陀螺系统微振动模态摄动分析与灵敏度计算*

刘　涛[1]　邓子辰[1,2]

(1.西北工业大学工程力学系,西安　710072)(2.大连理工大学工业装备结构分析国家重点实验室,大连　116024)

摘要　在状态空间下,将线性陀螺系统微振动问题导向哈密顿体系,可以得到一组加权共轭辛正交关系和模态展开定理.利用这种特点构造了陀螺系统模态摄动计算式与灵敏度计算式,从而解决了拉格朗日体系下陀螺系统模态摄动分析与灵敏度计算的困难,算例显示了文中计算方法的有效性.

关键词　陀螺系统,模态摄动分析,灵敏度计算

引言

惯性动力系统的模态摄动分析与灵敏度计算是结构动力优化设计与动力重分析的重要基础,这方面的研究工作一般集中在非转子动力系统,且在工程实践中已多有应用[1~3].而在陀螺系统微振动方面,由于模态分析和计算上的困难,这方面的工作显得不足.

众多学者对陀螺系统自由振动的计算进行了研究,比较有特色的有 Merovitch 的工作[4],他从状态空间出发,将特征向量实部与虚部分开,从而使问题归结为两个广义特征值方程.这一方法虽然简单但利用它构造模态摄动式与灵敏度计算式还存在诸多不便.钟万勰等人将陀螺系统自由振动问题导向哈密尔顿体系,建立了加权共轭辛正交关系,并得到了一系列模态计算列式[5].文献[6]将这一套计算体系推广到了小阻尼转子系统,得到了较理想的结果.

本文在已有工作的基础之上进行了推进,利用哈密顿体系下陀螺系统存在的加权共轭辛正交关系构造了陀螺系统模态摄动计算式与灵敏度计算式.该计算式表达简洁,算例表明计算精度亦能令人满意.

1　转子系统自由振动方程与加权共轭辛正交关系

无阻尼陀螺系统自由振动方程可写为

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{G}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = 0 \tag{1}$$

其中 $\boldsymbol{M},\boldsymbol{G},\boldsymbol{K}$ 分别是质量阵、陀螺项、刚度阵;$q(t)$ 为广义位移向量.

当陀螺系统自由振动问题导向哈密尔顿体系时,可以得到另一类特征值表达式

$$\boldsymbol{L}\dot{\boldsymbol{w}}_t = \boldsymbol{H}\boldsymbol{w}_t,\ \boldsymbol{L} = \begin{bmatrix} \boldsymbol{I}_n & \boldsymbol{0} \\ \boldsymbol{G}/2 & \boldsymbol{M} \end{bmatrix},$$

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I}_n \\ -\boldsymbol{K} & -\boldsymbol{G}/2 \end{bmatrix},$$

$$\boldsymbol{w}_t = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}\quad \boldsymbol{w}_t = \boldsymbol{w}e^{\mu t}, \tag{2}$$

相应特征方程为 $\mu \boldsymbol{L}\boldsymbol{w} = \boldsymbol{N}\boldsymbol{w}$.引入反对称单位阵 $\boldsymbol{J}$ 后,可得到

$$\boldsymbol{L}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{L} = \begin{bmatrix} \boldsymbol{G} & \boldsymbol{M} \\ -\boldsymbol{M} & \boldsymbol{0} \end{bmatrix},$$

$$\boldsymbol{N}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{N} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{K} \\ -\boldsymbol{K} & -\boldsymbol{G} \end{bmatrix},$$

$$\boldsymbol{L}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{L} = -\boldsymbol{N}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{L} = \begin{bmatrix} -\boldsymbol{K} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{M} \end{bmatrix} \tag{3}$$

