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哈密顿体系下矩形薄板自由振动的一般解*

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摘要　根据弹性薄板自由振动问题的基本方程,把问题引入到哈密顿对偶体系中.x方向模拟为时间,选取弯矩,等效剪力,转角和挠度为对偶向量,得到了在不同边界条件时关于x轴对称和反对称时的解析解.算例研究了四边固支薄板的自由振动情形,从而推广了哈密顿体系的应用范围,验证了哈密顿体系求解方法在自由振动问题中的有效性.

关键词　哈密顿体系,薄板,自由振动,本征向量

引言

弹性力学的哈密顿体系是由钟万勰[1~4]提出的理论,在研究弹性力学中的平面域、扇形域、薄板、中厚板问题取得了一定的成功.其主导思想是从现代控制论的数学问题与结构力学问题互相模拟[5]的角度引入哈密顿体系理论和辛几何的概念,其中对偶求解体系的辛正交关系和本征函数展开解法是区别于常规体系的重点.对于本征函数,需要考虑零本征值和非零本征值是否为重根的情况.对薄板和中厚板的哈密顿求解体系的相关文献参见[6~11],哈密顿体系已形成近年来研究的一个热点问题,如在断裂力学,空间粘性流体问题,电磁波导等领域[12~15]中得到应用.

薄板的自由振动问题是一个经典问题,在拉格朗日体系下许多学者研究过.如D.J.Gorman[16]用迭加法研究了各种不同边界条件组合下的这类问题.Huang Yan[17]用分离变量法求得了这一问题的一般解.对于角点的解,选择不同型式(多项式解、双正弦级数或两者结合),可避免多项式解或双正弦级数引起的耦合关系.

薄板自由振动问题中,经典算法针对高次偏微分方程进行分离变量法,这种传统的分离变量法会受到自共轭算子谱的限制,本征函数的正交性和完备性在理论上是不能保证的.而对于其它边界,一般给出的不是本征函数,为了提高精度,通行的补救方法是增加级数的项数或进行迭代.这种传统方法的数学框架是欧几里得空间,本文试图将哈密顿力学的辛算法引入到连续力学中.

迄今,还未见到利用哈密顿体系研究薄板的自由振动问题的文献.文[11,18]介绍了薄板静力问题哈密顿体系的一种构造方法,本文在其基础上,得到了薄板自由振动问题的对偶方程组,以及在不同边界条件时把x方向模拟为时间的解析解.算例考虑了四边固支方形薄板的自由振动情形.

1　弹性薄板弯曲自由振动问题的对偶体系

弹性薄板自由振动问题的控制方程为

$$D\nabla^4 w-\rho\omega^2 w=0 \tag{1}$$

其中 $\nabla^4=\left(\dfrac{\partial^4}{\partial x^4}+2\dfrac{\partial^4}{\partial x^2\partial y^2}+\dfrac{\partial^4}{\partial y^4}\right)$.

为了得到弹性薄板弯曲自由振动问题的哈密顿对偶方程,几何方程中用到

$$\phi_x=\frac{\partial w}{\partial x} \tag{2}$$

利用式(1)及如下关系

$$\begin{aligned}M_x&=D\left(\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^2 w}{\partial y^2}\right),\\ M_y&=D\left(v\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right),\\ Q_y&=\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y},\\ M_{xy}&=D(1-v)\frac{\partial^2 w}{\partial x\partial y}\end{aligned} \tag{3}$$

选4个对偶变量 $M_y,\bar{Q}_y=-\left(Q_y+\dfrac{\partial M_{xy}}{\partial x}\right),\phi_y,w$ 组成向量$\boldsymbol{v}$,可得

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$$\dot{M}_y=-\bar{Q}_y+2D(1-v)\frac{\partial^2\phi_y}{\partial x^2}$$
$$\dot{\bar{Q}}_y=v\frac{\partial^2 M_y}{\partial x^2}-D(1-v^2)\frac{\partial^4 w}{\partial x^4}+\rho\omega^2 w$$
$$\phi_y=-\frac{1}{D}M_y-v\frac{\partial^2 w}{\partial x^2}$$
$$\dot{w}=\phi_y \tag{4}$$

