

空间曲梁非线性动力学方程

赵跃宇¹ 冯锐² 劳文全² 王连华¹

(1. 湖南大学土木工程学院, 长沙 410082)(2. 湖南大学工程力学系, 长沙 410082)

摘要 基于有限变形原理, 采用微分几何的方法推导了不考虑剪切、转动惯量和翘曲影响的曲梁的三维变形的应力-应变关系, 然后利用 Hamilton 变分原理推导了三维空间曲梁在考虑三个位移自由度和三个转动自由度下的非线性动力学方程. 把得到的非线性动力学方程退化为面内圆弧拱的线性动力学方程, 并与已有结果进行了对比. 非线性动力学方程的建立为曲梁的非线性动力学分析做好了必要的准备.

关键词 空间曲梁, 动力学方程, 微分几何, 变分原理

引言

空间曲梁的非线性静力和动力分析在航空和土木工程中有着十分重要的意义, 在做结构分析时, 计算方法通常有两种: 一种是用分段的直线来近似代替曲线, 按直梁的计算理论来计算曲梁; 另一种是直接建立曲梁单元, 也就是假设单元存在初始曲率, 在这种情况下, 荷载所引起的梁截面上的轴力、弯矩、剪力和扭矩作用相互耦合, 致使单元的平衡微分方程非常复杂, 一般呈非线性的形式, 因而这种传统的曲梁计算理论十分繁冗, 具有相当的局限性. 在对曲梁单元早期的研究中, 曾遇到了应力锁定和计算结果不稳定等的严重问题, 以致使曲梁单元的计算精度有时还不如直梁单元. 后来发展了降阶积分、选择积分和混合有限元法等一些较为有效的计算方法, 使得曲梁单元的计算精度、计算稳定性和计算效率有了较大的提高.

由于曲梁单元的复杂性, 对曲梁的初始研究仅限于线性分析, 而对曲梁非线性静力和动力问题的研究却很少. Yeongby^[1] 讨论了空间曲梁单元的非线性有限元分析, 但是其中所列出来的空间曲梁单元的应变-位移关系式没有考虑初始曲率的影响. Rosen^[2], Bauchau^[3], Pai^[4] 在空间曲梁非线性方面做了不少研究, 并分析推导了空间曲梁的应变位移关系式, 虽然考虑了初始曲率的影响, 但空间

曲梁轴线在三维变形中的曲率位移关系却未能得到通顺流畅的证明. 尽管许多学者讨论了曲梁单元的强度计算和屈曲分析, 但是尚未见到很好的空间曲梁单元在三维变形中的应变-位移关系显式表达式, 更没有见到三维曲梁模型考虑 3 个位移分量和 3 个转动分量时的非线性动力学方程.

本文就是在考虑 3 个位移自由度和 3 个转动自由度的情况下, 采用非线性弹性力学的方法, 在 Ojalvo^[5], Wen^[6,7], 周文伟^[8] 等人研究的基础上, 通过哈密顿变分原理, 考虑轴向伸长, 不考虑剪切应变和转动惯量以及翘曲的影响, 建立空间曲梁的非线性动力学方程, 为以后进行曲梁的非线性动力学研究做出必要准备.

1 空间弹性曲梁的三维几何关系

考虑初始空间弹性曲梁形心主轴线上任意一点的 P 的矢量函数 $\bar{R}(\bar{s})$, 其中 \bar{s} 是弧长, 变形后, P 点的矢量函数为 $R(\bar{s})$, $\bar{e}_x, \bar{e}_y, \bar{e}_z$ 为未变形弹性曲梁中心轴线 s 上 P 点截面处形心主轴坐标系的单位向量, 则有

$$R(\bar{s}) = \bar{R}(\bar{s}) + u\bar{e}_x + v\bar{e}_y + w\bar{e}_z \quad (1)$$

其中 u, v, w 分别为 P 点在 $\bar{e}_x, \bar{e}_y, \bar{e}_z$ 方向上产生的线位移. 则沿着轴线方向的 Lagrang 应变为

$$\epsilon_z = \frac{1}{2} \left[\frac{dR}{ds} \frac{dR}{ds} - 1 \right] \quad (2)$$

对(1)两边求导

$$\frac{dR}{ds} = \frac{d\bar{R}}{ds} + \frac{du}{ds}e_x + u \frac{de_x}{ds} + \frac{dv}{ds}e_y + v \frac{de_y}{ds} + \frac{dw}{ds}e_z + w \frac{de_z}{ds} \quad (3)$$

