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# 基于微分/代数方程的多体系统动力学设计灵敏度分析的伴随变量方法*

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**摘要** 基于多体系统动力学微分/代数方程数学模型和通用积分形式的目标函数,建立了多体系统动力学设计灵敏度分析的伴随变量方法,避免了复杂的设计灵敏度计算,对于设计变量较多的多体系统灵敏度分析具有较高的计算效率.文中给出了通用公式以及具体的计算过程和验证方法,并将目标函数及其导数积分形式的计算转化为微分方程的初值问题,进一步提高了计算效率和精度.文末通过一曲柄 - 滑块机构算例对算法的有效性进行了验证.

**关键词** 多体系统动力学,微分/代数方程,灵敏度分析,伴随变量方法

## 引言

在采用大量高效的最优化设计方法解决动态系统最优设计问题时,往往需要计算系统目标函数对设计变量的导数.在多体系统动态最优化设计中,主要采用直接微分方法、伴随变量方法、有限差分方法[1].其中伴随变量方法对设计变量较多的系统具有较高的效率[2],该方法早期被应用于结构动力学与机械振动最优化设计[3],由于其计算效率方面的特色,近年被许多学者应用于多体系统最优化设计[1~7].其中 Haug[1]、Etman[2]、Serban[6]、Petzod[7]针对微分/代数方程提出灵敏度分析伴随变量方法,但[1]、[7]未充分利用多体通用动力学模型的特性,[2]、[6]的工作基于较简单的初始与终止条件.本文则基于通用的多体动力学 Euler-Lagrange 方程和通用目标函数及一般的初始条件定义建立了完善的多体动力学设计灵敏度伴随变量方法.为便于目标函数及其导数的计算,本文又将其积分转化为常微分方程的初值问题.

## 1 问题描述

设多体系统动态最优化设计的设计参数为 $\boldsymbol{b} = [b_1 \quad b_2 \quad \cdots \quad b_l]^{\mathrm{T}}$,系统广义坐标为 $\boldsymbol{q} = [q_1 \quad q_2 \quad \cdots \quad q_n]^{\mathrm{T}}$,受完整约束的多体系统动力学 Euler-Lagrange 方程为

$$\begin{cases} \boldsymbol{M}(\boldsymbol{q},\boldsymbol{b})\ddot{\boldsymbol{q}} + \boldsymbol{\Phi}_q^{\mathrm{T}}\boldsymbol{\lambda} = \boldsymbol{Q}(\dot{\boldsymbol{q}},\boldsymbol{q},\boldsymbol{b},t) & (1a) \\ \boldsymbol{\Phi}(\boldsymbol{q},\boldsymbol{b},t) = 0 & (1b) \end{cases}$$

假设初始时刻 $t^1$ 固定,系统运行终止时刻由下式确定(上标 2 表示终止时刻)

$$\Omega(\dot{\boldsymbol{q}}^2,\boldsymbol{q}^2,\boldsymbol{b},t^2) = 0 \tag{2}$$

且满足 $\dot{\Omega}\ \frac{\mathrm{d}\Omega}{\mathrm{d}t^2} = \Omega_{\dot{q}^2}\ddot{\boldsymbol{q}}^2 + \Omega_{q^2}\dot{\boldsymbol{q}}^2 + \Omega_{t^2} \neq 0$.

系统初始状态及速度附加条件为

$$\boldsymbol{\varphi}(\boldsymbol{q}^1,\boldsymbol{b},t^1) = \boldsymbol{0} \tag{3a}$$

$$\bar{\boldsymbol{\varphi}}(\dot{\boldsymbol{q}}^1,\boldsymbol{q}^1,\boldsymbol{b},t^1) = \boldsymbol{0} \tag{3a}$$

且满足$\begin{pmatrix} \boldsymbol{\Phi}^1_{q^1} \\ \boldsymbol{\varphi}_{q_1} \end{pmatrix}$与$\begin{pmatrix} \boldsymbol{\Phi}^1_{q_1} \\ \bar{\boldsymbol{\varphi}}^1_{q^1} \end{pmatrix}$非奇异.