式(4)写成矩阵形式如下

$$\dot{\boldsymbol{v}}=\boldsymbol{H}\boldsymbol{v} \tag{5}$$

其中

$$\boldsymbol{H}=\begin{bmatrix}0 & 1 & -2D(1-v)\frac{\partial^2}{\partial x^2} & 0\\ -v\frac{\partial^2}{\partial x^2} & 0 & 0 & D(1-v)\frac{\partial^4}{\partial x^4}-\rho\omega^2\\ \frac{1}{D} & 0 & 0 & v\frac{\partial^2}{\partial x^2}\\ 0 & 0 & -1 & 0\end{bmatrix} \tag{6}$$

其中 $\boldsymbol{H}$ 为哈密顿算子矩阵,因此其本征问题有如下的特点:本征值对应的负本征值及它们的共轭本征值也是其本征值;本征向量之间的共轭辛正交关系;全状态函数向量展开定理.进入哈密顿体系中,就可以采用分离变量法,设全状态向量为

$$v(x,y)=\xi(y)\psi(x) \tag{7}$$

把式(7)代入式(5)中推出

$$\xi(y)=e^{\mu y},H\psi(x)=\mu\psi(x) \tag{8}$$

其中 μ 是本征值,待求.全状态向量 $\boldsymbol{v}$ 还需要满足边界条件.

2 零本征解

零本征解满足方程

$$H\psi(x)=0 \tag{9}$$

当 $x=\pm a$ 两端简支或简支时,零本征向量为零向量.

对 $x=\pm a$ 两端自由时,说明有 6 个零本征值的本征解,即薄板的 3 个刚体位移及对应于薄板静力问题的纯弯曲、纯扭转和常剪力弯曲解.

3 解析解

展开本征方程(4)得

$$\mu M_y+\bar{Q}_y-2D(1-v)\frac{\partial^2\phi_y}{\partial x^2}=0$$
$$-v\frac{\partial^2 M_y}{\partial x^2}+\mu\bar{Q}_y+D(1-v)\frac{\partial^4 w}{\partial x^4}-\rho\omega^2 w=0$$
$$\frac{1}{D}M_y+\mu\phi_y+v\frac{\partial^2 w}{\partial x^2}=0$$
$$-\phi_y+\mu w=0 \tag{10}$$

式(4)的特征方程为

$$\det\begin{bmatrix}\mu & 1 & -2D(1-v)\lambda^2 & 0\\ -v\lambda^2 & \mu & 0 & D(1-v)\lambda^4-\rho\omega^2\\ \frac{1}{D} & 0 & 0 & v\lambda^2\\ 0 & 0 & -1 & \mu\end{bmatrix} \tag{11}$$

展开得

$$(\lambda^2+\mu^2)^2=\rho\omega^2/D \tag{12}$$

令 $R=\sqrt{\rho\omega^2/D}$,有

$$\lambda_{1,2}=\pm i\sqrt{\mu^2-R},\lambda_{3,4}=\pm i\sqrt{\mu^2+R} \tag{13}$$

由此得式(4)的通解为

$$M_y=A_1\cos\left(x\sqrt{\mu^2-R}\right)+B_1\sin\left(x\sqrt{\mu^2-R}\right)+C_1\cos\left(x\sqrt{\mu^2+R}\right)+D_1\sin\left(x\sqrt{\mu^2+R}\right)$$
$$\bar{Q}_y=A_2\cos\left(x\sqrt{\mu^2-R}\right)+B_2\sin\left(x\sqrt{\mu^2-R}\right)+C_2\cos\left(x\sqrt{\mu^2+R}\right)+D_2\sin\left(x\sqrt{\mu^2+R}\right)$$
$$\phi_y=A_3\cos\left(x\sqrt{\mu^2-R}\right)+B_3\sin\left(x\sqrt{\mu^2-R}\right)+C_3\cos\left(x\sqrt{\mu^2+R}\right)+D_3\sin\left(x\sqrt{\mu^2+R}\right)$$
$$w=A_4\cos\left(x\sqrt{\mu^2-R}\right)+B_4\sin\left(x\sqrt{\mu^2-R}\right)+C_4\cos\left(x\sqrt{\mu^2+R}\right)+D_4\sin\left(x\sqrt{\mu^2+R}\right) \tag{14}$$

如果把特征根取成 $\lambda_{1,2}=\pm\sqrt{\mu^2-R},\lambda_{3,4}=\pm\sqrt{-R-\mu^2}$,则解的形式可用双曲函数表示.