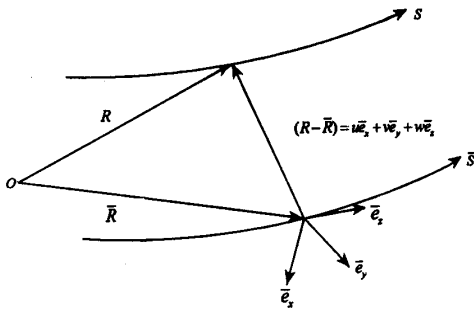


图 1 变形前后曲梁轴线位移图

Fig. 1 Pictorial relationship between reference-state and departure-state curved beam centerlines

这里考虑自然坐标,如图(2) $\bar{\rho}$ 为曲率半径 \bar{n} , $\bar{\tau}$, \bar{b} 分别为曲梁形心主轴线的法线,切线和次法线方向,根据微分几何的 Frenet-Serret 公式

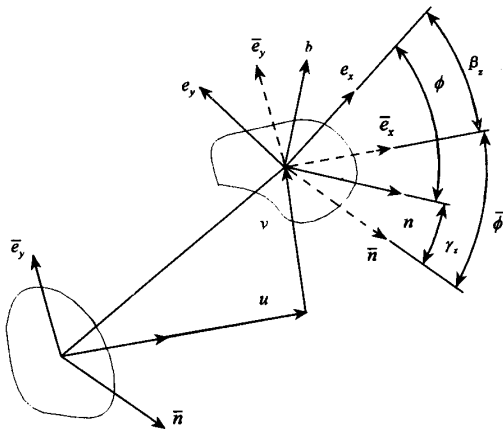


图 2 形心轴线坐标的变化关系

Fig. 2 Cross-sectional projections of reference and departure states

$$\begin{aligned} \frac{d\bar{\tau}}{ds} &= \frac{1}{\bar{\rho}}\bar{n} \\ \frac{d\bar{n}}{ds} &= -\frac{1}{\bar{\rho}}\bar{\tau} + \bar{\tau}b \\ \frac{d\bar{b}}{ds} &= -\bar{\tau}\bar{n} \end{aligned} \quad (4)$$

则它与主轴坐标的转换关系为

$$\begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = [\bar{\phi}] \begin{Bmatrix} \bar{n} \\ \bar{b} \\ \bar{\tau} \end{Bmatrix} \quad (5)$$

其中

$$[\bar{\phi}] = \begin{bmatrix} \cos\bar{\phi} & \sin\bar{\phi} & 0 \\ -\sin\bar{\phi} & \cos\bar{\phi} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (6)$$

$\bar{\phi}$ 为 e_x 与 \bar{n} 的夹角.这里定义曲梁的轴心主曲率分量分别为 $\bar{\kappa}_x, \bar{\kappa}_y, \bar{\kappa}_z$

$$\begin{aligned} \bar{\kappa}_x &= \frac{1}{\bar{\rho}}\sin\bar{\phi} \\ \bar{\kappa}_y &= \frac{1}{\bar{\rho}}\cos\bar{\phi} \\ \bar{\kappa}_z &= \bar{\tau} + \frac{d\bar{\phi}}{ds} \end{aligned} \quad (7)$$

由式(4)~(7)可以推出

$$\frac{de_x}{ds} = \bar{\kappa}_ze_y - \bar{\kappa}_ye_z \quad (8)$$

$$\frac{de_y}{ds} = -\bar{\kappa}_ze_x + \bar{\kappa}_xe_z \quad (9)$$

$$\frac{de_z}{ds} = \bar{\kappa}_ye_x - \bar{\kappa}_xe_y \quad (10)$$

于是式(3)可以表达为

$$\begin{aligned} \frac{dR}{ds} &= \frac{d\bar{R}}{ds} + \left(\frac{du}{ds} - \bar{\kappa}_zu + \bar{\kappa}_xw \right) e_x - \\ &\quad \left(-\frac{dv}{ds} - \bar{\kappa}_zv + \bar{\kappa}_xw \right) e_y + \\ &\quad \left(\frac{dw}{ds} - \bar{\kappa}_yu + \bar{\kappa}_xv \right) e_z \end{aligned} \quad (11)$$

根据文献[8]得到的关系,而 e_x, e_y, e_z 为变形后形心主轴单位坐标

$$\begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} = [T] \begin{Bmatrix} e_x \\ e_y \\ e_z \end{Bmatrix} \quad (12)$$