设系统设计目标函数具以下通用的积分形式

$$\psi = G(\dot{\boldsymbol{q}}^2,\boldsymbol{q}^2,\boldsymbol{b},t^2) + \int_{t^1}^{t^2} H(\ddot{\boldsymbol{q}},\dot{\boldsymbol{q}},\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{b},t)\mathrm{d}t \tag{4}$$

式(4) 两边对 $b$ 求导可得

$$\frac{\mathrm{d}\psi}{\mathrm{d}\boldsymbol{b}} = G_{\dot{q}^2}\dot{\boldsymbol{q}}^2_b + G_{q^2}\boldsymbol{q}^2_b + (\dot{G} + H^2)t^2_b + G_b + \int_{t^1}^{t^2}(H_{\ddot{q}}\ddot{\boldsymbol{q}}_b + H_{\dot{q}}\dot{\boldsymbol{q}}_b + H_{\lambda}\boldsymbol{\lambda}_b + H_b)\mathrm{d}t \tag{5}$$

其中

$$\dot{G}\ \frac{\mathrm{d}G}{\mathrm{d}t^2} = G_{\dot{q}^2}\ddot{\boldsymbol{q}}^2 + G_{q^2}\dot{\boldsymbol{q}}^2 + G_{t^2}$$

$$H^2\ H(\ddot{\boldsymbol{q}}^2,\dot{\boldsymbol{q}}^2,\boldsymbol{q}^2,\boldsymbol{\lambda}^2,\boldsymbol{b},t^2)$$

式(5) 对 $\ddot{\boldsymbol{q}}_b$ 及 $\dot{\boldsymbol{q}}_b$ 所在项进行分部积分可得

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$$\frac{\mathrm{d}\psi}{\mathrm{d}\boldsymbol{b}} = (G_{\ddot{q}^2} + H^2_{\ddot{q}^2})\dot{\boldsymbol{q}}^2_b + (G_{q^2} - \frac{\mathrm{d}}{\mathrm{d}t^2}H^2_{\ddot{q}^2} + H^2_{\dot{q}^2})\boldsymbol{q}^2_b - H^1_{\dot{q}^1}\dot{\boldsymbol{q}}^1_b + (\frac{\mathrm{d}}{\mathrm{d}t^1}H^1_{\dot{q}^1} - H^1_{q^1})\boldsymbol{q}^1_b + (\dot{G} + H^2)t^2_b + G_b + \int_{t^1}^{t^2}[(\frac{\mathrm{d}^2}{\mathrm{d}t^2}H_{\ddot{q}} - \frac{\mathrm{d}}{\mathrm{d}}H_{\dot{q}} + H_q)\boldsymbol{q}_b + H_\lambda\boldsymbol{\lambda}_b + H_b]\mathrm{d}t \tag{6}$$

本文采用伴随变量方法来计算式(6)，通过引进一系列伴随变量，消去未知量 $\dot{\boldsymbol{q}}^2_b,\boldsymbol{q}^2_b,\dot{\boldsymbol{q}}^1_b,\dot{\boldsymbol{q}}^1_b,\boldsymbol{q}_b,\boldsymbol{\lambda}_b$.

## 2 伴随变量方法

式(1a)两边对 $b$ 求导可得

$$\boldsymbol{M}\ddot{\boldsymbol{q}}_b - \boldsymbol{Q}_{\dot{q}}\dot{\boldsymbol{q}}_b + [(\boldsymbol{M}\ddot{\boldsymbol{q}})_q + (\boldsymbol{\Phi}^{\mathrm{T}}_q\boldsymbol{\lambda})_q - \boldsymbol{Q}_q]\boldsymbol{q}_b + \boldsymbol{\Phi}^{\mathrm{T}}_q\boldsymbol{\lambda}_b + [(\boldsymbol{M}\ddot{\boldsymbol{q}})_b + (\boldsymbol{\Phi}^{\mathrm{T}}_q\boldsymbol{\lambda})_b - \boldsymbol{Q}_b] = \boldsymbol{0} \tag{7}$$

记

$$\boldsymbol{\Pi} = \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{\Phi}^{\mathrm{T}}_q\boldsymbol{\lambda} - \boldsymbol{Q} = \boldsymbol{0} \tag{8}$$

则式(7)可以简写为

$$\boldsymbol{M}\ddot{\boldsymbol{q}}_b - \boldsymbol{Q}_{\dot{q}}\dot{\boldsymbol{q}}_b + \boldsymbol{\Pi}_b\boldsymbol{q}_b + \boldsymbol{\Phi}^{\mathrm{T}}_q\boldsymbol{\lambda}_b + \boldsymbol{\Pi}_b = \boldsymbol{0} \tag{9}$$

引入伴随变量 $\boldsymbol{\mu}$，将其转置后左乘以式(9)，并在 $[t^1,t^2]$ 上积分可得

$$\int_{t^1}^{t^2}\boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{M}\ddot{\boldsymbol{q}}_b - \boldsymbol{Q}_{\dot{q}}\dot{\boldsymbol{q}}_b + \boldsymbol{\Pi}_q\boldsymbol{q}_b + \boldsymbol{\Phi}^{\mathrm{T}}_q\boldsymbol{\lambda}_b + \boldsymbol{\Pi}_b)\mathrm{d}t \tag{10}$$