进一步有如下辛正交关系以及共轭加权辛归一关系[5]

$$\begin{cases} \boldsymbol{w}_i^{\mathrm{T}}(\boldsymbol{L}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{L})\boldsymbol{w}_j = 0 \\ \boldsymbol{w}_i^{\mathrm{T}}(\boldsymbol{N}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{L})\boldsymbol{w}_j = 0 \\ \boldsymbol{w}_j^{\mathrm{T}}(\boldsymbol{L}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{N})\boldsymbol{w}_i = 0 \end{cases} \quad \mu_i \neq -\mu_j \tag{4}$$

$$\begin{cases} \boldsymbol{w}_i^{\mathrm{T}}(\boldsymbol{L}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{L})\boldsymbol{w}_j = 1 \\ \boldsymbol{w}_i^{\mathrm{T}}(\boldsymbol{N}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{L})\boldsymbol{w}_j = \mu_i \end{cases} \quad \mu_i \neq -\mu_j \tag{5}$$

式(5)中 μ_i,μ_j 为互为共轭的特征值,可以按下式编排

a) $\mathrm{Im}\,(\mu_i) \geqslant 0, i = 1,2,\ldots,n$

b) $\mathrm{Im}\,(\mu_{n+i}) \leqslant 0, \mu_i = -\mu_j, i = 1,2,\ldots,n$　　(6)

另外,改变特征向量比例可使

$$\boldsymbol{w}_i^{\mathrm{T}}\boldsymbol{w}_i = \boldsymbol{w}_{n+i}^{\mathrm{T}}\boldsymbol{w}_{n+i} \tag{7}$$

2004-04-13 收到第 1 稿,2004-10-20 收到修改稿.

* 国家自然科学基金(10372084)、高校博士点基金(20010699016)、陕西省自然科学基金(2002A17)及大连理工大学工业装备结构分析国家重点实验室开发基金资助项目

2　模态摄动计算

由特征方程 $\mu L w = N w$,当 N,L 发生摄动:$N = N_0 + \varepsilon N_1$,$L = L_0 + \varepsilon L_1$,其中 ε 为小参数.摄动后特征方程变为

$$\mu_i(L_0 + \varepsilon L_1) w_i = (N_0 + \varepsilon N_1) w_i \tag{8}$$

其中

$$\mu_i = \mu_{0i} + \varepsilon\mu_{1i} + \varepsilon^2\mu_{2i} + \cdots$$
$$w_i = w_{0i} + \varepsilon w_{1i} + \varepsilon^2 w_{2i} + \cdots \tag{9}$$

将式(8)和(9)代入式(7),可得

$$(\mu_{0i} + \varepsilon\mu_{1i} + \varepsilon^2\mu_{2i} + \cdots)(L_0 + \varepsilon L_1)(w_{0i} + \varepsilon w_{1i} + \varepsilon^2 w_{2i} + \cdots) = (N_0 + \varepsilon N_1)(w_{0i} + \varepsilon w_{1i} + \varepsilon^2 w_{2i} + \cdots) \tag{10}$$

将式(10)展开后,取 ε 同次幂相等,有

$$1):\varepsilon^0:\mu_{0i} L_0 w_{0i} = N_0 w_{0i} \tag{11}$$

$$2):\varepsilon^1:\mu_{0i} L_1 w_{0i} + \mu_{0i} L_0 w_{1i} + \mu_{1i} L_0 w_{0i} = N_0 w_{1i} + N_1 w_{0i} \tag{12}$$

…

这样对于一阶项剩下的工作就是求出 w_{1i} 和 μ_{1i} 的计算式了.对式(12)左乘 $w_{0n+i}^T L_0^T J$

$$\mu_{0i} w_{0n+i}^T L_0^T J L_1 w_{0i} + \mu_{0i} w_{0n+i}^T L_0^T J L_0 w_{1i} + \mu_{1i} w_{0n+i}^T L_0^T J L_0 w_{0i} = w_{0n+i}^T L_0^T J N_0 w_{1i} + w_{0n+i}^T L_0^T J N_1 w_{0i} \tag{13}$$