其中

$$[T] = \begin{bmatrix} 1 & \beta_z & -\beta_y \\ -\beta_z & 1 & \beta_x \\ \beta_y & -\beta_x & 1 \end{bmatrix} \quad (13)$$

其中 $\beta_x, \beta_y, \beta_z$ 为绕形心主轴转动的3个独立的转角位移.因为 $\frac{dR}{ds} = e_x, \frac{d\bar{R}}{ds} = \bar{e}_z$,所以(11)可写为

$$\begin{aligned} e_x &= \bar{e}_z + \left(\frac{du}{ds} - \bar{\kappa}_zu + \bar{\kappa}_xw \right) \bar{e}_x - \\ &\quad \left(-\frac{dv}{ds} - \bar{\kappa}_zv + \bar{\kappa}_xw \right) \bar{e}_y + \\ &\quad \left(\frac{dw}{ds} - \bar{\kappa}_yu + \bar{\kappa}_xv \right) \bar{e}_z \end{aligned} \quad (14)$$

因为 $\epsilon = \frac{dw}{ds} - \bar{\kappa}_yu + \bar{\kappa}_xv$ 比1小很多,所以可由式

(12) ~ (14) 得

$$\beta_x = -\frac{dv}{ds} - \bar{\kappa}_z u + \bar{\kappa}_x w \quad (15)$$

$$\beta_y = \frac{du}{ds} - \bar{\kappa}_z v + \bar{\kappa}_y w \quad (16)$$

应用式(15) ~ (17), 可以把式(2)展开为

$$\epsilon_z = \frac{1}{2} [(1 + \epsilon)^2 + \beta_x^2 + \beta_y^2 - 1] \quad (17)$$

也可以写成

$$\epsilon_z = \frac{1}{2} [(1 + \frac{dw}{ds} - \bar{\kappa}_y u + \bar{\kappa}_x v)^2 + \beta_x^2 + \beta_y^2 - 1] \quad (18)$$

2 空间曲梁的位移 - 应变关系

考虑空间的任意曲梁如图(3), 其轴线为 S , 轴线上任意一点 P 的曲率半径为 $\bar{\rho}(s)$, x, y, z 分别为曲梁形心主轴坐标系, 相对应的线位移为 u, v, w , 转角位移为 $\beta_x, \beta_y, \beta_z$, 则曲梁在 S 截面上任意一点 (ζ, η) 的沿着主轴线上的应变为

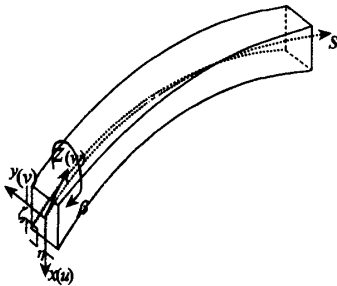


图3 空间任意截面的曲梁模型

Fig. 3 Spatial arbitrary cross sectional curved beam model

$$\epsilon_{z, \zeta, \eta} = \epsilon_{z, o} + \eta \kappa_x - \zeta \kappa_y \quad (19)$$

其中

$$\epsilon_{z, o} = \frac{1}{2} [(1 + \epsilon)^2 + \beta_x^2 + \beta_y^2 - 1] \quad (20)$$

$$\kappa_x = \bar{\kappa}_x + \bar{\kappa}_y \beta_z - \bar{\kappa}_z \beta_y + \frac{d\beta_x}{ds} \quad (21)$$

$$\kappa_y = -\bar{\kappa}_x \beta_z + \bar{\kappa}_y - \bar{\kappa}_z \beta_x + \frac{d\beta_y}{ds} \quad (22)$$

$$\epsilon = \frac{dw}{ds} - \bar{\kappa}_y u + \bar{\kappa}_x v \quad (23)$$

3 动力学方程的建立

考虑只有保守力而无外力的自由振动, 根据 Hamilton 变分原理

$$\delta \int_{t_1}^{t_2} (T - U) dt = 0 \quad (24)$$

这里 $U = U_e + U_i$ 为曲梁的应变能, 不考虑剪切应变

$$U_e = \int_V \frac{E\epsilon^2}{2} dV = \int_s \int_A \frac{E\epsilon^2}{2} dA ds \quad (25)$$

$$U_i = \int_s \frac{GK_t}{2} \kappa_z^2 ds \quad (26)$$

曲梁的动能为

$$T = \frac{1}{2} \int_V \{ mA \{ (\frac{\partial u}{\partial t})^2 + (\frac{\partial v}{\partial t})^2 + (\frac{\partial w}{\partial t})^2 \} + mI_x (\frac{\partial \beta_x}{\partial t})^2 + mI_y (\frac{\partial \beta_y}{\partial t})^2 + mI_z (\frac{\partial \beta_z}{\partial t})^2 \} dV \quad (27)$$