式(10)对及所在项进行分部积分可得

$$\int_{t^1}^{t^2}\{[\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{M}) + \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{Q}_{\dot{q}}) + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\Pi}_q]\boldsymbol{q}_b + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\Phi}^{\mathrm{T}}_q\boldsymbol{\lambda}_b + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\Pi}_b\}\mathrm{d}t + \{\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{M}\dot{\boldsymbol{q}}_b - [\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{M}) + \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{Q}_{\dot{q}}]\boldsymbol{q}_b\}\Big|_{t^1}^{t^2} = \boldsymbol{0} \tag{11}$$

式(1b)两边对求导可得

$$\boldsymbol{\Phi}_q\boldsymbol{q}_b + \boldsymbol{\Phi}_b = \boldsymbol{0} \tag{12}$$

引入伴随变量 $\boldsymbol{v}$，将其转置后左乘以(12)式，并 $[t^1,t^2]$ 在上积分可得

$$\int_{t^1}^{t^2}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{\Phi}_q\boldsymbol{q}_b + \boldsymbol{v}^{\mathrm{T}}\boldsymbol{\Phi}_b)\mathrm{d}t = \boldsymbol{0} \tag{13}$$

初始时刻和终止时刻的式(1b)满足

$$\boldsymbol{\Phi}^i(\boldsymbol{q}^i,\boldsymbol{b},t^i) = \boldsymbol{0} \tag{14a}$$

$$\dot{\boldsymbol{\Phi}}^i = \boldsymbol{\Phi}^i_{q^i}\dot{\boldsymbol{q}}^i + \boldsymbol{\Phi}^i_{t^i} = \boldsymbol{0},\ i = 1,2 \tag{14b}$$

式(14a)两边对 $b$ 求导，并引进新的伴随变量 $\boldsymbol{\eta}^i(i=1,2)$ 可得

$$\boldsymbol{\eta}^{1\mathrm{T}}\boldsymbol{\Phi}^1_{q^1}\boldsymbol{q}^1_b + \boldsymbol{\eta}^{1\mathrm{T}}\boldsymbol{\Phi}^1_b = \boldsymbol{0} \tag{15a}$$

$$\boldsymbol{\eta}^{2\mathrm{T}}\boldsymbol{\Phi}^2_{q^2}\boldsymbol{q}^2_b + \boldsymbol{\eta}^{2\mathrm{T}}\dot{\boldsymbol{\Phi}}^2 t^2_b + \boldsymbol{\eta}^{2\mathrm{T}}\boldsymbol{\Phi}^2_b = \boldsymbol{0} \tag{15b}$$

式(14b)两边对 $b$ 求导，引进新的伴随变量 $\zeta(i=1,2)$ 可得

$$\zeta^{1\mathrm{T}}\boldsymbol{\Phi}^1_{q^1}\dot{\boldsymbol{q}}^1_b + \zeta^{1\mathrm{T}}\dot{\boldsymbol{\Phi}}^1_{q^1}\boldsymbol{q}^1_b + \zeta^{1\mathrm{T}}\dot{\boldsymbol{\Phi}}^1_b = \boldsymbol{0} \tag{16a}$$

$$\zeta^{2\mathrm{T}}\boldsymbol{\Phi}^2_{q^2}\dot{\boldsymbol{q}}^2_b + \zeta^{2\mathrm{T}}\dot{\boldsymbol{\Phi}}^2_{q^2}\boldsymbol{q}^2_b + \zeta^{2\mathrm{T}}(\boldsymbol{\Phi}^2_{q^2}\ddot{\boldsymbol{q}}^2 + \ddot{\boldsymbol{\Phi}}^2)t^2_b + \zeta^{2\mathrm{T}}\dot{\boldsymbol{\Phi}}^2_b = \boldsymbol{0} \tag{16b}$$

式(2)两边对 $b$ 求导，引进新的伴随变量 $\tau$ 可得

$$\tau\boldsymbol{\Omega}_{\dot{q}^2}\dot{\boldsymbol{q}}^2_b + \tau\boldsymbol{\Omega}_{q^2}\boldsymbol{q}^2_b + \tau\dot{\boldsymbol{\Omega}}t^2_b + \tau\boldsymbol{\Omega}_b = \boldsymbol{0} \tag{17}$$

式(3a)及式(3b)两边对 $b$ 求导，引进新的伴随变量 $\boldsymbol{\alpha},\boldsymbol{\beta}$ 可得

$$\boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\varphi}_{q^1}\boldsymbol{q}^1_b + \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\varphi}_b = \boldsymbol{0} \tag{18a}$$

$$\boldsymbol{\beta}^{\mathrm{T}}\bar{\boldsymbol{\varphi}}_{\dot{q}^1}\dot{\boldsymbol{q}}^1_b + \boldsymbol{\beta}^{\mathrm{T}}\bar{\boldsymbol{\varphi}}_{q^1}\boldsymbol{q}^1_b + \boldsymbol{\beta}^{\mathrm{T}}\bar{\boldsymbol{\varphi}}_b = \boldsymbol{0} \tag{18b}$$