考虑以摄动前模态特征向量为基底,将 w_{1i}, w_{1n+i} 按下式展开

$$w_{1i} = \sum_{k=1}^{n}(a_k w_{0j} + b_k w_{0n+j}) \tag{14}$$

$$w_{1n+i} = \sum_{k=1}^{n}(a_k^* w_{0k} + b_k^* w_{0n+k}) \tag{15}$$

结合式(5)的正交关系,可得

$$\mu_{1i} = \mu_{0i} w_{0n+i}^T L_0^T J L_1 w_{0i} - w_{0n+i}^T L_0^T J N_1 w_{0i} \tag{16}$$

这样就得到了 μ_{1i} 的计算式,同理也能得到 μ_{1n+i} 的计算式.

对式(12)左乘 $w_{0k}^T L_0^T J(k = 1,\cdots,n)$ 并结合式(14),(4)可得

$$b_k = \frac{w_{0k}^T L_0^T J N_1 w_{0i} - \mu_{0i} w_{0k}^T L_0^T J L_1 w_{0i}}{(\mu_{0i} + \mu_{0k})} \tag{17}$$

同理

$$b_k^* = \frac{w_{0k}^T L_0^T J N_1 w_{0n+i} - \mu_{0n+i} w_{0k}^T L_0^T J L_1 w_{0n+i}}{(\mu_{0n+i} + \mu_{0k})} \quad (k = 1,\cdots,n, k \neq i) \tag{18}$$

显然改变左乘项可以依次得到式(14)中 $2n-1$ 个系数,对式(12)左乘 $w_{0n+k}^T L_0^T J(k = 1,\ldots,n, k \neq i)$ 得

$$a_k = \frac{w_{0n+k}^T L_0^T J N_1 w_{0i} - \mu_{0i} w_{0n+k}^T L_0^T J L_1 w_{0i}}{(\mu_{0k} - \mu_{0i})} \tag{19}$$

$$a_k^* = \frac{w_{0n+k}^T L_0^T J N_1 w_{0n+i} - \mu_{0n+i} w_{0n+k}^T L_0^T J L_1 w_{0n+i}}{(-\mu_{0n+k} - \mu_{0n+i})} \quad (k = 1,\cdots,n) \tag{20}$$

当 $a_k = a_i$ 时,由式(5)可知

$$(w_{0n+i}^T + \varepsilon w_{1n+i}^T + \cdots)[(L_0^T + \varepsilon L_1^T) J (L_0 + \varepsilon L_1)](w_{0i} + \varepsilon w_{1i} + \cdots) = -1 \tag{21}$$

展开上式,取同次幂相等并结合式(5)有

$$a_i + b_i^* = w_{0n+i}^T L_0^T J L_1 w_{0i} + w_{0n+i}^T L_1^T J L_0 w_{0i} \tag{22}$$

由式(7)有

$$(w_{0i}^T + \varepsilon w_{1i}^T + \cdots)(w_{0i} + \varepsilon w_{1i} + \cdots) = (w_{0n+i}^T + \varepsilon w_{1n+i}^T + \cdots) \times (w_{0n+i} + \varepsilon w_{1n+i} + \cdots)$$

取 ε 二阶幂相等,并引入式(14),(15),可得下列方程

$$b_i^* - a_i = [(b_i - a_i^*) w_{0i}^T w_{0n+i} + \sum_{\substack{k=1\\k\neq i}}^{n} w_{0i}^T (a_k w_{0k} + b_k w_{0n+k}) - \sum_{\substack{k=1\\k\neq i}}^{n} w_{0n+i}^T (a_k^* w_{0k} + b_k^* w_{0n+k})] / w_{0i}^T w_{0i} \tag{23}$$

联立式(22)与(23)可以解出 a_i, b_i^*.

显然对于式(12)中更高阶 ε 项展开式,可以用类似的方法得到未知项计算式.