可以看出 A 与 C 组的解是关于 x 轴对称的解,而 B,D 组关于 x 轴为反对称.

3.1 关于 x 轴对称

式(14)中只取含 A,C 的解代入式(4),可列出对称时的方程为

$$\begin{bmatrix}\mu & 1 & -2D(1-v)\mu_1^2 & 0\\ v\mu_1^2 & \mu & 0 & D(1-v^2)\mu_1^4-\omega^2\\ \frac{1}{D} & 0 & \mu & -v\mu_1^2\\ 0 & 0 & -1 & \mu\end{bmatrix}\times\begin{Bmatrix}A_1\\A_2\\A_3\\A_4\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\end{Bmatrix} \tag{15}$$

和

$$\begin{bmatrix} \mu & 1 & -2D(1-v)\mu_2^2 & 0 \\ v\mu_2^2 & \mu & 0 & D(1-v^2)\mu_2^4-\omega^2 \\ \frac{1}{D} & 0 & \mu & -v\mu_2^2 \\ 0 & 0 & -1 & \mu \end{bmatrix} \times \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{16}$$

这里记 $\mu_1=\sqrt{\mu^2-R}$，$\mu_2=\sqrt{\mu^2+R}$ 方程(15)的行列式为零，因此可以求解，有

$$\begin{cases} A_1=[(v-1)\mu^2-vR]A_4 \\ A_2=D\mu[(v-1)\mu^2-(v-2)R]A_4 \\ A_3=\mu A_4 \end{cases} \tag{17}$$

由方程(16)的行列式是零，可得

$$\begin{cases} C_1=[(v-1)\mu^2-vR]C_4 \\ C_2=D\mu[(v-1)\mu^2+(v-2)R]C_4 \\ C_3=\mu C_4 \end{cases} \tag{18}$$

其中 A_4，C_4 为待定常数.

下面考虑边界条件：

1) 简支边界条件为

$$w=0, M_x=-D(1-v^2)\frac{\partial^2 w}{\partial x^2}+vM_y=0 \quad x=\pm a \text{ 时} \tag{19}$$

把系数关系式及一般解(只取 A，C 项)代入式(19)，整理得

$$\begin{cases} A_4\cos\left(a\sqrt{\mu^2-R}\right)+ \\ \quad C_4\cos\left(a\sqrt{\mu^2+R}\right)=0 \\ A_4[(2v^2-v-1)\mu^2+(1-2v^2)R]\times \\ \quad \cos\left(a\sqrt{\mu^2-R}\right)+C_4[(2v^2-v-1)\mu^2- \\ \quad (1-2v^2)R]\cos\left(a\sqrt{\mu^2+R}\right)=0 \end{cases} \tag{20}$$

为了使此方程组有解，应有 A_4，C_4 系数行列式为零，故超越方程为

$$\cos\left(a\sqrt{\mu^2-R}\right)\cos\left(a\sqrt{\mu^2+R}\right)=0 \tag{21}$$

对不同的 R 值，式(21)有实根 $\pm\mu a$.

