求一类非线性偏微分方程精确解的简化试探函数法*

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摘要 利用试探函数法,将一个难于求解的非线性偏微分方程化为一个易于求解的代数方程,然后用待定系数法确定相应的常数,简洁地求得了一类非线性偏微分方程的精确解.将此方法应用到 Burgers 方程、KdV 方程和 KdV - Burgers 方程,所得结果与已有结果完全吻合.本方法可望进一步推广用于求解其它非线性偏微分方程.

关键词 非线性偏微分方程,试探函数法,解析解

引言

随着科学技术的发展,非线性问题的研究在自然科学和社会科学领域的作用越来越重要. 物理、化学、生物、工程技术,甚至经济研究等都存在着大量的非线性问题,这些问题可用非线性常微分方程或非线性偏微分方程来描述,因此,如何求解这些非线性方程成为广大数学和科技工作者致力于研究的一个重要课题.近年来,人们提出和发展了许多求解非线性方程的有效方法,如齐次平衡法[1,2]、双曲正切函数展开法[3~6]、试探函数法[7~9]、非线性变换法[10]、sine ~ cosine 法[11]、Jacobi 椭圆函数展开法[12~14]等等,并用这些方法求解了许多非线性方程.然而,非线性方程(尤其是非线性偏微分方程)的求解是很困难的,而且没有统一而普适的方法,以上一些方法也只能具体应用于某个或某些非线性方程的求解,因此,远不能说解非线性方程的任务已经完成,继续寻找一些有效可行的方法仍是一项十分重要的工作.

作为一种有益的探索和尝试,我们在文[15]中基于“Hopf-Cole 变换法”的思想简洁地求得了一类非线性偏微分方程的精确解析解,在文[16]中利用文献[15]中所引入的一个变换给出了属于这一类方程中最简单的 Burgers 方程的一种直接求解方法.由于这类方程在非线性物理学中的地位和作用十分重要,因此探讨对它的多种解法是很有必要的,这不但具有重要的理论意义,而且具有重要的实际意义.为此,本文利用文献[9]中所提出的“试探函数法”,研究求这类非线性偏微分方程新的解析解.

1 基本思想

考虑如下一类非线性偏微分方程

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\alpha\frac{\partial^2 u}{\partial x^2}+\beta\frac{\partial^3 u}{\partial x^3}+\gamma\frac{\partial^4 u}{\partial x^4}+\cdots+\delta\frac{\partial^n u}{\partial x^n}=0 \tag{1}$$

为了求解上述方程,选取试探函数为

$$u=\frac{a\mathrm{e}^{b\xi}}{(1+\mathrm{e}^{b\xi})^d} \tag{2}$$

其中 $\xi=kx-\omega t$, k(波数), ω(频率)为已知常数, a,b,d 为待定常数.

记 u 的阶数为

$$O(u)=d \tag{3}$$

则易于求得

$$O(\frac{\partial u}{\partial x})=d+1 \tag{4}$$

$$O(\frac{\partial^2 u}{\partial x^2})=d+2 \tag{5}$$

$$\cdots$$

$$O(\frac{\partial^n u}{\partial x^n})=d+n \tag{6}$$

令方程(1)中的最高阶偏导数项与其最高幂次的非线性项平衡得

$$O(\frac{\partial^n u}{\partial x^n})=O(u\frac{\partial u}{\partial x}) \tag{7}$$

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即

$$d + n = d + d + 1 \tag{8}$$

由上式得

$$d = n - 1 \tag{9}$$

确定了 u 的阶数d后,再将式(2)代入式(1)就可根据具体的非线性偏微分方程确定 a,b,从而得到所求方程的解,下面根据这个思想来具体求解几个非线性偏微分方程.

