4^2 + \dot{q}_5^2) + \frac{1}{2}\frac{2}{5}ma^2(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2\dot{q}_1\dot{q}_3\cos q_2),$$
$$\tilde{L} = \frac{1}{2}ma^2\left\{\frac{7}{5}\dot{q}_2^2 + \dot{q}_3^2\sin^2 q_2 + \frac{2}{5}(\dot{q}_1^2 + \dot{q}_3^2 + 2\dot{q}_1\dot{q}_3\cos q_2)\right\},$$
$$\dot{q}_4 = a(\dot{q}_2\sin q_1 - \dot{q}_3\sin q_2\cos q_1),$$
$$\dot{q}_5 = -a(\dot{q}_2\cos q_1 + \dot{q}_3\sin q_2\sin q_1) \tag{15}$$

广义 Noether 等式(13)给出

$$\tilde{L}\dot{\xi}_0 + ma^2\dot{q}_3\left(\dot{q}_3\cos q_2 - \frac{2}{5}\dot{q}_1\right)\xi_2\sin q_2 + \frac{2}{5}ma^2(\dot{q}_1 + \dot{q}_3\cos q_2)(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + \frac{7}{5}ma^2\dot{q}_2(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0) + ma^2[\dot{q}_3\sin^2 q_2 + \frac{2}{5}(\dot{q}_3 + \dot{q}_1\cos q_2)](\dot{\xi}_3 - \dot{q}_3\dot{\xi}_0) - ma^2\dot{q}_3(\dot{q}_1 + \dot{q}_3\cos q_2)(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0)\dot{q}_3\sin q_2 - ma^2\dot{q}_2(\dot{q}_1 + \dot{q}_3\cos q_2)\times(\dot{\xi}_3 - \dot{q}_3\dot{\xi}_0)\dot{q}_2\sin q_2 + \dot{G} = 0 \tag{16}$$

(16)式有如下解

$$\xi_0 = 1, \xi_1 = \xi_2 = \xi_3 = 0, G = 0 \tag{17}$$
$$\xi_1 = 1, \xi_0 = \xi_2 = \xi_3 = 0, G = 0 \tag{18}$$

又由文[4]可知

$$\xi_4 = \xi_5 = 0 \tag{19}$$

它们对应系统的 Noether 对称性.相应的守恒量分别为

$$I = \tilde{L} - \frac{\partial\tilde{L}}{\partial\dot{q}_\sigma}\dot{q}_\sigma = const \tag{20}$$

$$I = \frac{\partial\tilde{L}}{\partial\dot{q}_1} = \frac{2}{5}ma^2(\dot{q}_1 + \dot{q}_3\cos q_2) = const \tag{21}$$

它们分别代表系统的机械能守恒和对固定铅锤轴的动量矩守恒.

将式(17),(19)代入式(11),(12),可知前者能满足后者.因此式(17),(19)也是问题的形式不变性.将式(18),(19)代入式(11),(12),可以验证前者不能满足后者,因此,式(18),(19)不是形式不变性.

参　考　文　献

1　赵跃宇,梅凤翔.力学系统的对称性与不变量.北京:科学出版社,1999 (Zhao Yueyu, Mei Fengxiang. Symmetries and Invariants of Mechanical Systems. Beijing:Science Press, 1999(in Chinese))

2　梅凤翔.完整力学系统的三类对称性与三类守恒量.动力学与控制学报,2004,2(1):28～31(Mei Fengxiang. Three kinds of symmetries and three kinds of conserved quantities for holonomic systems. *Journal of dynamics and control*, 2004,2(1):28～31(in Chinese))

3　许志新,梅凤翔.广义 Chaplygin 系统的 Noether 对称性.北京理工大学学报, 2002, 22(2):143～146 (XU Zhixin, MEI Fengxiang. Noether Symmetry of Generalized Chaplygin System. *Journal of Bejing Institute of Technology*, 2002, 22(2):143～146(in Chinese))