其中 E 为弹性模量, G 为剪切模量, m 为曲梁的线密度, A 为截面面积, I_x, I_y, I_z 分别为 x, y, z 轴的截面惯性矩, 从而得到非线性动力学方程为

$$-\frac{EA}{2} [(1 + \frac{\partial w}{\partial s} - u\bar{\kappa}_y + v\bar{\kappa}_x)^2 + \beta_x^2 + \beta_y^2 - 1] (1 + \frac{\partial w}{\partial s} - u\bar{\kappa}_y + v\bar{\kappa}_x) \bar{\kappa}_y + mA \frac{\partial^2 u}{\partial t^2} = 0 \quad (28)$$

$$\frac{EA}{2} [(1 + \frac{\partial w}{\partial s} - u\bar{\kappa}_y + v\bar{\kappa}_x)^2 + \beta_x^2 + \beta_y^2 - 1] (1 + \frac{\partial w}{\partial s} - u\bar{\kappa}_y + v\bar{\kappa}_x) \bar{\kappa}_x + mA \frac{\partial^2 v}{\partial t^2} = 0 \quad (29)$$

$$-EA \{ \frac{1}{2} [\frac{\partial^2 w}{\partial s^2} - \frac{\partial(u\bar{\kappa}_y)}{\partial s} + \frac{\partial(v\bar{\kappa}_x)}{\partial s}] (1 + \frac{\partial w}{\partial s} - u\bar{\kappa}_y + v\bar{\kappa}_x)^2 + \beta_x^2 + \beta_y^2 - 1 \} + (\beta_x \frac{\partial \beta_x}{\partial s} + \beta_y \frac{\partial \beta_y}{\partial s}) (1 + \frac{\partial w}{\partial s} - u\bar{\kappa}_y + v\bar{\kappa}_x) + mA \frac{\partial^2 w}{\partial t^2} = 0 \quad (30)$$

$$\frac{EA}{2} [(1 + \frac{\partial w}{\partial s} - u\bar{\kappa}_y)^2 + \beta_x^2 + \beta_y^2 - 1] \beta_x + EI_x \bar{\kappa}_z [-\bar{\kappa}_x \beta_z + \bar{\kappa}_y + \beta_x \bar{\kappa}_z + \frac{\partial \beta_y}{\partial s}] - EI_y [\frac{\partial \bar{\kappa}_x}{\partial s} + \frac{\partial(\beta_x \bar{\kappa}_y)}{\partial s} - \frac{\partial(\beta_y \bar{\kappa}_x)}{\partial s} + \frac{\partial^2 \beta_x}{\partial s^2}] - GK_t \bar{\kappa}_y [\beta_y \bar{\kappa}_x - \beta_x \bar{\kappa}_y + \bar{\kappa}_z + \frac{\partial \beta_z}{\partial s}] + mI_x \frac{\partial^2 \beta_x}{\partial t^2} = 0 \quad (31)$$

$$\frac{EA}{2} [(1 + \frac{\partial w}{\partial s} - u\bar{\kappa}_y)^2 + \beta_x^2 + \beta_y^2 - 1] \beta_y - EI_x \bar{\kappa}_z [\bar{\kappa}_x + \beta_x \bar{\kappa}_y - \beta_y \bar{\kappa}_z + \frac{\partial \beta_x}{\partial s}] +$$

$$EI_x \left[-\frac{\partial(\beta_x \bar{\kappa}_x)}{\partial s} + \frac{\partial \bar{\kappa}_y}{\partial s} + \frac{\partial(\beta_x \bar{\kappa}_z)}{\partial s} + \frac{\partial^2 \beta_y}{\partial s^2} \right] + GK \bar{\kappa}_x [\beta_y \bar{\kappa}_x - \beta_x \bar{\kappa}_y - \bar{\kappa}_z + \frac{\partial \beta_x}{\partial s}] + m I_y \frac{\partial^2 \beta_y}{\partial t^2} = 0 \quad (32)$$

$$EI_y \bar{\kappa}_y [\bar{\kappa}_x + \beta_x \bar{\kappa}_y - \beta_y \bar{\kappa}_z + \frac{\partial \beta_x}{\partial s}] - EI_y \bar{\kappa}_x [-\beta_x \bar{\kappa}_x + \bar{\kappa}_y + \beta_x \bar{\kappa}_z + \frac{\partial \beta_y}{\partial s}] - GK_t \left[\frac{\partial(\beta_x \bar{\kappa}_x)}{\partial s} - \frac{\partial(\beta_x \bar{\kappa}_y)}{\partial s} + \frac{\partial \bar{\kappa}_z}{\partial s} + \frac{\partial^2 \beta_x}{\partial s^2} \right] + m I_x \frac{\partial^2 \beta_x}{\partial t^2} = 0 \quad (33)$$