2 应用举例

2.1 Burgers 方程

一维冲击波传播可用 Burgers 方程来描述,它的一般形式为

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \alpha\frac{\partial^2 u}{\partial x^2} = 0 \tag{10}$$

这里 $n = 2$,由式(9)得 $d = 1$,因此它的试探函数为

$$u = \frac{a e^{b\xi}}{1 + e^{b\xi}} \tag{11}$$

根据上式容易求得

$$\frac{\partial u}{\partial t} = \frac{-ab\omega e^{b\xi}}{(1 + e^{b\xi})^2} \tag{12}$$

$$\frac{\partial u}{\partial x} = \frac{abk e^{b\xi}}{(1 + e^{b\xi})^2} \tag{13}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{ab^2k^2(e^{b\xi} - e^{2b\xi})}{(1 + e^{b\xi})^3} \tag{14}$$

将式(11)～式(14)代入式(10)得下列代数方程

$$-(\omega + \alpha bk^2)e^{b\xi} + (ak + \alpha bk^2 - \omega)e^{2b\xi} = 0 \tag{15}$$

显然 $e^{b\xi} \neq 0$,故由上式得

$$\omega + \alpha bk^2 = 0 \tag{16}$$

$$ak + \alpha bk^2 - \omega = 0 \tag{17}$$

又由文献[15]知,k,ω 之间有下列关系

$$\omega = -\alpha k^2 \tag{18}$$

由式(16)～式(18)解得

$$a = -2\alpha k \qquad b = 1 \tag{19}$$

将式(19)代入式(11)得

$$u = -\frac{2\alpha k e^{\xi}}{1 + e^{\xi}} = -\frac{2\alpha k e^{(kx-\omega t)}}{1 + e^{(kx-\omega t)}} \tag{20}$$

利用双曲正切函数的定义,可将式(20)化为

$$u = -\alpha k\left[1 + \tanh\frac{1}{2}(kx - \omega t)\right] \tag{21}$$

2.2 KdV 方程

KdV 方程的一般形式为

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} = 0 \tag{22}$$

这里 $n = 3$,由式(9)得 $d = 2$,因此它的试探函数为

$$u = \frac{a e^{b\xi}}{(1 + e^{b\xi})^2} \tag{23}$$

根据上式不难求得

$$\frac{\partial u}{\partial t} = \frac{ab\omega(e^{2b\xi} - e^{b\xi})}{(1 + e^{b\xi})^3} \tag{24}$$

$$\frac{\partial u}{\partial x} = \frac{abk(e^{b\xi} - e^{2b\xi})}{(1 + e^{b\xi})^3} \tag{25}$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{ab^3k^3(e^{b\xi} - 11e^{2b\xi} + 11e^{3b\xi} - e^{4b\xi})}{(1 + e^{b\xi})^5} \tag{26}$$

将式(23)～式(26)代入式(22)得下列代数方程

$$(\beta b^2k^3 - \omega)e^{b\xi} + (ak - \omega - 11\beta b^2k^3)e^{2b\xi} - (ak - \omega - 11\beta b^2k^3)e^{3b\xi} - (\beta b^2k^3 - \omega)e^{4b\xi} = 0 \tag{27}$$

显然 $e^{b\xi} \neq 0$,故由上式得

$$\beta b^2k^3 - \omega = 0 \tag{28}$$

$$ak - \omega - 11\beta b^2k^3 = 0 \tag{29}$$

又由文献[15]知,k,ω 之间有下列关系

$$\omega = 4\beta k^3 \tag{30}$$

由式(28)～式(30)解得

$$a = 48\beta k^2 \qquad b = 2 \tag{31}$$

将式(31)代入式(23)得

$$u = \frac{48\beta k^2 e^{2\xi}}{(1 + e^{2\xi})^2} = \frac{48\beta k^2 e^{2(kx-\omega t)}}{[1 + e^{2(kx-\omega t)}]^2} \tag{32}$$