求出本征根后，可得本征根对应的本征函数向量

$$\boldsymbol{\psi}_n=\begin{cases} D\left\{[(v-1)\mu_n^2-vR]\cos\left(x\sqrt{\mu_n^2-R}\right)- \right. \\ \quad [(v-1)\mu_n^2+vR]\cos\left(a\sqrt{\mu_n^2-R}\right)\times \\ \quad \left.\cos\left(x\sqrt{\mu_n^2+R}\right)/\cos\left(a\sqrt{\mu^2+R}\right)\right\} \\ D\mu_n\left\{[(v-1)\mu_n^2-(v-2)R]\cos\left(x\sqrt{\mu_n^2-R}\right)- \right. \\ \quad [(v-1)\mu_n^2+(v-2)R]\cos\left(a\sqrt{\mu_n^2-R}\right)\times \\ \quad \left.\cos\left(x\sqrt{\mu_n^2+R}\right)/\cos\left(a\sqrt{\mu_n^2+R}\right)\right\} \\ \mu_n\left\{\cos\left(x\sqrt{\mu_n^2-R}\right)-\cos\left(a\sqrt{\mu_n^2-R}\right)\times \right. \\ \quad \left.\cos\left(x\sqrt{\mu_n^2+R}\right)/\cos\left(a\sqrt{\mu^2+R}\right)\right\} \\ \cos\left(x\sqrt{\mu_n^2-R}\right)-\cos\left(a\sqrt{\mu_n^2-R}\right)\times \\ \quad \cos\left(x\sqrt{\mu_n^2+R}\right)/\cos\left(a\sqrt{\mu^2+R}\right) \end{cases} \tag{22}$$

2) 固支边界条件为

$$w=0, \mathrm{d}w/\mathrm{d}x=0 \quad x=\pm a \text{ 时} \tag{23}$$

把系数关系式及一般解(只取 A，C 项)代入式(23)，整理得

$$\begin{cases} A_4\cos\left(a\sqrt{\mu^2-R}\right)+C_4\cos\left(a\sqrt{\mu^2+R}\right)=0 \\ A_4\sqrt{\mu^2-R}\sin\left(a\sqrt{\mu^2-R}\right)+C_4\sqrt{\mu^2+R}\times \\ \quad \sin\left(a\sqrt{\mu^2+R}\right)=0 \end{cases} \tag{24}$$

为了使此方程组有解，应有 A_4，C_4 系数行列式为零，故超越方程为

$$\sqrt{\mu^2-R}\tan\left(a\sqrt{\mu^2-R}\right)=\sqrt{\mu^2+R}\times \tan\left(a\sqrt{\mu^2+R}\right) \tag{25}$$

对不同的 R 值，令 $\mu a\pm\alpha\pm\mathrm{i}\beta$，有 μ，$-\mu$ 都是式(25)的根.

求出本征根后，可得本征根对应的本征函数向量

$$\boldsymbol{\psi}_n=\begin{cases} D\left\{[(v-1)\mu_n^2-vR]\cos\left(x\sqrt{\mu_n^2-R}\right)- \right. \\ \quad [(v-1)\mu_n^2+vR]\cos\left(a\sqrt{\mu_n^2-R}\right)\times \\ \quad \left.\cos\left(x\sqrt{\mu_n^2+R}\right)/\cos\left(a\sqrt{\mu_n^2+R}\right)\right\} \\ D\mu_n\left\{[(v-1)\mu_n^2-(v-2)R]\cos\left(x\sqrt{\mu_n^2-R}\right)- \right. \\ \quad [(v-1)\mu_n^2+(v-2)R]\cos\left(a\sqrt{\mu_n^2-R}\right)\times \\ \quad \left.\cos\left(x\sqrt{\mu_n^2+R}\right)/\cos\left(a\sqrt{\mu_n^2+R}\right)\right\} \\ \mu_n\left\{\cos\left(x\sqrt{\mu_n^2-R}\right)-\cos\left(a\sqrt{\mu_n^2-R}\right)\times \right. \\ \quad \left.\cos\left(x\sqrt{\mu_n^2+R}\right)/\cos\left(a\sqrt{\mu^2+R}\right)\right\} \\ \cos\left(x\sqrt{\mu_n^2-R}\right)-\cos\left(a\sqrt{\mu_n^2-R}\right)\times \\ \quad \cos\left(x\sqrt{\mu_n^2+R}\right)/\cos\left(a\sqrt{\mu_n^2+R}\right) \end{cases} \tag{26}$$

此本征向量即式(26)与简支边界条件时的本征函数向量式(22)具有相同的形式.