式(6)右部累减式(11)、(13)、(15a)、(15b)、(16a)、(16b)、(17)、(18a)、(18b)的左部，在所得结果中分别令 $\dot{\boldsymbol{q}}^2_b,\boldsymbol{q}^2_b,\dot{\boldsymbol{q}}^1_b,\boldsymbol{q}^1_b,\boldsymbol{q}_b,\boldsymbol{\lambda}_b,t^2_b$ 相关项系数为零，可得如下伴随变量方程

$$\boldsymbol{M}^2\boldsymbol{\mu}^2 + \boldsymbol{\Phi}^{2\mathrm{T}}_{q^2}\zeta^2 + \boldsymbol{\Omega}^{\mathrm{T}}_{\dot{q}^2}\boldsymbol{\tau} = H^{2\mathrm{T}}_{\ddot{q}^2} + G^{\mathrm{T}}_{\dot{q}^2} \tag{19a}$$

$$\boldsymbol{M}^2\dot{\boldsymbol{\mu}}^2 + (\dot{\boldsymbol{M}}^2 + \boldsymbol{Q}^{2\mathrm{T}}_{\dot{q}^2})\boldsymbol{\mu}^2 - \boldsymbol{\Phi}^{2\mathrm{T}}_{q^2}\boldsymbol{\eta}^2 - \dot{\boldsymbol{\Phi}}^{2\mathrm{T}}_{q^2}\zeta^2 - \boldsymbol{\Omega}^{\mathrm{T}}_{q^2}\boldsymbol{\tau} = \frac{\mathrm{d}}{\mathrm{d}t^2}H^{2\mathrm{T}}_{\ddot{q}^2} - H^{2\mathrm{T}}_{\dot{q}^2} - G^{\mathrm{T}}_{q^2} \tag{19b}$$

$$\boldsymbol{M}^1\boldsymbol{\mu}^1 - \boldsymbol{\Phi}^{1\mathrm{T}}_{q^1}\zeta^1 - \bar{\boldsymbol{\varphi}}^{\mathrm{T}}_{\dot{q}^1}\boldsymbol{\beta} = H^{1\mathrm{T}}_{\dot{q}^1} \tag{19c}$$

$$\boldsymbol{M}^1\dot{\boldsymbol{\mu}}^1 + (\dot{\boldsymbol{M}}^1 + \boldsymbol{Q}^{1\mathrm{T}}_{\dot{q}^1})\boldsymbol{\mu}^1 + \boldsymbol{\Phi}^{1\mathrm{T}}_{q^1}\boldsymbol{\eta}^1 + \dot{\boldsymbol{\Phi}}^{1\mathrm{T}}_{q^1}\zeta^1 + \boldsymbol{\varphi}^{\mathrm{T}}_{q^1}\boldsymbol{\alpha} + \bar{\boldsymbol{\varphi}}^{\mathrm{T}}_{q^1}\boldsymbol{\beta} = \frac{\mathrm{d}}{\mathrm{d}t^1}H^{1\mathrm{T}}_{\dot{q}^1} - H^{1\mathrm{T}}_{q^1} \tag{19d}$$

$$(\ddot{\boldsymbol{M}} + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Q}^{\mathrm{T}}_{\dot{q}} + \boldsymbol{\Pi}^{\mathrm{T}}_q)\boldsymbol{\mu} + (2\dot{\boldsymbol{M}} + \boldsymbol{Q}^{\mathrm{T}}_{\dot{q}})\dot{\boldsymbol{\mu}} + \boldsymbol{M}\ddot{\boldsymbol{\mu}} + \boldsymbol{\Phi}^{\mathrm{T}}_q\boldsymbol{v} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}H^{\mathrm{T}}_{\ddot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}H^{\mathrm{T}}_{\dot{q}} + H^{\mathrm{T}}_q \tag{19e}$$

$$\boldsymbol{\Phi}_q\boldsymbol{\mu} = H^{\mathrm{T}}_\lambda \tag{19f}$$

$$\dot{\boldsymbol{\Phi}}^{2\mathrm{T}}\boldsymbol{\eta}^2 + [\boldsymbol{\Phi}^2_{q^2}\ddot{\boldsymbol{q}}_{q^2}\ddot{\boldsymbol{q}}^2 + \ddot{\boldsymbol{\Phi}}^2]^{\mathrm{T}}\zeta^2 + \dot{\boldsymbol{\Omega}}^{\mathrm{T}}\tau = \dot{\boldsymbol{G}}^{\mathrm{T}} + H^{2\mathrm{T}} \tag{19j}$$