3　灵敏度分析

由特征方程 $\mu_i L w_i = N w_i$ 得

$$\left(\frac{\partial N}{\partial d_j} - \frac{\partial \mu_i}{\partial d_j} L - \mu_i \frac{\partial L}{\partial d_j}\right) w_i + (N - \mu_i L)\frac{\partial w_i}{\partial d_j} = 0 \tag{24}$$

其中 d_j 为设计变量,可以是转子几何设计尺寸,材料特性等.式(24)左乘 $w_{n+i}^T L^T J$,并令 $B_{n+i} = w_{n+i}^T \left(L^T J \frac{\partial N}{\partial d_j} - \mu_i L^T J \frac{\partial L}{\partial d_j}\right) w_i$ 得

$$\frac{\partial \mu_i}{\partial d_j} = B_{n+i} - w_{n+i}^T (L^T J N - \mu_i L^T J L)\frac{\partial w_i}{\partial d_j} \tag{25}$$

显然,要求出 $\frac{\partial \mu_i}{\partial d_j}$ 必须先计算出 $\frac{\partial w_i}{\partial d_j}$.

式(24)左乘 $w_k^T L^T J$ 可得

$$w_k^{\mathrm T}(L^{\mathrm T}J\frac{\partial N}{\partial d_j}-\frac{\partial \mu_i}{\partial d_j}L^{\mathrm T}JL-\mu_i L^{\mathrm T}J\frac{\partial L}{\partial d_j})w_i+$$

$$w_k^{\mathrm T}(L^{\mathrm T}JN-\mu_i L^{\mathrm T}JL)\frac{\partial w_i}{\partial d_j}=0 \qquad (26)$$

令 $B_k=w_k^{\mathrm T}(L^{\mathrm T}J\frac{\partial N}{\partial d_j}-\mu_i L^{\mathrm T}J\frac{\partial L}{\partial d_j})w_i$，仍然用基底展开 $\frac{\partial w_i}{\partial d_j}$，即

$$\frac{\partial w_i}{\partial d_j}=\sum_{k=1}^{n}(g_k w_k+f_k w_{n+k}) \qquad (27)$$

$$\frac{\partial w_{n+i}}{\partial d_j}=\sum_{k=1}^{n}(g_k^* w_k+f_k^* w_{n+k}) \qquad (28)$$

得

$$f_k=\frac{B_k}{\mu_i+\mu_k} \quad (k=1,\cdots,n) \qquad (29)$$

式(24)左乘 $w_{n+k}^{\mathrm T}L^{\mathrm T}J$，$B_{n+k}=w_{n+k}^{\mathrm T}(L^{\mathrm T}J\frac{\partial N}{\partial d_j}-\mu_i L^{\mathrm T}J\frac{\partial L}{\partial d_j})w_i$，则

$$g_k=\frac{-B_{n+k}}{\mu_{n+k}+\mu_i} \quad (k=1,\cdots,n,k\neq i) \qquad (30)$$

同理可以依次得到式(27)，(28)中 $2n-1$ 个系数.

当 $g_k=g_i$ 时由式(5)可得

$$\frac{\partial w_i^{\mathrm T}}{\partial d_j}(L^{\mathrm T}JL)w_{n+i}+w_i^{\mathrm T}\frac{\partial(L^{\mathrm T}JL)}{\partial d_j}w_{n+i}+$$

$$w_i^{\mathrm T}(L^{\mathrm T}JL)\frac{\partial w_{n+i}}{\partial d_j}=0 \qquad (31)$$

由式(27)，(28)得

$$g_i+f_i^*=-w_i^{\mathrm T}\frac{\partial L^{\mathrm T}}{\partial d_j}LLw_{n+i}-$$

$$w_i^{\mathrm T}L^{\mathrm T}J\frac{\partial L}{\partial d_j}w_{n+i} \qquad (32)$$

由式(7)可得

$$\frac{\partial w_i^{\mathrm T}}{\partial d_j}w_i=\frac{\partial w_{n+i}^{\mathrm T}}{\partial d_j}w_{n+i} \qquad (33)$$