3) 自由边界条件为

$$M_x = 0, \bar{Q}_x = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-v)\mu^2\frac{\partial w}{\partial x}\right] = 0 \quad x = \pm a \text{ 时} \tag{27}$$

把系数关系式及一般解(只取 A, C 项)代入式(27),整理得

$$\begin{cases} A_4[(2v^2 - v - 1)\mu^2 + (1 - 2v^2)R]\times \\ \quad \cos\left(a\sqrt{\mu^2 - R}\right) + C_4[(2v^2 - v - 1)\mu^2 - \\ \quad (1 - 2v^2)R]\cos\left(a\sqrt{\mu^2 + R}\right) = 0 \\ A_4\sqrt{\mu^2 - R}[(1 - v)\mu^2 + R]\times \\ \quad \sin\left(a\sqrt{\mu^2 - R}\right) + C_4\sqrt{\mu^2 + R}\times \\ \quad [(1 - v)\mu^2 - R]\sin\left(a\sqrt{\mu^2 + R}\right) = 0 \end{cases} \tag{28}$$

为了使此方程组有解,应有 A_4, C_4 系数行列式为零,故超越方程为

$$\frac{\sqrt{\mu^2 - R}}{\sqrt{\mu^2 + R}} \cdot \frac{(1 - v)\mu^2 + R}{(1 - v)\mu^2 - R}\times \frac{\tan\left(a\sqrt{\mu^2 - R}\right)}{\tan\left(a\sqrt{\mu^2 + R}\right)} = \frac{(2v^2 - v - 1)\mu^2 + (1 - 2v^2)R}{(2v^2 - v - 1)\mu^2 - (1 - 2v^2)R} \tag{29}$$

对不同的 R 值,令 $\mu a = \pm\alpha \pm \mathrm{i}\beta$,有 $\mu, -\mu$ 都是式(29)的根.

求出本征根后,可得本征根对应的本征函数向量.

对以上 3 种边界条件,相应问题(4)的解为

$$\boldsymbol{v}_n = \exp(\mu_n y)\boldsymbol{\psi}_n \tag{30}$$

利用本征函数向量的辛共轭性质,可以按展开定理写出通解.

当另两侧边为固支、简支或自由时,边界条件的变分方程可按如下方式表达.

(a) 当另两侧边为简支时,边界条件为

$$w = 0, M_y = 0 \quad y = \pm b \text{ 时} \tag{31}$$

用变分方程表示为

$$\int_{-a}^{a}(w\delta\bar{Q}_y + M_y\delta\phi_y)_{y=-b}^{y=b}\mathrm{d}x = 0 \tag{32}$$

(b) 当另两侧边为固支时,边界条件为

$$w = 0, \phi_y = 0 \quad y = \pm b \text{ 时} \tag{33}$$

用变分方程表示为

$$\int_{-a}^{a}(w\delta\bar{Q}_y + \phi_y\delta M_y)_{y=-b}^{y=b}\mathrm{d}x = 0 \tag{34}$$

(c) 当另两侧边为自由时,边界条件为

$$M_y = 0, \bar{Q}_y = 0 \quad y = \pm b \text{ 时} \tag{35}$$

用变分方程表示为

$$\int_{-a}^{a}(\bar{Q}_y\delta w + M_y\delta\phi_y)_{y=-b}^{y=b}\mathrm{d}x = 0 \tag{36}$$

3.2 关于 x 轴反对称

式(18)中只取含 B, D 的解代入式(19),可列出反对称时的方程为

$$\begin{bmatrix} \mu & 1 & 2D(1-v)\mu_1^2 & 0 \\ v\mu_1^2 & \mu & 0 & D(1-v^2)\mu_1^4 - \omega^2 \\ \frac{1}{D} & 0 & \mu & -v\mu_1^2 \\ 0 & 0 & -1 & \mu \end{bmatrix}\times \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{37}$$