4　葛伟宽.Chaplygin 系统的 Noether 对称性与形式不变形.物理学报,2002,51(5):939～942 (Ge WeiKuan. Noether symmetry and form invariance of the Chaplygin system. *Acta physica sinica*, 2002, 51(5):939～942(in Chinese))

5　李仁杰,乔永芬,等.变质量完整系统 Gibbs－Appell 方程的形式不变性.物理学报, 2002, 51(1):1～5 (Li Renjie, Qiao Yongfen, Meng Jun. Form invariance of Gibbs-Appell equations for variable mass holonomic systems. *Acta physica sinica*,2002,51(2):1～5(in Chinese))

6　乔永芬,张耀良,韩广才.非完整系统 Hamilton 正则方程的形式不变性.物理学报,2003,52(5):1051～1056 (Qiao Yongfen,Zhang Yaoliang. Form invariance of Hamilton 's canonical equations of a nonholonomic mechanical system. *Acta physica sinica*, 2003, 52(5):1051～1056(in Chinese))

7　方建会,薛庆忠,等.非保守力学系统 Nielsen 方程的形式不变性.物理学报,2002,51(10):2183～2185 (Fang Jianhui, Xue Qingzhong, Zhao Songqing. Form invariance of Nielsen equation of a nonconservative mechanical system. *Acta* physica sinica ,2002,51(10):2183 ~ 2185(in Chinese))

8 陈向炜,梅凤翔,等.约束Birkhoff系统的形式不变性.应用数学和力学,2002,23(1):47 ~ 51 (Chen Xiangwei, Luo Shaokai, Mei Fengxiang. A Form Invariance of Constrained Birkhoffian System. *Applied Mathematics and Mechanics*, 2002, 23(1):47 ~ 51(in Chinese))

9 赵跃宇,金波,许文喜.关于非完整系统虚位移的不确定性问题与非线性问题.动力学与控制学报,2003,1(1):20 ~ 24 (Zhao Yueyu ,JinBo , Xu wenxi . On the unqualitative and nonlinear problems of virtual displacement in nonholonomic system. *Journal of Dynamics and Control*, 2003,1(1):20 ~ 24 (in Chinese))

10 金波,赵跃宇,周海兵.本质线性非完整系统的Hamilton的原理.动力学与控制学报,2004,2(1):32 ~ 36 (Jin Bo ,Zhao Yueyu ,Zhou Haibing. Hamilton principle of lntrinsical linear nonholonomic system. *Journal of Dynamics and Control*, 2004,2(1):32 ~ 36 (in Chinese))

11 吴惠彬.Poincaré-Chetaev变量下广义Routh方程的对称性与守恒量.动力学与控制学报,2004,2(1):37 ~ 39 (Wu Huibin . Symmetries and conserved quantities for generalized Routh 's equations in Poincaré-Chetaev variables. *Journal of Dynamics and Control*, 2004,2(1):37 ~ 39(in Chinese))

12 秦茂昌,梅凤翔,许学军.Fokker – Planck方程的守恒量.动力学与控制学报,2005,3(1):7 ~ 9 (Qin Maochang , Mei Fengxiang ,Xu Xuejun . The conserved quantities of Fokker-Planck equation. *Journal of Dynamics and Control*, 2005,3(1):7 ~ 9 (in Chinese))

FORM INVARIANCE AND NOETHER SYMMETRY OF GENERAL CHAPLYGIN SYSTEM*

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Abstract The form invariance of general Chaplygin system was studied. Using the infinitesimal transformations of general Chaplygin equation, the definition and criterion of the form invariance of general Chaplygin system were given. The criterion of Noether symmetry of general Chaplygin system was given, and the relation between the form invariance and the Noether symmetry was studied. The results show that the form invariance of general Chaplygin system isn't identical with Noether symmetry. In some situation, the form invariance may be equal to the Noether symmetry, and in another instance they are different. One example was given to illustrate the application of the result.

Key words general Chaplygin system, form invariance, Noether symmetry, conserved quantity

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