此时目标函数关于设计参数的导数为

$$\frac{\mathrm{d}\psi}{\mathrm{d}\boldsymbol{b}} = G_b - \boldsymbol{\eta}^{1\mathrm{T}}\boldsymbol{\Phi}^1_b - \boldsymbol{\eta}^{2\mathrm{T}}\boldsymbol{\Phi}^2_b - \zeta^{1\mathrm{T}}\dot{\boldsymbol{\Phi}}^1_b - \xi^{2\mathrm{T}}\dot{\boldsymbol{\Phi}}^2_b\tau\boldsymbol{\Omega}_b - \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\varphi}_b - \boldsymbol{\beta}^{\mathrm{T}}\bar{\boldsymbol{\varphi}}_b + \int_{t^1}^{t^2}(H_b - \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\Pi}_b - \boldsymbol{v}^{\mathrm{T}}\boldsymbol{\Phi}_b)\mathrm{d}t \tag{20}$$

由伴随变量方程解出伴随变量 $\boldsymbol{\mu},\boldsymbol{v},\boldsymbol{\eta}^1,\boldsymbol{\eta}^2,\zeta^1,\zeta^2,\boldsymbol{\tau},\boldsymbol{\alpha},\boldsymbol{\beta}$ 代入式(20)即可求出 $\frac{\mathrm{d}\psi}{\mathrm{d}\boldsymbol{b}}$.

## 3　计算过程

给定设计参数 $b$,可以利用下面的步骤来求目标函数关于设计参数的灵敏度.

步骤一 求 $\ddot{q},\dot{q},q,\lambda$;

初始时刻已知时,由约束方程及其速度极约束可求得初始值 $q^1,\dot{q}^1$,采用约束违约自动稳定方法,可求解下面微分/代数方程可得 $\ddot{q},\dot{q},q$ 及 $\lambda$

$$\begin{pmatrix} \boldsymbol{M} & \boldsymbol{\Phi}_q^{\mathrm{T}} \\ \boldsymbol{\Phi}_q & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \ddot{q} \\ \lambda \end{pmatrix} = \begin{pmatrix} Q(\dot{q},q,b,t) \\ \gamma - 2\alpha\dot{\boldsymbol{\Phi}} - \beta^2\boldsymbol{\Phi} \end{pmatrix} \tag{21}$$

其中 $\gamma = -(\boldsymbol{\Phi}_q\dot{q})_q\dot{q} - 2\boldsymbol{\Phi}_{qt}\dot{q} - \boldsymbol{\Phi}_{tt}$.

步骤二 求伴随变量 $\boldsymbol{\eta}^2,\zeta^2,\tau$;

式(19a)和 $t^2$ 时刻的式(19b))联立可得

$$\begin{pmatrix} \boldsymbol{M}^2 & \boldsymbol{\Phi}_{q^2}^{2\mathrm{T}} \\ \boldsymbol{\Phi}_{q^2}^2 & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \boldsymbol{\mu}^2 \\ \zeta^2 \end{pmatrix} = -\begin{pmatrix} \boldsymbol{\Omega}_{\dot{q}^2}^{\mathrm{T}} \\ \boldsymbol{0} \end{pmatrix}\boldsymbol{\tau} + \begin{pmatrix} H_{\ddot{q}^2}^{2\mathrm{T}} + G_{\dot{q}^2}^{\mathrm{T}} \\ H_\lambda^{2\mathrm{T}} \end{pmatrix} \tag{22}$$

式(19e)关于时间 $t$ 的一阶、二阶导数为

$$\boldsymbol{\Phi}_q\dot{\boldsymbol{\mu}} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Phi}_q\right)\boldsymbol{\mu} = \frac{\mathrm{d}}{\mathrm{d}t}H_\lambda^{\mathrm{T}} \tag{23a}$$

$$\boldsymbol{\Phi}_q\ddot{\boldsymbol{\mu}} + 2\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Phi}_q\right)\dot{\boldsymbol{\mu}} + \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{\Phi}_q\right)\boldsymbol{\mu} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}H_\lambda^{\mathrm{T}} \tag{23b}$$