和

$$\begin{bmatrix} \mu & 1 & 2D(1-v)\mu_2^2 & 0 \\ v\mu_2^2 & \mu & 0 & D(1-v^2)\mu_2^4 - \omega^2 \\ \frac{1}{D} & 0 & \mu & -v\mu_2^2 \\ 0 & 0 & -1 & \mu \end{bmatrix}\times \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{38}$$

方程(37)的行列式为零,因此可以求解,有

$$\begin{cases} B_1 = [(v - 1)\mu^2 - vR]B_4 \\ B_2 = D\mu[(v - 1)\mu^2 - (v - 2)R]B_4 \\ B_3 = \mu B_4 \end{cases} \tag{39}$$

方程(38)的行列式也是零,其解为

$$\begin{cases} D_1 = [(v - 1)\mu^2 + vR]D_4 \\ D_2 = D\mu[(v - 1)\mu^2 + (v - 2)R]D_4 \\ D_3 = \mu D_4 \end{cases} \tag{40}$$

其中 B_4, D_4 为待定常数.

下面考虑边界条件,这里以固支边界条件为例

$$w = 0, \mathrm{d}w/\mathrm{d}x = 0 \quad x = \pm a \text{ 时} \tag{41}$$

把系数关系式及一般解(只取 B, D 项)代入式(41),整理得

$$\begin{cases} B_4\sin\left(a\sqrt{\mu^2-R}\right)+D_4\sin\left(a\sqrt{\mu^2+R}\right)=0 \\ B_4\sqrt{\mu^2-R}\cos\left(a\sqrt{\mu^2-R}\right)+D_4\sqrt{\mu^2+R}\times \\ \qquad \cos\left(a\sqrt{\mu^2+R}\right)=0 \end{cases} \tag{42}$$

为了使此方程组有解,应有 B_4,D_4 的系数行列式为零,故超越方程为

$$\sqrt{\mu^2-R}\cot\left(a\sqrt{\mu^2-R}\right)=\sqrt{\mu^2+R}\times \cot\left(a\sqrt{\mu^2+R}\right) \tag{43}$$

对不同的 R 值,令 $\mu a=\pm\alpha\pm i\beta$,有 μ, $-\mu$ 都是式(43)的根.

求出本征根后,可得本征根对应的本征函数向量

$$\boldsymbol{\psi}_n=\begin{cases} D\left\{\left[(v-1)\mu_n^2-vR\right]\sin\left(x\sqrt{\mu_n^2-R}\right)-\right. \\ \quad \left[(v-1)\mu_n^2+vR\right]\sin\left(a\sqrt{\mu_n^2-R}\right)\times \\ \quad \left.\sin\left(x\sqrt{\mu_n^2+R}\right)/\sin\left(a\sqrt{\mu_n^2+R}\right)\right\} \\ D\mu_n\left\{\left[(v-1)\mu_n^2-(v-2)R\right]\sin\left(x\sqrt{\mu_n^2-R}\right)-\right. \\ \quad \left[(v-1)\mu_n^2+(v-2)R\right]\sin\left(a\sqrt{\mu_n^2-R}\right)\times \\ \quad \left.\sin\left(x\sqrt{\mu_n^2+R}\right)/\sin\left(a\sqrt{\mu^2+R}\right)\right\} \\ \mu_n\left\{\sin\left(x\sqrt{\mu_n^2-R}\right)-\sin\left(a\sqrt{\mu_n^2-R}\right)\times\right. \\ \quad \left.\sin\left(x\sqrt{\mu_n^2+R}\right)/\sin\left(a\sqrt{\mu^2+R}\right)\right\} \\ \sin\left(x\sqrt{\mu_n^2-R}\right)-\sin\left(a\sqrt{\mu_n^2-R}\right)\times \\ \quad \sin\left(x\sqrt{\mu_n^2+R}\right)/\sin\left(a\sqrt{\mu^2+R}\right) \end{cases} \tag{44}$$

而相应问题(4)的解为

$$\boldsymbol{v}_n=\exp(\mu_n y)\boldsymbol{\psi}_n \tag{45}$$

利用本征函数向量的辛共轭性质,可以按展开定理写出通解.