利用双曲正割函数的定义,可将(32)式化为

$$u = 12\beta k^2 \mathrm{sech}^2(kx - \omega t) \tag{33}$$

2.3 KdV-Burgers 方程

在工程上某些问题可以用 KdV-Burgers 方程来描述,例如,无碰撞等离子体的冲击波研究. KdV-Burgers 的一般形式为

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} = 0 \tag{34}$$

为了求得方程(3)的与实际情况相符的解,我们对前面的试探函数形式(2)作一点改动,取为

$$u = u_0 + \frac{a e^{b\xi}}{(1 + e^{c\xi})^d} \tag{35}$$

其中 a,b,c,d 为待定常数,d 仍可根据前面的平衡法求得为2,即

$$u = u_0 + \frac{a e^{b\xi}}{(1 + e^{c\xi})^2} \tag{36}$$

根据上式容易求得

$$\frac{\partial u}{\partial t} = -\frac{a\omega[b e^{b\xi} + (b - 2c)e^{(b+c)\xi}]}{(1 + e^{c\xi})^3} \tag{37}$$

$$\frac{\partial u}{\partial x} = \frac{ak[b e^{b\xi} + (b - 2c)e^{(b+c)\xi}]}{(1 + e^{c\xi})^3} \tag{38}$$

$$\frac{\partial^2 u}{\partial x^2} = ak^2[b^2 e^{b\xi} + (2b^2 - 4bc - 2c^2)e^{(b+c)\xi} + (b^2 - 4bc + 4c^2)e^{(b+c)\xi}]/(1 + e^{c\xi})^4 \tag{39}$$

$$\frac{\partial^3 u}{\partial x^3} = ak^3[b^3 e^{b\xi} + (3b^3 - 10b^2 c - 2bc^2 - 2c^3)e^{(b+c)\xi}]/(1 + e^{c\xi})^5 + ak^3[(3b^3 - 12b^2 c + 10bc^2 + 14c^3)e^{(b+2c)\xi} + (b^3 - 6b^2 c + 12bc^2 - 8c^3) \times e^{(b+3c)\xi}]/(1 + e^{c\xi})^5 \tag{40}$$

仿照前面类似的方法,并借助于数学符号计算软件 Mathematica 可求得

$$a = -\frac{12\alpha^2}{25\beta}, b = 0, c = -\frac{\alpha}{5\beta k}, u_0 = \frac{12\alpha^2}{25\beta} \tag{41}$$

将式(41)代入式(36)得

$$u = \frac{12\alpha^2}{25\beta} - \frac{12\alpha^2}{25\beta}\left[\frac{1}{1 + e^{-\frac{\alpha\xi}{5\beta k}}}\right]^2 \tag{42}$$

利用公式

$$\frac{1}{1 + e^x} = \frac{1}{2}(1 - \tanh\frac{x}{2}),$$
$$\tanh(-x) = -\tanh x \tag{43}$$

式(42)化为

$$u = \frac{3\alpha^2}{25\beta}\{4 - [1 + \tanh\frac{\alpha}{10\beta k}(kx - \omega t)]^2\} \tag{44}$$

以上一些结果与文献[15]的结果完全一样,说明本文的方法是可行的.

3　结论

试探函数法是一种行之有效的用于求解非线性偏微分方程的好方法,本文对已有的试探函数法进行了一些简化的改进,从而简洁地求得了一类非线性偏微分方程的精确解析解.这种方法有望进一步推广用于求解其它非线性偏微分方程.

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A SIMPLIFIED TRIAL FUNCTION METHOD FOR SEEKING THE EXACT SOLUTIONS TO A CLASS OF NONLINEAR PDES *

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Abstract By utilizing the trial function method, a class of nonlinear partial differential equations (PDEs for short) that are hard to be solved by the usual ways can be reduced to a set of algebraic equations, which can be easily solved, and their related coefficients can be easily determined by the undetermined coefficients method. Then, the exact analytical solutions to the class of nonlinear PDEs were successfully derived. Moreover, the method was applied to the Burgers equation, the KdV equation and the KdV－Burgers equation and the results were in very good agreement with those given in the reference. The method may be generalized to construct the solutions of other nonlinear PDEs.

Key words nonlinear partial differential equation, trial function method, analytical solution

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