式(19b)和 $t^2$ 时刻的式(23a)联立可得

$$\begin{pmatrix} \boldsymbol{M}^2 & \boldsymbol{\Phi}_{q^2}^{2\mathrm{T}} \\ \boldsymbol{\Phi}_{q^2}^2 & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \dot{\boldsymbol{\mu}}^2 \\ -\boldsymbol{\eta}^2 \end{pmatrix} = \left(\boldsymbol{\Omega}_{q^2}^{\mathrm{T}}\right)\boldsymbol{\tau} + \begin{pmatrix} \dot{\boldsymbol{\Phi}}_{q^2}^{2\mathrm{T}}\zeta^2 - (\dot{\boldsymbol{M}}^2 + \boldsymbol{Q}_{\dot{q}^2}^{2\mathrm{T}})\boldsymbol{\mu}^2 + \frac{\mathrm{d}}{\mathrm{d}t^2}H_{\ddot{q}^2}^{2\mathrm{T}} - \\ H_{\dot{q}^2}^{2\mathrm{T}} - G_{q^2}^{\mathrm{T}} - \left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Phi}_{q^2}^2\right)\boldsymbol{\mu}^2 + \frac{\mathrm{d}}{\mathrm{d}t}H_\lambda^{2\mathrm{T}} \end{pmatrix} \tag{24}$$

线性方程组(22)、(24)的解是关于 $\tau$ 的函数. 将求得的 $\boldsymbol{\eta}^2,\zeta^2$ 代入式(19j)中可求出伴随变量 $\tau$,从而可以求出 $\boldsymbol{\eta}^2,\zeta^2$ 及 $\boldsymbol{\mu}^2,\dot{\boldsymbol{\mu}}^2$.

步骤三 求伴随变量 $\boldsymbol{\mu},\boldsymbol{v}$;

式(19f)与式(23b)联立可得

$$\begin{pmatrix} \boldsymbol{M} & \boldsymbol{\Phi}_q^{\mathrm{T}} \\ \boldsymbol{\Phi}_q & \boldsymbol{0} \end{pmatrix}\begin{pmatrix} \ddot{\boldsymbol{\mu}} \\ \boldsymbol{v} \end{pmatrix} = \begin{pmatrix} \{-2\dot{\boldsymbol{M}} + \boldsymbol{Q}_{\dot{q}}^{\mathrm{T}})\dot{\boldsymbol{\mu}} - (\ddot{\boldsymbol{M}} + \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Q}_{\dot{q}}^{\mathrm{T}} + \boldsymbol{\Pi}_q^{\mathrm{T}})\boldsymbol{\mu} + \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2}H_{\ddot{q}}^{\mathrm{T}} - \frac{\mathrm{d}}{\mathrm{d}t}H_{\dot{q}}^{\mathrm{T}} + H_q^{\mathrm{T}}\} - 2\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Phi}_q\right)\dot{\boldsymbol{\mu}} - \\ \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\boldsymbol{\Phi}_q\right)\boldsymbol{\mu} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}H_\lambda^{\mathrm{T}} \end{pmatrix} \tag{25}$$

$\boldsymbol{\mu}^2,\dot{\boldsymbol{\mu}}^2$ 确定后,式(25)构成典型的微分/代数方程,向后积分可求得伴随变量 $\boldsymbol{\mu},\boldsymbol{v}$ 及 $\boldsymbol{\mu}^1,\dot{\boldsymbol{\mu}}^1$.

步骤四 求伴随变量 $\zeta^1,\boldsymbol{\beta}$;

式(19c)可以写成

$$\left(\boldsymbol{\Phi}_{q^1}^{1\mathrm{T}} \quad \bar{\boldsymbol{\varphi}}_{\dot{q}^1}^{\mathrm{T}}\right)\begin{pmatrix} \zeta^1 \\ \boldsymbol{\beta} \end{pmatrix} = \boldsymbol{M}^1\boldsymbol{\mu}^1 - H_{\dot{q}^1}^{1\mathrm{T}} \tag{26}$$

该线性方程组系数矩阵非奇异,解之可得 $\zeta^1,\boldsymbol{\beta}$.

步骤五 求伴随变量 $\boldsymbol{\eta}^1,\boldsymbol{\alpha}$;

式(19d)可以写成

$$\left(\boldsymbol{\Phi}_{q^1}^{1\mathrm{T}} \quad \boldsymbol{\varphi}_{q^1}^{\mathrm{T}}\right)\begin{pmatrix} \boldsymbol{\eta}^1 \\ \boldsymbol{\alpha} \end{pmatrix} = -\boldsymbol{M}^1\dot{\boldsymbol{\mu}}^1 - (\dot{\boldsymbol{M}}^1 + \boldsymbol{Q}_{\dot{q}^1}^{1\mathrm{T}})\boldsymbol{\mu}^1 - \dot{\boldsymbol{\Phi}}_{q^1}^{1\mathrm{T}}\zeta^1 - \bar{\boldsymbol{\varphi}}_{q^1}^{\mathrm{T}}\boldsymbol{\beta} + \frac{\mathrm{d}}{\mathrm{d}t^1}H_{\ddot{q}^1}^{1\mathrm{T}} - H_{\dot{q}^1}^{1\mathrm{T}} \tag{27}$$

该线性方程组系数矩阵非奇异,解之可得 $\boldsymbol{\eta}^1,\boldsymbol{\alpha}$.