从而有

$$g_i+f_i^*=[\sum_{\substack{k=1\\k\neq i}}^{n}(g_k^* w_k^{\mathrm T}+f_k^* w_{n+k}^{\mathrm T})w_{n+i}+$$

$$(g_i^*-f_i)w_i^{\mathrm T}w_{n+i}-$$

$$\sum_{\substack{k=1\\k\neq i}}^{n}(g_k w_k^{\mathrm T}+f_k w_{n+k}^{\mathrm T})w_i]/w_i^{\mathrm T}w_i \qquad (34)$$

联立(32)与(34)可解出 g_i,f_i^*.

4 算例

考虑有如下陀螺系统

$$M=m^2 I_2, G=2m^2\theta J, K=m^2\theta^2\begin{bmatrix}1&0\\0&2\end{bmatrix}$$

其中 θ 为系统旋转角速度，$m^2=1\ \mathrm{kg}^2$，$\theta=2.5$ rad/s. 相应特征解见表1.

表1 系统的特征解

Table 1 The eigen-solutions of the system

μ	6.4718i	−6.4718i	1.3657i	−1.3657i
	0.140	0.140	−1.140	−1.140
w	−0.156i	0.156i	−0.732i	0.732i
	0.908i	−0.908i	−1.56i	1.56i
	1.00	1.00	1.00	1.00

1) 考虑系统 m^2 变为1.1025，显然改变 m^2，不影响模态. 此时有

$$\varepsilon L_1=L-L_0=$$

$$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0.256&0.103&0\\-0.256&0&0&0.103\end{bmatrix}$$

$$\varepsilon N_1=N-N_0=$$

$$\begin{bmatrix}0&0&0&0\\0&0&0&0\\-0.641&0&0&-0.256\\0&-1.281&0.256&0\end{bmatrix}$$

代入式(15)得一阶摄动量见表2.

表2 1阶摄动量

Table 2 One-order perturbation

$\varepsilon\mu_1$	−1.2190e−007	−1.2190e−007	3.9270e−006	3.9270e−006
	−0.0079−0.0409i	−0.0079−0.0409i	−0.986i	−0.986i
εw_1	0.0132−0.0047i	0.0132−0.0047i	−0.735	−0.735
(1.0e−005)	0.1447−0.0174i	0.1447−0.0174i	−1.458	−1.458
	0.0137−0.0701i	0.0137−0.0701i	0.782i	0.782i

上表可以看出仅考虑一阶摄动时，m 摄动量引起的特征向量、特征值的改变是可以忽略的，这和上面分析的结果达成了一致.

2) 计算 $\frac{\partial \mu}{\partial \theta},\frac{\partial w}{\partial \theta}$

$$\frac{\partial N}{\partial \theta} = \begin{bmatrix} 0 & 0 \\ -\frac{\partial k}{\partial \theta} & -\frac{\partial G}{2\partial \theta} \end{bmatrix},$$

$$\frac{\partial L}{\partial \theta} = \begin{bmatrix} 0 & 0 \\ \frac{\partial L}{2\partial \theta} & 0 \end{bmatrix}$$

进而可以得到$\frac{\partial \mu}{\partial \theta}$和$\frac{\partial w}{\partial \theta}$,见表 3.

表 3　$\frac{\partial \mu}{\partial \theta}$和$\frac{\partial w}{\partial \theta}$的计算值

Table 3　The numerical results of $\frac{\partial \mu}{\partial \theta}$ and $\frac{\partial w}{\partial \theta}$

$\frac{\partial \mu}{\partial \theta}$	- 0.1403 + 0.0002i	- 0.1403 + 0.0002i	1.1403 - 0.6229i	1.1403 - 0.6229i
$\frac{\partial w}{\partial \theta}$	- 1.1374i	- 1.1374i	- 3.0744i	- 3.0744i
	- 0.8416	0.8416	1.9930	- 1.9930
	- 1.8758	1.8758	3.9351	- 3.9351
	1.0754i	1.0754i	2.4577i	2.4577i