另两侧边边界条件的变分方程与前面叙述的表达式相同.

4 算例

四边固支的方形各向同性薄板($-a/2\leqslant x\leqslant a/2$, $-b/2\leqslant y\leqslant b/2$, $\nu=0.3$).这里只考虑关于 x 轴对称的情形,按两对边固支的矩形薄板公式(26)计算,对于反对称情形可类似求解.为了验证已知振动频率解,关于 x 轴对称时,第一阶频率 ω,文[19]取 $35.84\ \frac{1}{L^2}\sqrt{\frac{D}{\rho}}$,文[16]取 $36.15\ \frac{1}{L^2}\sqrt{\frac{D}{\rho}}$(其中 L 为边长).这里 $a=b=2$,ω 取成 $9\sqrt{\frac{D}{\rho}}$,则有 $R=3$.超越方程式(25)变为

$$\sqrt{\mu^2-3}\tan\left(\sqrt{\mu^2-3}\right)=\sqrt{\mu^2+3}\times \tan\left(\sqrt{\mu^2+3}\right) \tag{46}$$

表1 对应的非零本征值

Table 1 Corresponding non-zero eigenvalues

n	1	2	3	4	5
$\mathrm{Re}(\mu_n a)$	1.999943	5.34915	8.5346	18.0046	21.1532
$\mathrm{Im}(\mu_n a)$	0.904214	1.52442	1.76516	2.13958	2.22005

从表1中可看出,每一个本征值都有其辛共轭本征值以及它们的复共轭本征值,共4个本征值.从方程(46)可知,非零本征根均为单根.

根据展开定理,得对偶向量的表达式

$$\boldsymbol{v}=\sum_{n=1}^{\infty}\left(f_n v_n+f_{-n}v_{-n}+\bar{f}_n\bar{v}_n+\bar{f}_{-n}\bar{v}_{-n}\right) \tag{47}$$

另两侧边的边界条件即 $y=\pm b/2$ 边固支,两端边界条件的变分式为

$$\int_{-1}^{1}\left(w\delta\bar{Q}_y+\phi_y\delta M_y\right)_{y=-1}^{y=1}\mathrm{d}x=0 \tag{48}$$

求解时在式(47)中取 μ_1,μ_2(即 $n=2$),对于形成的实型正则方程组,$f_i,f_{-i},\bar{f}_i,\bar{f}_{-i}$($i=1,\cdots,n$)有解,故系数行列式为零.验算知 $R=3$ 时可以满足此条件.对于其他频率,可类似处理.为求出各阶频率,需要对各区间分别验证.

5 结论

本文基于哈密顿体系导出一套本征函数向量展开求解问题的方法,将传统自由振动问题进行降阶形成对偶方程组,利用分离变量法把问题转化为本征值问题的求解.通过本征函数、辛正交系、展开求解等手段可以求出通解,然后根据边界条件求出自由振动问题的频率.对各区间分别验证可求出各阶频率.当频率取零时,问题退化为薄板静力问题,有重根存在,通解形式也发生改变.值得提出的是,哈密顿体系下薄板的稳定问题可通过类似方法解决,只是哈密顿算子矩阵有所改变.

算例表明:本文方法所采用的解具有收敛速度快、精度高的特点,算例结果同其他方法解的结果吻合得很好.且本法从理性角度求解,具有很大的优越性.哈密顿体系的一套方法可推广到其他学科

和领域.

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A GENERAL SOLUTION OF FREE VIBRATION FOR RECTANGULAR THIN PLATES IN HAMILTON SYSTEMS*

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Abstract Based on the basic equations of free vibration of thin plate, the Hamilton canonical equations were obtained. By variable selection of moment, equivalent shear force, rotation angel and deflection forming dual variables, the analytical solutions for free vibration of thin plate were obtained under different boundary conditions, which were divided into symmetrical and asymmetrical about x-axial. The computational example of a quadrilateral rectangular plate bending was given, which demonstrated the effectiveness of the proposed method, thus extending the application of Hamilton system.

Key words Hamilton system, thin plate, free vibration, eigenfunction vector

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