步骤六 求 $\frac{\mathrm{d}\psi}{\mathrm{d}\boldsymbol{b}}$.

将求得的伴随变量 $\boldsymbol{\mu},\boldsymbol{v},\boldsymbol{\eta}^1,\boldsymbol{\eta}^2,\zeta^1,\zeta^2,\boldsymbol{\tau},\boldsymbol{\alpha},\boldsymbol{\beta}$ 代入式(20)即可得到目标函数关于设计变量的一阶灵敏度 $\frac{\mathrm{d}\psi}{\mathrm{d}\boldsymbol{b}}$.

## 4　目标函数及其导数的计算

为计算简便,以下将目标函数及其导数的计算转化为标准的微分方程初值问题.

设

$$A(t) = G + \int_{t^1}^{t} H\mathrm{d}s \tag{28}$$

则

$$A(t^1) = G,\ \frac{\mathrm{d}A}{\mathrm{d}t} = H \tag{29}$$

求解该微分方程初值问题可得 $t^2$ 时刻的函数值 $A(t^1)$,而它恰好是目标函数 $\psi(\boldsymbol{b})$.

同理,设

$$\boldsymbol{B}(t) = G_b - \boldsymbol{\eta}^{1\mathrm{T}}\boldsymbol{\Phi}_b^1 - \boldsymbol{\eta}^{2\mathrm{T}}\boldsymbol{\Phi}_b^2 - \boldsymbol{\zeta}^{1\mathrm{T}}\dot{\boldsymbol{\Phi}}_b^1 - \boldsymbol{\zeta}^{2\mathrm{T}}\dot{\boldsymbol{\Phi}}_b^2 - \boldsymbol{\tau}\boldsymbol{\Omega}_b - \boldsymbol{\alpha}^{\mathrm{T}}\boldsymbol{\varphi}_b - \boldsymbol{\beta}^{\mathrm{T}}\bar{\boldsymbol{\varphi}}_b + \int_{t^1}^{t}(H_b - \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\Pi}_b - \boldsymbol{v}^{\mathrm{T}}\boldsymbol{\Phi}_b)\mathrm{d}t \tag{30}$$

则

$$B(t^1) = G_b - \boldsymbol{\eta}^{1T}\boldsymbol{\Phi}_b^1 - \boldsymbol{\eta}^{2T}\boldsymbol{\Phi}_b^2 - \boldsymbol{\zeta}^{1T}\dot{\boldsymbol{\Phi}}_b^1 - \boldsymbol{\zeta}^{2T}\dot{\boldsymbol{\Phi}}_b^2 - \boldsymbol{\tau}\boldsymbol{\Omega}_b - \boldsymbol{\alpha}^T\boldsymbol{\varphi}_b - \boldsymbol{\beta}^T\bar{\boldsymbol{\varphi}}_b \tag{31}$$

$$\frac{d\boldsymbol{B}}{dt} = H_b - \boldsymbol{\mu}^T\boldsymbol{\Pi}_b - \boldsymbol{\nu}^T\boldsymbol{\Phi}_b$$

求解该微分方程初值问题可得 $t^2$ 时刻的函数值 $\boldsymbol{B}(t^2)$,而它恰好是目标函数的导数$\frac{d\psi}{d\boldsymbol{b}}$.

## 5 结果验证

对设计参数 $b$ 给定一个微小扰动$\delta b$,得到一个新的设计变量 $b^*$,即

$$\boldsymbol{b}^* = \boldsymbol{b} + \delta\boldsymbol{b} \tag{32}$$

利用泰勒展开式可得目标函数 $\psi$ 的一阶近似值

$$\psi(\boldsymbol{b}^*)\psi(\boldsymbol{b}) + \frac{d\psi(\boldsymbol{b})}{d\boldsymbol{b}}\delta\boldsymbol{b} \tag{33}$$

因此,在扰动 $\delta\boldsymbol{b}$ 下,目标函数的改变量为

$$\Delta\psi\psi(\boldsymbol{b}^*) - \psi(\boldsymbol{b})\frac{d\psi(\boldsymbol{b})}{d\boldsymbol{b}}\delta\boldsymbol{b}\delta\psi \tag{34}$$

可以通过分别计算 $\Delta\psi$ 和 $\delta\psi$,比较它们的值是否接近来验证结果的可靠性.如果 $\Delta\psi$ 和 $\delta\psi$ 非常接近,则认为结果是可靠的.