5　结论

本文利用哈密尔顿体系下陀螺系统存在的加权共轭辛正交关系构造了陀螺系统模态摄动计算式与灵敏度计算式,算例证明了该式的有效性.传统的模态摄动分析与灵敏度计算一般都是在拉格朗日体系下进行的,解决了一些工程实际问题[1~3].但在很多情况下,拉格朗日体系下推导的计算列式并不总令人满意,如重频的灵敏度分析问题等.从哈密尔顿体系入手寻求这些难题的解决方案是一个努力方向,本例只是哈密尔顿体系下模态摄动分析与灵敏度计算的一个尝试,进一步的工作还将在重频的灵敏度分析,复模态求导等方面展开.

参　考　文　献

1　林家浩.结构动力优化中的灵敏度分析.振动与冲击,1985,3(1):1~6(Lin Jiahao. Sensitivity analysis in structural dynamic optimization. *Journal of Vibration and Shock*,1985,3(1):1~6(in Chinese))

2　林家浩.结构动力优化设计发展综述.力学进展,1983,13 (1):423~431(Lin Jiahao. Review on the development in structural dynamic optimum design. *Advances In Mechanics*,1983,13 (1):423~431(in Chinese))

3　张令弥等.特征向量导数计算各种模态法的比较和发展.应用力学学报,1994,11(3):68~73(Zhang Lingmi. Calculation of eignvectors derivatives using modal methods: assessment and advance. *Chin Journ of Appl Mech*,1994,11(3):68~73(in Chinese))

4　Meirovitch L. Computational Methods in Structural Dynamics. Sijthoff & Noordhoff, Netherands: Alphen ann den Rijn,1980

5　钟万勰等.陀螺系统与反对称矩阵辛本征解的计算.计算结构力学及其应用,1993,10(3):237~253(Zhong Wangxie etc. Computation of gyroscopic system and the symplectic eignsolution of anti - symmetric matrix. *Computational Structural Mechanics And Applications*,1993,10(3):237~253(in Chinese))

6　欧阳华江,邓子辰.小阻尼转子系统动力学的本征解.应用力学学报,199613(1):108~112(Ouyang Huajiang, Deng Zichen. Dynamic eigensolution for slightly damped rotor system. *Chin Journ of Appl Mech*,1996 13(1):108~112(in Chinese))

MODAL PERTURBATION ANALYSIS AND SENSIBILITY COMPUTATION FOR LINEAR GYROSCOPIC SYSTEM *

Liu Tao[1] Deng Zichen[1,2]

(1. *Department of Engineering Mechanics, Northwestern Polytechnical University, Xi'an* 710072, *China*)

(2. *State Key Laboratory of Structural Analysis of Industrial Equipment, Dalian University of Technology, Dalian* 116023, *China*)

Abstract Based on the dynamic equation of linear gyroscopic system, the differential equation in Lagrange system was transformed into Hamilton system, and then the weighted adjoint symplectic orthogonal relations between the eigenvectors and the expansion theorem for arbitrary state vector were given in state space. And based on the above relations, the equations for modal perturbation analysis and the sensibility computation of eigenvectors were established, and a new effective algorithm for modal perturbation analysis and the sensibility computation was proposed, which can eliminate the traditional difficulties in perturbation analysis and sensibility computation in Lagrange system. An example showed the effectiveness of the numerical method.

Key words gyroscopic system, modal perturbation analysis, sensibility computation

Received 13 April 2004, revised 20 October 2004.

* The project supported by the National Natural Science Foundation of China(10372084); Doctoral Program Foundation of Education Ministry of China(20010699016); Natural Science Foundation of Shanxi Province of China(2002A17) and Open Foundation of State Key Laboratory of Structural Analysis of Industrial Equipment of Dalian University of Technology.