这样就得到六个自由度的空间曲梁非线性动力学方程. 利用以上公式我们可以进行简化, 只考虑面内圆弧拱的动力学方程, 半径为 R , 通过简化我们可以得到

$$EA \left\{ \frac{1}{R} u' \dot{w}' - \frac{1}{R^2} u u'' - \dot{w}'' + \frac{1}{R} u' - u \ddot{u} - \frac{1}{R} u \dot{w}' - \frac{1}{R} u \ddot{w} - \frac{1}{R^3} u w \right\} - EI_x \left\{ \frac{1}{R} u'' + \frac{1}{R^2} w'' \right\} + m \ddot{w} + c \dot{w} = 0 \quad (34)$$

$$EA \left\{ u' \dot{w}'' - \frac{1}{R} u'^2 - u'' \dot{w}' - \frac{1}{R} u u'' - u' \frac{w}{R^2} - \frac{1}{R^2} u w' + \frac{1}{R} \dot{w}' - \frac{u}{R^2} + \frac{1}{2R} u'^2 + \frac{1}{R^2} u' w \right\} - EI_x \left\{ u^{(4)} + \frac{1}{R} w'' \right\} - m \ddot{u} - c \dot{u} = 0 \quad (35)$$

上面方程的是 m 单位长度质量, $w(s, t)$ 是切向位移, $u(s, t)$ 是径向位移, c 是粘性阻尼系数, ' 表示对弧长 s 求导, '' 表示对时间 t 求导. 为了简化的需要, 上面方程只列出了平方非线性的项, 去掉了高次非线性. 为了进一步得到验证, 把上面的动力学方程去掉非线性项, 得到面内圆弧拱的线性动力学方程如下

$$EA \left\{ -\dot{w}'' + \frac{1}{R} u' \right\} - EI_x \left\{ \frac{1}{R} u'' + \frac{1}{R^2} w'' \right\} + m \ddot{w} + c \dot{w} = 0 \quad (36)$$

$$EA \left\{ \frac{1}{R} \dot{w}' - \frac{u}{R^2} \right\} - EI_x \left\{ u^{(4)} + \frac{1}{R} w'' \right\} - m \ddot{u} + c \dot{u} = 0 \quad (37)$$

以上线性方程与文献[9]中的平面圆弧拱的动力学方程完全一致, 说明推导出的方程的正确性.

4 结论

在对曲梁进行非线性动力学特性分析时, 动力学方程的建立是必不可少的. 本文建立了空间曲梁的六个位移自由度的非线性动力学方程, 它考虑了初始曲率对应变和位移的影响, 同时考虑了轴向应变和扭转应变, 不考虑剪切, 转动惯量和翘曲影响. 建立的非线性动力学方程是最基本的方程, 它可以通过简化到面内和面外分别考虑动力学特性. 因此本文是曲梁非线性动力学分析的基础工作, 为以后的深入研究做了必要的准备.

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NONLINEAR KINEMATIC EQUATION OF SPATIAL CURVED BEAM

Zhao Yueyu¹ Feng Rui² Lao Wenquan² Wang Lianhua¹

(1. *Civil engineering of hunan university, hunan, changsha 410082, China*)

(2. *Mechanics engineering Department of hunan university, hunan, changsha 410082, China*)

Abstract Based on the theory of large deformation and using the differential geometry. The paper derived the relationship of strain and displacement for spatial curved beams, taking into account the longitudinal strains and torsion strains, but neglecting the effects of shear, moment of inertia and warping. The Hamilton variation principle was used to derive the nonlinear dynamics equations of spatial curved beam under three-displacement-freedom and three-rotation-freedom, which can be degenerated to linear dynamics equations of planar circular arch, and their results were compared with the results in literatures. The nonlinear dynamics equations provide a reliable foundation for analysis of the nonlinear dynamics of curved beam.

Key words spatial curved beam, nonlinear kinematic equation, differential geometry, variation principle