## 6 算例

图 1 为一曲柄 - 滑块系统,该系统由匀质的曲柄、连杆和理论滑块构成,曲柄、连杆的长度分别为 $l_1$,$l_2$ 质量分别为 $m_1$,$m_2$,滑块质量为 $m_3$.

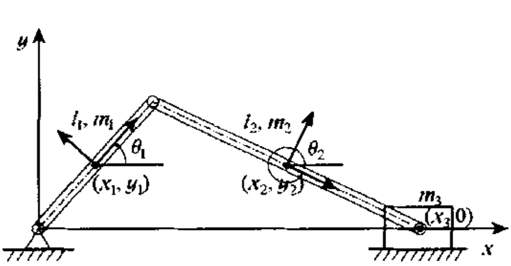


图 1 曲柄 - 滑块系统

Fig.1 A slider-crank mechanism

设系统状态变量为 $\boldsymbol{q} = [x_1, y_1, \theta_1, x_2, y_2, \theta_2, x_3]^T$,系统设计参数为 $b = [l_1, l_2, m_1, m_2, m_3]^T$,系统动力学方程的广义质量阵为

$$\boldsymbol{M} = diag[m_1, m_2, J_1, m_2, m_2, J_2, m_3],$$

$$J_1 = \frac{1}{12}m_1 l_1^2, J_2 = \frac{1}{12}m_2 l_2^2$$

当系统仅受重力作用时,其广义力列阵为

$$\boldsymbol{Q} = [0, -m_1 g, 0, 0, -m_2 g, 0, 0]^T$$

系统约束方程为

$$\boldsymbol{\Phi} = \begin{pmatrix} x_1 - \frac{l_1}{2}\cos\theta_1 \\ y_1 - \frac{l_1}{2}\sin\theta_1 \\ x_2 - l_1\cos\theta_1 - \frac{l_2}{2}\cos\theta_2 \\ y_2 - l_1\sin\theta_1 - \frac{l_2}{2}\sin\theta_2 \\ x_3 - l_1\cos\theta_1 - l_2\cos\theta_2 \\ l_1\sin\theta_1 + l_2\sin\theta_2 \end{pmatrix} = \boldsymbol{0}$$

取 $m_1 = m_2 = 1$ kg,$m_3 = 2$ kg,$l_1 = 1$ m,$l_2 = \sqrt{3}$ m,设 $\theta_2^1 = 30°$,$\boldsymbol{\Psi} = \int_0^1 (x_3 - x_3^1)^2 dt$,则利用本文方法求得

$\boldsymbol{\Psi} = 0.2306$,$\frac{d\boldsymbol{\Psi}}{d\boldsymbol{b}}$[−0.1287 −0.3242 0.2321 0.1183 − 0.1599]

给定微小扰动 $\delta b_i = 0.001$,$(i = 1, \cdots, 5)$,可得

$\boldsymbol{\Psi} = 0.2304$,$\frac{d\boldsymbol{\Psi}}{d\boldsymbol{b}}$ = [− 0.1285 − 0.3232 0.2318 0.1182 − 0.1597]

计算得

$$\Delta\psi = -2.1209 \times 10^{-4}, \delta\psi = -2.6235 \times 10^{-4}$$

二者值非常接近,因此该结果可靠.

## 7 结束语

对本文问题,当系统用 $n$ 个广义坐标描述,且有 $k$ 个设计变量时,采用直接微分方法需求解 $n \times k$ 个状态设计灵敏度的微分/代数方程组,当采用本文方法时,仅需求解 $n$ 个伴随变量的微分/代数方程组,其余伴随变量方程为简单的线性代数方程,从而具有较高的效率.为进一步提高精度,后继研究工作为二阶灵敏度分析的伴随变量方法.

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# ADJIONT VARIABLE METHOD FOR SENSITIVITY ANALYSIS OF MULTIBODY SYSTEM DYNAMICS DESCRIBED BY DIFFERENTIAL/ALGEBRAIC EQUATIONS*

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**Abstract**　An adjoint variable method was established for sensitivity analysis of multibody system dynamics described by differential/algebraic equations based on general objective function, which had high efficiency for systems with many design parameters and avoided the complex derivatives computation of the generalized coordinates with respect to design parameters. The adjoint variable equations for the first order sensitivity analysis and design sensitivity formulations were derived, and the detailed processes of the design sensitivity algorithm were presented. For the purpose of simplification, the objective function and its first derivative were transformed into an initial value problem of ordinary differential equation with one variable. An example of slide-crank system was given to validate the method presented.

**Key words**　Multibody system dynamics, Differential/algebraic equations, Sensitivity analysis, Adjoint variable method.

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